

Module 1

Introduction to signals & systems - classification of signals - Basic operations on signals - Elementary signals - Concept of system - Properties of systems - Stability, inevitability - time invariance - Linearity - Causality - memory - Convolution - Impulse response - Representation of LTI systems - Differential equation representations of LTI systems.

Signals

A signal may be a function of time, temperature, position, pressure, distance etc.

A function of one or more independent variables which contains some information is called a signal. In electrical sense, the signal can be voltage or current. The voltage or current is the function of time as an independent variable.

Classification of Signals

Based upon their nature and characteristics in time domain, signals broadly classified as

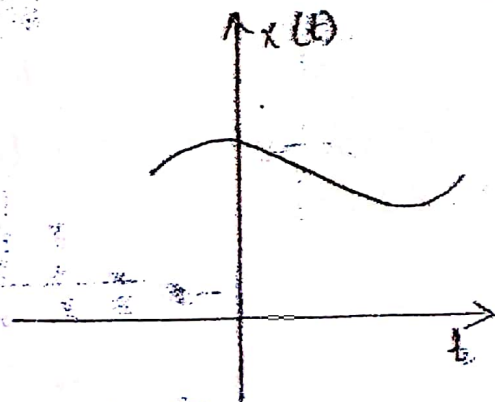
- A) Continuous - time signal
- B) Discrete - time signal

- A continuous time signal is a signal whose amplitude varies continuously with time. i.e., both the amplitude & time are continuous over their respective intervals.

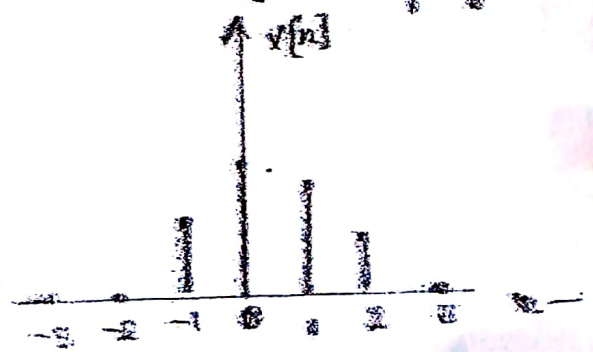
- Continuous time signal arise naturally when a physical waveform like light wave or acoustic wave is converted into an electrical signal.

- Discrete - time signal is defined at discrete intervals of time.

- A discrete - time signal $x[n]$ can be derived from continuous time signal by sampling it at a fixed interval.
 Eg: Rainfall in all the months of a year, number of goals produced by a company in a year.



(Continuous-time signal)



(discrete-time signal)

- If a discrete time signal $x(n)$ can take on only a finite number of distinct values, the signal is digital signal.

C. Deterministic and Random signals.

Deterministic

— Deterministic signals are those signals whose can be completely specified in time.

— Signals are deterministic if their behaviour is known and can be represented eg. by a formula.

The deflection voltage of an oscilloscope is a deterministic signal, for its behaviour is known and can be represented by a sawtooth wave.

— Random (Stochastic signal)

We cannot define the amplitude values of a voice signal by means of formulae or graphical elements in addition their continued behaviour is not known.

It is impossible to specify their behaviour in terms of functions, such signals are described by expected values

Such as Mean, Variance & many others.

1' Representation of Discrete-time Signals

- * Graphical Representation
- * Functional Representation

$$x[n] = \begin{cases} \left(\frac{1}{2}\right)^n & ; n \geq 0 \\ 0 & ; n < 0 \end{cases}$$

- * Tabular Representation

n	-3	-2	-1	0	1	2
x[n]	0	-1	0	1	2	0

- * Sequence Representation

$$x[n] = \{ \dots, 0, 1, 2, 1, 0, 2, 0, 0, \dots \}$$

↑ denotes $n = 0$ term

When $n = 0$ is indicated, first term corresponds to $n = 0$

B Analog & Digital Signals

- If a continuous time signal $x(t)$ can take on any values in the continuous interval (a, b) where a may be $-\infty$ and b may be $+\infty$, then the continuous time signal $x(t)$ is called Analog signal.

D.

(3)

D. Even & odd signals

DEPARTMENT OF ELECTRICAL AND ELECTRONICS ENGINEERING

A signal $x(t)$ or $x(n)$ referred to as even signal if

$$x(-t) = x(t)$$

$$x(-t), x(t)$$

$$x(-n) = x(n)$$

$$x(-n) - -x(n)$$

A signal $x(t)$ or $x(n)$ referred to as odd signal if

$$x(-t) = -x(t)$$

$$x(-n) = -x(n)$$

Consider an arbitrary continuous-time signal $x(t)$
To find even & odd parts of this signal

$$x(t) = x_e(t) + x_o(t)$$

$$x(-t) = x_e(-t) + x_o(-t)$$

$$x(t) + x(-t) = 2x_e(t)$$

$$x_e(t) = \frac{1}{2} [x(t) + x(-t)]$$

$$x_o(t) = \frac{1}{2} [x(t) - x(-t)]$$

1) Find even & odd components of

$$x(t) = e^{jt}$$

$$x_e(t) = \frac{1}{2} [x(t) + x(-t)] = \frac{1}{2} [e^{jt} + e^{-jt}] = \cos t$$

$$x(t) - x(-t) = \frac{1}{2} [x(t) - x(-t)] = \frac{1}{2} [e^{j\pi t} - e^{-j\pi t}] = j \sin t$$

2] Show that the product of two even signals or of two odd signals is an even signal while the product of an even & odd signal is an odd signal.

$$\text{even} \cdot \text{even} = \text{even} \quad \text{odd} \cdot \text{odd} = \text{even}$$

$$(1)h \cdot (1)h = (1)h$$

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$$(1)h \cdot (1)h = (1)h$$

F. Periodic & Non periodic signals

→ A continuous-time signal $x(t)$ is said to be periodic with period T , if there is a positive non zero value of T for which

$$x(t+T) = x(t) \text{ for all } t$$

→ A signal which repeats after every time interval T is called a periodic signal. A signal $x(t)$ is called periodic if & only if $x(t+T) = x(t)$ for all ' t '.

' T ' denotes time & ' T ' is a constant.

' T ' is the fundamental period. or $f = \frac{1}{T}$.

Angular frequency, $\omega = 2\pi f = \frac{2\pi}{T}$.

For Discrete-time signal; $x[n]$ is said to be periodic Satisfies the condition;

$$x(n+N) = x[n] \text{ for all integers } n$$

Non-periodic signals

- A signal $x(t)$ for which there is no value of ' T '

Satisfying the condition $x(t+T) = x(t)$ is called

Non-periodic or aperiodic signal.

- A signal is said to be Non-periodic, if it doesn't repeat.

1. The sum of two continuous-time periodic signals may not be periodic.

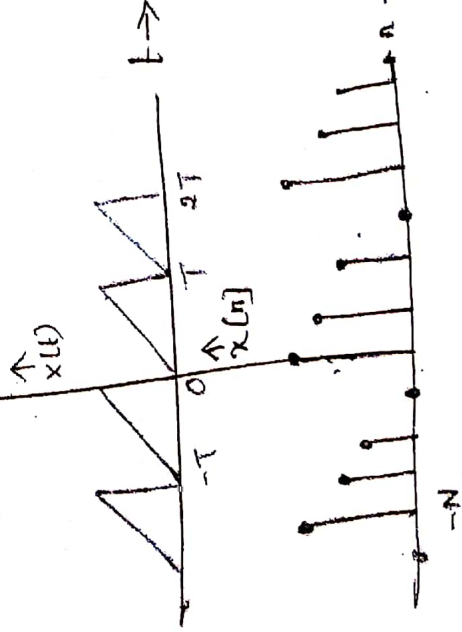
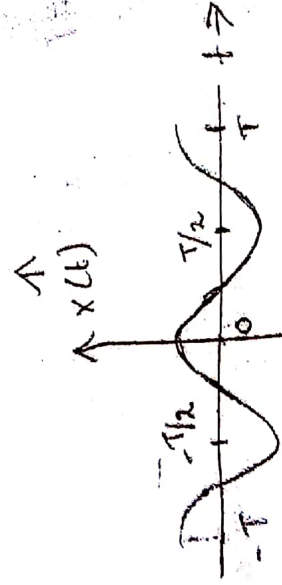
2. The sum of two periodic sequences is always periodic.

3. The sum of two periodic signals is periodic only if the ratio of their respective periods is a rational number.

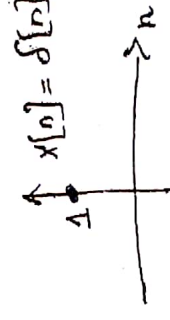
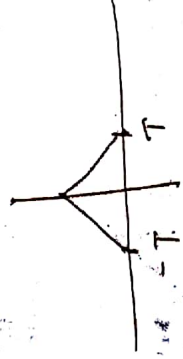
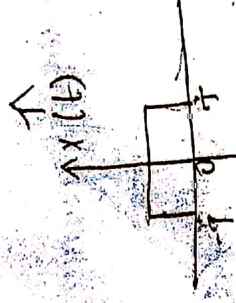
$$\frac{T_1}{T_2} = \text{rational number.}$$

4. The fundamental period is the LCM of T_1 & T_2 .

5. If ratio of T_1/T_2 is an irrational number, then signals $x_1(t)$ & $x_2(t)$ do not have a common period & $x(t)$ cannot be periodic.



periodic signal



$$x(t) = e^{-at}$$

periodic or Not

periodic or Not
Signal
3.14 = $\frac{314}{100}$
rational

$$x(t+T) = e^{-a(t+T)} = e^{-at} \cdot e^{-aT}$$

As $T \rightarrow \infty$

$$= e^{-at} \cdot e^{-a\infty}$$

$$x(t+T) = e^{-at} \cdot 0$$

$$= 0$$

$x(t)$

18 = rational
12 = irrational

π , e

$$\pi = 3.1415926536$$

i.e., signal having period $T = \infty$ is Non-periodic signal

2) $x(t) = \sin(15\pi t)$

periodic or Not

Fundamental period, $T = \frac{2\pi}{\omega} = \frac{2\pi}{15\pi} = \frac{2}{15} \text{ sec}$

$$\omega = \frac{2\pi}{T} \quad T = \frac{2\pi}{\omega} = \frac{2\pi}{15\pi} = \frac{2}{15} \sqrt{\pi}$$

3) $x(t) = \sin(\sqrt{2}\pi t)$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{2}\pi} = \frac{\sqrt{2}}{1} \text{ is periodic}$$

4) $x(t) = \cos(10\pi t)$

$$\omega = 10\pi$$

$$2\pi f = 10\pi$$

$$f = \frac{10}{2} = 5$$

rational number
periodic signal

$$x(t) = \cos\left(10\pi \frac{30}{105} t\right)$$

$$\omega = 2\pi f$$

$$f = \frac{30}{105}$$

$$f = \frac{30}{2 \times 105} = \frac{1}{7}$$

Rational number - periodic

$$x(n) = \cos(3\pi n)$$

$$2\pi f = 3\pi$$

$$f = \frac{3}{2}$$

Rational number - periodic

$$x(t) = 3 \cos(5t + \pi/6)$$

$$\omega = 5$$

$$2\pi f = 5$$

$$f = \frac{5}{2\pi}$$

$$T = \frac{2\pi}{5}$$

Non-periodic

$$x(t) = 3 \cos\left(5t + \frac{\pi}{6}\right)$$

$$2\pi f = 5$$

$$f = \frac{5}{2\pi}$$

Non-periodic

$$\cos\left(\frac{n\pi}{8}\right)$$

$$x(n) = \cos\left(\frac{n\pi}{8}\right)$$

$$f_c = \frac{1}{16\pi}$$

$$2\pi f_c = \frac{1}{8}$$

non-periodic

$$x(t) = \sin \pi t \cdot u(t)$$

$$\sin \pi t$$

$$T = \frac{2\pi}{\pi} = 2$$

Rational

$$T_2 = 2 \text{ units } h/w = 0.5$$

periodic

$$T_1 = 1, T_2 = 2$$

periodic

$$x(t) = \cos 2t + \sin 3t$$

$$T_1 = \frac{2\pi}{2} = \pi$$

$$T_2 = \frac{2\pi}{3}$$

Rational

Non-periodic

$$x(t) = 1 + \sin\left(\frac{5\pi t}{8} + \frac{\pi}{4}\right)$$

$$2\pi f = \frac{5\pi}{8}$$

$$f = \frac{5}{16}$$

F. Energy & Power Signals

Consider a resistance 'R' excited by a voltage, $V(t)$ producing a current $i(t)$. Assume that $R = 1 \Omega$.

$$i(t) = V(t) = x(t)$$

The instantaneous power expended in $R = 1 \Omega$ given by

$$p(t) = x^2(t) \text{ watts} \quad (1)$$

The total Energy, E and the average power, P of the signal $x(t)$ are

$$E = \int_{-\infty}^{\infty} x^2(t) dt \quad (2)$$

$$P = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t) dt \quad (3)$$

If $x(t)$ is complex, eq (2) & (3) modified as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt \quad (4)$$

$$P = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt \quad (5)$$

If signal $x(t)$ is periodic, then equation (5) modified as

$$P = \frac{1}{T} \int_0^T |x(t)|^2 dt \quad (6)$$

For discrete time signal $x(n)$

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$P = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N |x(n)|^2$$

If $x(n)$ is periodic, then average power is given as

$$P = \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2$$

Conclusion:

1. $x(n)$ or $x(n)$ is defined to be an Energy signal if and only if the total Energy content of the signal is a finite quantity i.e., $0 < E < \infty$.

All Energy signals will have zero average power i.e., $P = 0$.

2. $x(n)$ or $x(n)$ defined to be a power signal if & only if the average power content of the signal is a finite quantity i.e., $0 < P < \infty$. All power signals will have $E = \infty$.

3. Power & Energy signals are mutually exclusive

However there could be certain class of signals which do not satisfy both the classified as in Q.2) Such signals are neither energy nor power signals

Q.1) Calculate the Energy of the signal

$$x(t) = \begin{cases} 1, & -\frac{G}{2} \leq t \leq \frac{G}{2} \\ 0, & \text{otherwise} \end{cases}$$



$$E = \int_{-\infty}^{\infty} |x(t)|^2 \cdot dt$$

$$= \int_{-\frac{G}{2}}^{\frac{G}{2}} 1 \cdot dt$$

The average power in $x(t)$ is

$$P = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 \cdot dt$$

$$P = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\frac{G}{2}}^{\frac{G}{2}} 1 \cdot dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{T} (G) = 0$$

The eg: shows if a signal has finite duration & it is bounded, then the power of the signal is zero.

Also $0 < E < \infty$; $x(t)$ is an energy signal.

Q.2. Calculate the average power of the signal

$$x(t) = A \cos(\omega t + \phi) \quad -\infty \leq t \leq \infty$$

$$\text{Average power} = P = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

Sinusoidal signal $x(t)$ is periodic with $T = \frac{2\pi}{\omega}$.

$$T = \frac{2\pi}{\omega}$$

$$P = \frac{1}{T} \int_{-T/2}^{T/2} |A \cos(\omega t + \phi)|^2 dt$$

$$= \frac{\omega}{2\pi} \int_0^{2\pi/\omega} A^2 \cos^2(\omega t + \phi) dt$$

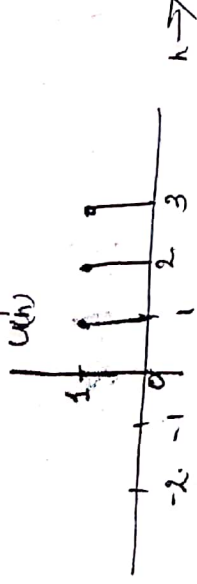
$$= \frac{A^2 \omega}{2\pi} \int_0^{2\pi/\omega} \frac{1}{2} [\cos(2\omega t + 2\phi) + 1] dt$$

$$= \frac{A^2 \omega}{2\pi} \left[\frac{1}{2} \sin(2\omega t + 2\phi) + t \right]_0^{2\pi/\omega} = \frac{A^2 \omega}{2\pi} \cdot \frac{2\pi}{\omega}$$

$$P = \frac{A^2}{2} < \infty, \quad x(t) \text{ is a power signal}$$

$$\text{Energy } E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} \frac{A^2}{2} [1 + \cos(2\omega t + 2\phi)] dt$$

Q.3) Find the average power of the signal, $x(n) = u(n)$



$$\begin{aligned}
 P &= \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N |x(n)|^2 \\
 &= \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=0}^N 1 \\
 &= \lim_{N \rightarrow \infty} \frac{N+1}{2N+1} = \frac{1}{2}
 \end{aligned}$$

Q.4)

Determine whether the following signals are Energy signals, power signals or neither & find the normalized energy & power

(i) $x(t) = e^{at} u(t)$

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$x(t) = e^{-at}$$

$$e^{-at} = \begin{cases} e^{-at}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$|x(t)|^2 = (e^{-at})^2 = e^{-2at}, \quad t \geq 0$$

$$\begin{aligned} E &= \int_{-\infty}^{\infty} e^{-2at} dt = \int_0^{\infty} e^{-2at} dt \\ &= \left(\frac{e^{-2at}}{-2a} \right)_0^{\infty} = \frac{0 - e^0}{-2a} = \frac{0 - 1}{-2a} \\ &= \frac{1}{2a} \text{ J} \end{aligned}$$

average power

$$\begin{aligned} P &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^{T/2} (e^{-2at})^2 dt \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \left(\frac{e^{-2at}}{-2a} \right)_0^{T/2} \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \left(\frac{e^{-aT} - 1}{-2a} \right) \end{aligned}$$

Since $\lim_{T \rightarrow \infty} \frac{e^{-aT} - 1}{-2a} = 0$, $x(t)$ is energy signal

$$u(t) = A u(t)$$

$$\text{Given } u(t) = A u(t) = \begin{cases} A, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$|u(t)|^2 = A^2, \quad t \geq 0$$

$$E = \int_{-\infty}^{\infty} |u(t)|^2 dt = \int_0^{\infty} A^2 dt$$

$$= A^2 (\infty - 0)$$

$\therefore$ Average power

$$P = \lim_{T \rightarrow \infty}$$

$$= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^{T/2} A^2 dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{T} A^2 (t)_0^{T/2}$$

$$= \lim_{T \rightarrow \infty} \frac{1}{T} A^2 \cdot T/2$$

$$= \lim_{T \rightarrow \infty} \frac{A^2}{2}$$

$$= \underline{\underline{\frac{A^2}{2}}}$$

Since Energy is infinite & power is finite
it is power signal.

$$x[n] = u[n]$$

$$\text{Power } x[n] = u[n] = 1 \quad n > 0$$

$$0 \quad n < 0$$

$$x[n]^2 = 1 \quad n > 0$$

Normalized Energy

$$E = \sum_{n=-\infty}^{\infty} x[n]^2$$

$$= \sum_{n=0}^{\infty} (1)^2$$

$$= 1 + 1 + \dots + 1 = \infty$$

Average power:

$$P = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N |x[n]|^2$$

$$= \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=0}^N (1)^2$$

$$= \lim_{N \rightarrow \infty} \left(\frac{1}{2N+1} \right) (N+1) = \lim_{N \rightarrow \infty} \frac{1}{2} \left(\frac{N+1}{N+1} \right)$$

$$= \frac{1}{2} \quad (\text{using L'Hospital's rule})$$

Since Energy is infinite & Power is finite, $x[n]$ is a power signal.

Ques: $x(t) = t \quad x(t) = x(t)$

$$= \begin{cases} t & ; t \geq 0 \\ 0 & ; t < 0 \end{cases}$$

$$|x(t)|^2 = t^2$$

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \cdot dt = \int_0^{\infty} t^2 \cdot dt = \left(\frac{t^3}{3} \right)_0^{\infty} = \infty$$

Average power, $P = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \cdot dt$

$$= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^{T/2} t^2 \cdot dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{T} \cdot \left(\frac{t^3}{3} \right)^{T/2}_0 = \frac{1}{T} \cdot \frac{1}{3} \cdot \frac{T^3}{8} = \infty$$

Since $E = \infty$; $P = \infty$; $x(t)$ is neither energy nor power signal.

$$x[n] = 2e^{j4n}$$

$x[n]$ is a complex signal

$$e^{*}[n] = 2e^{-j4n}$$

$$|x[n]|^2 = x[n] \cdot x^{*}[n] = 2e^{j4n} \cdot 2e^{-j4n} = 4$$

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=-\infty}^{\infty} 4 = \infty$$

$$\text{Average power} = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N |x[n]|^2$$

$$= \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N 4$$

$$= \lim_{N \rightarrow \infty} \frac{1}{2N+1} \cdot 4(2N+1) = 4 < \infty$$

$E = \infty$ $P = 4$ $\therefore x[n]$ is a power signal

$$x[n] = (-0.5)^n u[n]$$

$$\text{Given } x[n] = (-0.5)^n u[n] = \begin{cases} (-0.5)^n; & n \geq 0 \\ 0; & n < 0 \end{cases}$$

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=0}^{\infty} (0.25)^n$$

$$= \frac{1}{1-0.25} = \frac{4}{3} < \infty$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Average power:

$$P = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N |x(n)|^2$$

$$P = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=0}^N (0.25)^n \left(\frac{1}{1 - 0.25^{N+1/4}} \right)$$

$$= \underline{\underline{0}}$$

Since $E = f_{\text{in}} \& P = 0$

$$\text{Ques: } x(t) = e^{-2t}$$

$$|x(t)|^2 = e^{-4t} \quad E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} e^{-4t} dt = \left(\frac{e^{-4t}}{-4} \right)_{-\infty}^{\infty}$$

$$= \underline{\underline{\infty}}$$

$$P = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} e^{-4t} dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{T} \left(\frac{e^{-4t}}{-4} \right)_{-T/2}^{T/2} = \lim_{T \rightarrow \infty} \frac{1}{T} \left[\frac{e^{-2T}}{-4} - \frac{e^{2T}}{-4} \right]$$

$$P = \underline{\underline{0}}$$

$$E = \infty \text{ \& } P = 0;$$

$x(t)$ is neither energy nor power signal.

Causal & Non Causal Signals

Causal signals

A signal is said to be causal, if ~~it~~ is defined for ~~$t \geq 0$~~ input $t \leq 0$ ($n \leq n_0$)

$x(t) = 0$ for $t < 0$ { Present value of o/p signal depend on present or past values or combination of present & past value of input signal }
 eg: $x(n) = x(n) + x(n-1)$ ie, It does not depend on future values of that input.

Non-Causal signals

A system is aorge of continuous time aures
 eg: $y(t) = R \cdot i(t)$
 ie, physically realizable

Exponential Signal

Response of the system to an input depends on the future value of that input.

$$y(t) = x(t) - x(t+1)$$

Image processing

$$y(t) = x(t+1) \quad y(n) = x(n)$$

$$y(n-2) = x(n)$$

$$y(n) = x(n) - x(n+1)$$

$$y(t) = x(-t)$$

$$y(t) = x(-2) \quad y(-2) = x(4)$$

The o/p depends on future inputs

Basic operations on Signals

A. A) operations performed on dependent variables

i) Amplitude Scaling:

Let $x(t)$ be Continuous-time signal where C is Scaling factor.

$$y(t) = C x(t)$$

A physical Eg: of a device that performs Amplitude Scaling is Electronic Amplifier.

ii) Addition:

$x_1(t)$, $x_2(t)$ denotes a pair of continuous time signals

$$y(t) = x_1(t) + x_2(t)$$

A physical Eg: of a device that performs addition is a frequency mixer, which combines low frequency & high frequency signals.

(iii) Multiplication

Let $x_1(t)$ & $x_2(t)$ a pair of continuous time signals.

$$y(t) = x_1(t) x_2(t)$$

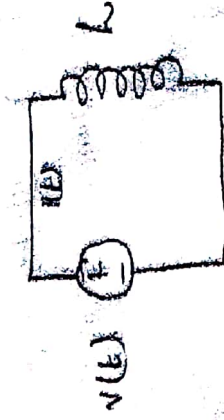
Multiplicative operation encountered in Analog communication - where an audio frequency signal is multiplied by a high frequency sinusoid known as carrier

The input signal is known as Amplitude Modulated wave

Differentiation:

Let $x(t)$ denote a continuous-time signal.

$$y(t) = \frac{dx(t)}{dt}$$



A physical device that performs differentiation is an inductor.

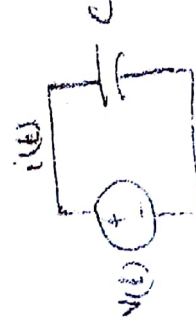
$$v(t) = L \frac{di(t)}{dt}$$

Integration:

$x(t)$ denotes a continuous-time signal. The

$$y(t) = \int_{-\infty}^t x(\tau) d\tau$$

This type of operation is encountered in a capacitor.

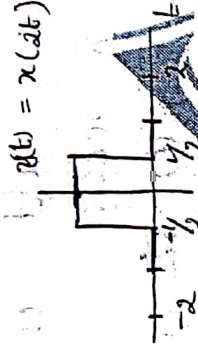
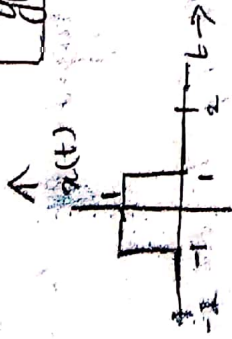


3) Operation performed on Independent Variable

1) Time Scaling

Let $x(t)$ be a continuous-time signal.

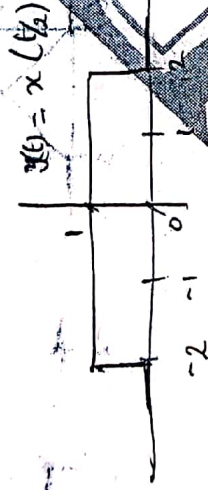
$$y(t) = x(at)$$



$$\begin{aligned} t &= 1 \\ x(1) &= 1 \\ y(1) &= x(2) \\ y(1) &= 0 \\ t &= 1/2 \\ y(1/2) &= x(1) \\ y(1/2) &= 1 \end{aligned}$$

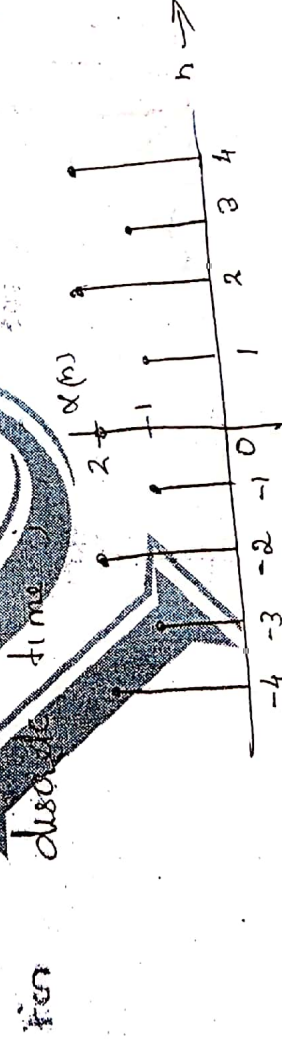
Rectangular pulse, $x(t)$

Compression by a factor of 2

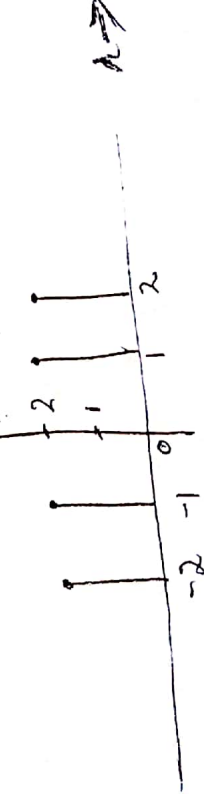


Eg: when $a > 1$, $y(t)$ is the compressed version of $x(t)$.

when $a < 1$, $y(t)$ is the expanded version of $x(t)$.



$$y(n) = x(2n)$$



Q.1 Time Shifting:

Let $x(t)$ be a continuous-time signal.

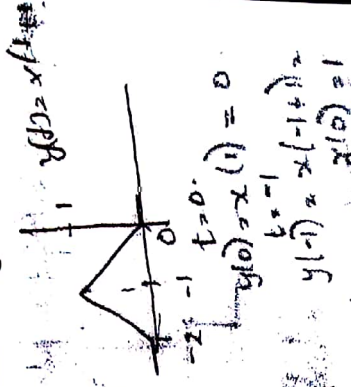
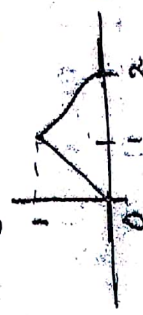
Replacing t by $(t+b)$ results in $y(t) = x(t+b)$

If $b < 0$; $x(t)$ is shifted to the right by an amount $|b|$ sec.

If $b > 0$; $x(t)$ is translated to the left by $|b|$ sec.



$$y(t) = x(t+b)$$

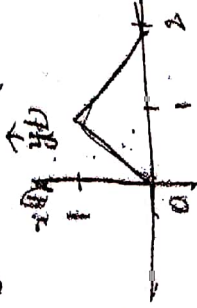


Reflection:

Let $x(t)$ denote a continuous-time signal.

The reflected version of $x(t)$ is obtained by replacing t with $-t$.

$$y(t) = x(-t)$$



1. For an even signal $x(t)$, the signal & its reflected version $x(-t)$ are identical.

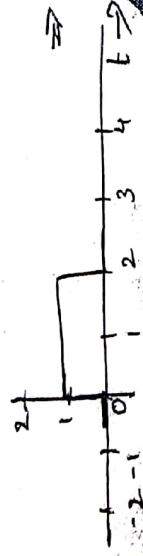
2. For an odd signal $x(t)$, the signal & its reflected version $x(-t)$ are negative of each other.

1. A continuous-time signal $x(t)$



$\Rightarrow x(t)$ translated to right by 2.

(b) $x(2t)$



$\Rightarrow x(t)$ compressed by a factor of 2.

(c) sketch of $x(t/2)$



$\Rightarrow x(t)$ expanded by a factor of 2

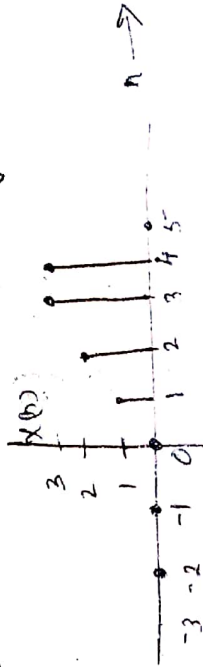
d) sketch of $x(-t)$



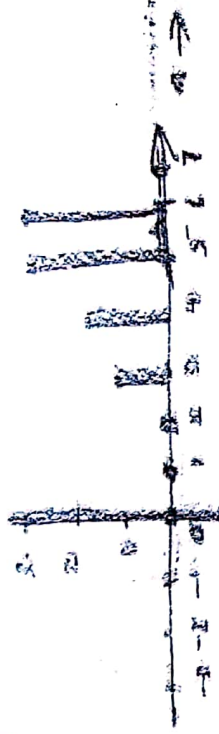
$\Rightarrow x(t)$ reflected or folded about $t=0$

Ans A discrete-time signal $x[n]$ is shown in given fig. Sketch and label each of the following signals.

- a) $x(n-2)$
- b) $x(2n)$
- c) $x(-n)$



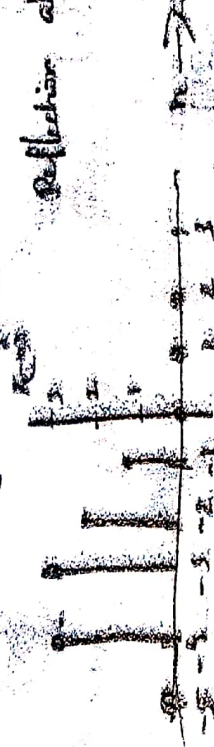
Transmitted in right half



$x[n]$



Q. 1) sketch $x[n]$

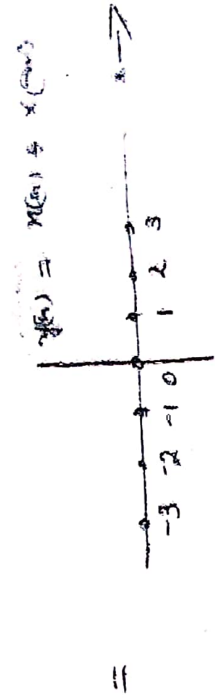
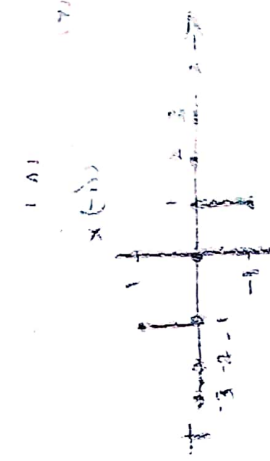
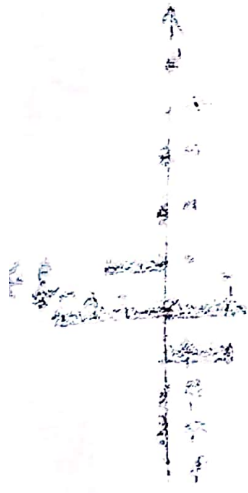


Reflection about origin

Q. 2) A discrete-time signal $x[n]$ is defined by

$$x[n] = \begin{cases} 1 & n = 1 \\ -1 & n = -1 \\ 0 & \text{of } |n| > 1 \end{cases}$$

Find the composite signal $y[n]$ defined in terms of $x[n]$ by $y[n] = x[n] + x[-n]$

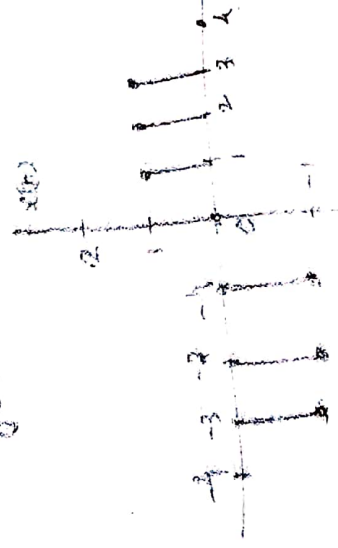


i.e., $y[n] = 0$ for all n

An a discrete-time signal $x[n]$ is described by

$$x[n] = \begin{cases} 1 & n = 1, 2, 3 \\ -1 & n = -1, -2, -3 \\ 0 & n = 0, |n| > 3 \end{cases}$$

Find $y[n] = x[2n+2]$



Elementary Signals

i) Exponential Signals:

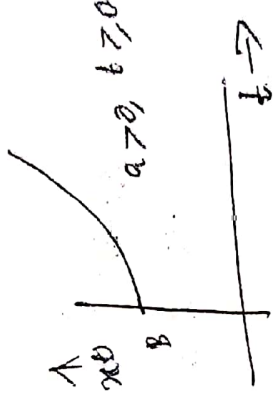
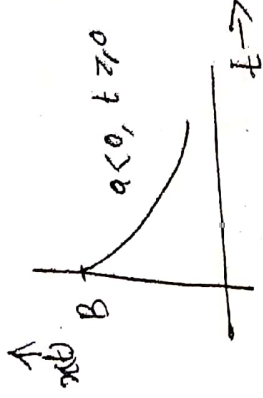
A real exponential signal is written as

$$x(t) = B e^{at}$$

where B & a are real parameters.

The parameter ' B ' is the amplitude of the signal

$x(t)$ at $t = 0$



ii) Sinusoidal Signals:

Let continuous time sinusoid is;

$$x(t) = A \cos(\omega t + \phi)$$

' A ' = Amplitude

ω = Angular frequency in rad/sec

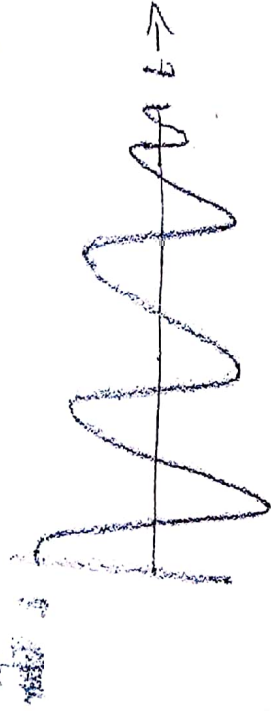
ϕ = phase angle in radians.

$$\begin{aligned} x(t+T) &= A \cos[\omega(t+T) + \phi] \\ &= A \cos[\omega t + \omega T + \phi] \\ &= A \cos[\omega t + \phi] \\ &= x(t) \end{aligned}$$

which defines the condition for a periodic signal.
(continuous time sinusoidal signal is always periodic)

10. Continuous time domain signals

$$x(t) = R \sin(\omega t + \theta) \quad \omega > 0$$

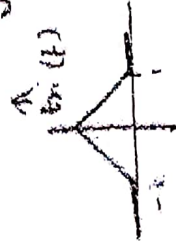


(10) Basic Signals:

$$\text{rect}(t) = \begin{cases} 1 & |t| < 0.5 \\ 0 & \text{elsewhere} \end{cases}$$



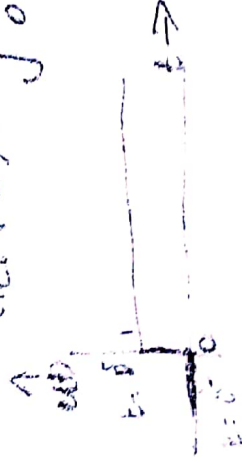
$$\text{tri}(t) = \begin{cases} 1 - |t| & |t| \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$



(10) Unit step function:

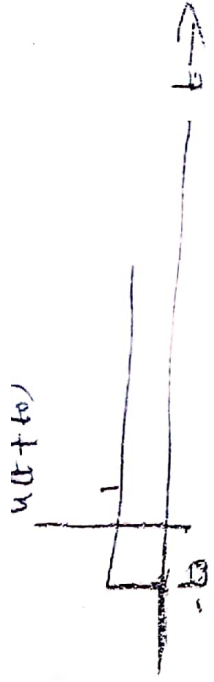
$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

Shifted unit step function:
 $u(t + t_0) = \begin{cases} 1 \\ 0 \end{cases}$



$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

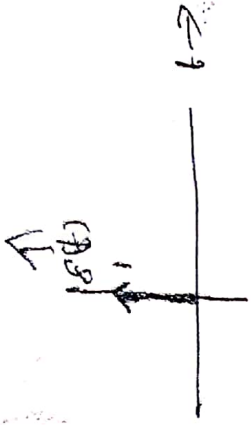
$$\begin{aligned} t + t_0 &> 0 \\ t + t_0 &< 0 \end{aligned}$$



(v) Unit Impulse function:

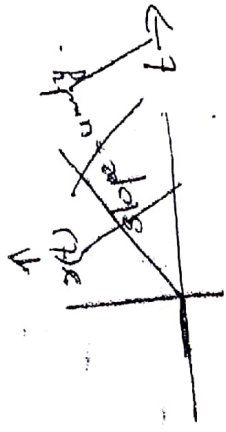
$$\delta(t) = 0 ; t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t) \cdot dt = 1 ; t = 0$$



Ramp function:

$$r(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$



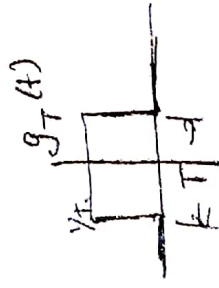
(i) Product property:

$$f(t) \delta(t) = f(0) \delta(t)$$

$$f(t) \delta(t - t_0) = f(t_0) \delta(t - t_0)$$

(ii) Shifting property:

$$\int_{t=-\infty}^{\infty} f(t) \cdot \delta(t - t_0) \cdot dt = f(t_0)$$



(iii)

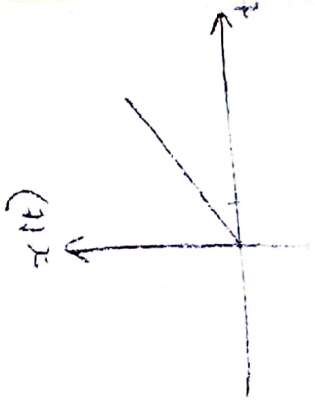
Time Scaling property:

$$\delta(at) = \frac{1}{a} \delta(t) ; a > 0$$

(vi) Ramp Function | Ramp Signal.

$$r(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

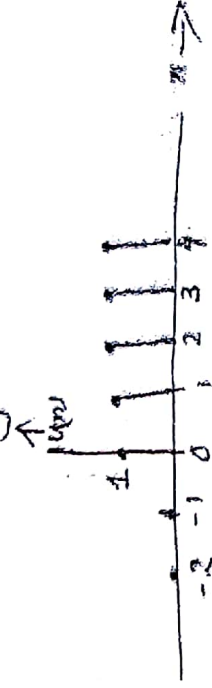
$$r(t) = t \cdot u(t)$$



Discrete Time Signals

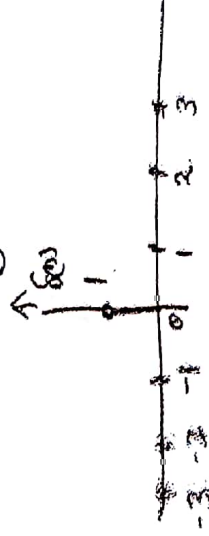
Step Function

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



Impulse function

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$



$$\delta(n) \delta(n-k) = \delta(n-k)$$

(i) Product property

(ii) Shifting property

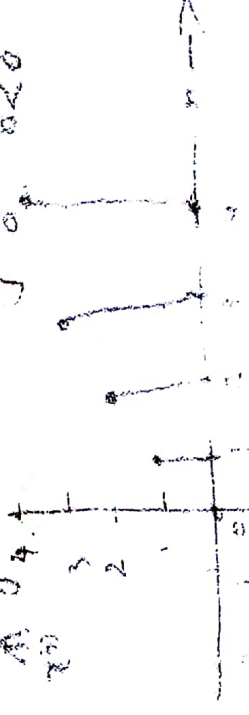
$$\sum_{k=-\infty}^{\infty} x(n-k) = x(n)$$

3. Ramp function

function

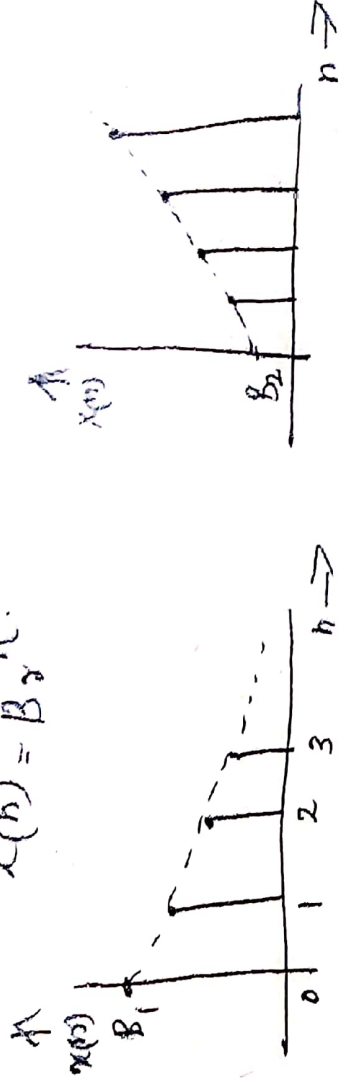
Discrete time version of a ramp

$$r(n) = \begin{cases} n & n \geq 0 \\ 0 & n < 0 \end{cases}$$



4) Exponential Function

$$x(n) = B r^n$$



If $0 < r < 1$; $x(n)$ is called decaying exponential sequence & if $r > 1$, then $x(n)$ is known as growing exponential sequence.

5) Sinusoidal Signals:

The discrete time version of a Sinusoidal

Signal is written as:-

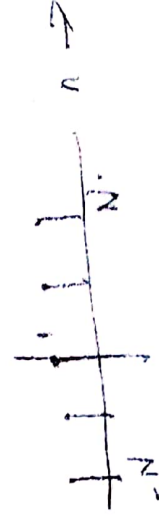
$$x(n) = A \cos(\Omega n + \phi)$$

6) Exponentially damped sinusoidal signals.

$$x(n) = B r^n \sin(\Omega n + \phi)$$

7) Pulse Signals:

$$\text{rect}\left(\frac{n}{2N}\right) = \begin{cases} 1 & |n| \leq N \\ 0 & \text{elsewhere} \end{cases}$$



Superposition of systems:

1. Linearity

- A discrete time system described by

$T\{\cdot\}$ is linear, if the response to a weighted

sum of inputs $x_1(n)$ & $x_2(n)$ is a weighted

sum of the responses of two inputs separately

for all weights & all acceptable signal $x_1(n)$ & $x_2(n)$.

$$T[a_1 x_1(n) + a_2 x_2(n)] = a_1 T[x_1(n)] + a_2 T[x_2(n)]$$

A system is linear if it obeys the principle of superposition.

2. Causality:

A system is said to be Causal, if the present value of the output signal depends only on the present or past value or a combination of present & past value of the input signal.

$$y(n) = x(n) + x(n-1)$$

3. Time Invariance:

A System is said to be time invariant, if a time delay or time advance of input signal leads to an identical time shifts in the output signal.

$$T\{x(n-n_0)\} = y(n-n_0)$$

4. Memory:

A system is said to possess memory, if its output signal depends on the past values of the input signal.

A system is said to be Memoryless if its output signal depends only on the present value of input signal.

for eg: Resistor is Memoryless, since its ~~out~~ current $i(t)$ through it at any time ' t ' depends on the Voltage across it at the same t .
ie, $i(t) = \frac{V(t)}{R}$

for eg: Inductor has Memory

$$i(t) = \frac{1}{L} \int_{-\infty}^t v(\tau) d\tau$$

Stability:

A discrete-time system is BIBO stable if every bounded input sequence produces a bounded output sequence.

If $|x(n)| \leq M_x < \infty$, then for BIBO stability

the output $y(n)$ obeys condition

$$|y(n)| \leq M_y < \infty$$

$$y(n) = \frac{1}{3} [x(n) + x(n-1) + x(n-2)]$$
$$|y(n)| \leq \frac{1}{3} [|x(n)| + |x(n-1)| + |x(n-2)|]$$

$$|y(n)| \leq \frac{1}{3} [M_x + M_x + M_x]$$

$$\Rightarrow |y(n)| \leq M_x < \infty \quad (\text{Bounded output})$$

Hence the system under investigation is BIBO stable.

Linearity:

A system is said to be invertible if the input to the system uniquely determines the output.

In other words, given two inputs $x_1(n)$ & $x_2(n)$ with $x_1(n) \neq x_2(n)$, it must be true that

$$y_1(n) \neq y_2(n)$$

For a system described by $T\{x(n)\} = ax(n) + b$
 Check for

- a) Stability
- b) Causality

- c) Linearity
- d) Time Invariance
- e) Memory.

Soln: $|x(n)| \leq M_x < \infty$ (Bounded Input)

$T\{x(n)\} = y(n) = ax(n) + b$

$|y(n)| = |ax(n) + b|$

$|y(n)| \leq |a||x(n)| + |b|$

$\therefore |y(n)| \leq (|a| M_x + |b|) < \infty$

$\therefore y(n)$ is bounded, the s/m under investigation is stable.

$\Rightarrow y(n) = ax(n_0) + b$

The output at $n = n_0$ depends on the input $n = n_0$.

The present output is produced by present input only. Then the system is causal.

c) $T\{a_1 x_1(n) + a_2 x_2(n)\} = \cancel{ax(n)} + b$

$a[a_1 x_1(n) + a_2 x_2(n)] + b$

$= a[a_1 T\{x_1(n)\} + a_2 T\{x_2(n)\}] + b$

Impulse Response -

- The response of a system when the input is an impulse function is defined as impulse response or unit sample response of the system.
- Impulse response completely characterizes the behavior of any LTI system.



- Consider an LTI system represented by $T\{\cdot\}$.
The output $y[n] = T\{x[n]\}$.

$$x[n] = \dots + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \dots$$

$\Downarrow$

This equation in compact form;

$$x[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot \delta[n-k]$$

The response of the system to an arbitrary input

$$\begin{aligned} y[n] &= T\{x[n]\} \\ &= T\left\{\sum_{k=-\infty}^{\infty} x[k] \delta[n-k]\right\} \\ &= \sum_{k=-\infty}^{\infty} x[k] T\{\delta[n-k]\} \end{aligned}$$

Impulse Response

The response of a system when the input is an impulse function is defined as impulse response or unit sample response of the system.

- Impulse response completely characterizes the behavior of any LTI system.



- Consider an LTI system represented by $T\{\cdot\}$.
The output $y(n) = T\{x(n)\}$.

$$x(n) = \dots + x(-2)\delta(n+2) + x(-1)\delta(n+1) + x(0)\delta(n) \\ + x(1)\delta(n-1) + x(2)\delta(n-2) + \dots$$

$\Downarrow$
This equation in compact form;

$$x(n) = \sum_{k=-\infty}^{\infty} x(k) \cdot \delta(n-k)$$

The response of the system to an arbitrary input

$$\begin{aligned} y(n) &= T\{x(n)\} \\ &= T\left\{\sum_{k=-\infty}^{\infty} x(k) \delta(n-k)\right\} \\ &= \sum_{k=-\infty}^{\infty} x(k) T\{\delta(n-k)\} \end{aligned}$$

$$\Rightarrow a_0 \frac{d^n y(t)}{dt^n} + a_1 \frac{d^{n-1} y(t)}{dt^{n-1}} + \dots + a_{n-1} \frac{dy(t)}{dt} + a_n y(t) = b_0 \frac{d^m x(t)}{dt^m} + b_1 \frac{d^{m-1} x(t)}{dt^{m-1}} + \dots + b_m x(t)$$

A LTI discrete time system can be represented using difference equation

The difference equation for N^{th} order system is

$$\sum_{k=0}^N a_k y(n-k) = \sum_{k=0}^M b_k x(n-k)$$

Causal & Non-Causal Signal

Causal Signal

A signal $x(t)$ said to be Causal if $x(t) = 0$ for $t < 0$

Non-Causal Signal

A signal $x(t)$ is said to be Non-Causal if $x(t) \neq 0$ for $t < 0$

1. $x(t) = 2u(-t)$.

ie; $u(-t) = 1$; $t \leq 0$

$t < 0$ $u(t) = 1$

$x(t) = 2u(-t) = 2 \times 1 = \underline{\underline{2}}$ $t \leq 0$

Non-Causal

2. $x(t) = \sin 2t$

$\sin 2t$ or $\cos 2t$ exists from $-\infty$ to $+\infty$.
 $t < 0$; This exists $x(t) \neq 0$

Non-Causal

3. $x(t) = \sin 2t \cdot u(t)$.

$u(t) = 1$ $t \geq 0$
 $= 0$ $t < 0$

For $t < 0$ $u(t) = 0$ $x(t) = 0$; Causal

Energy Signal (Non-periodic signal)

$x(t)$ or $x(n)$ defined to be an Energy signal if only if total Energy content of the signal is a finite quantity. ($0 < E < \infty$)
(i.e., Average power = 0).

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \cdot dt \quad \text{Joules}$$

$$P = 0.$$

In discrete;

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2.$$

Power Signal (Periodic signal)

A signal is said to be a power signal if only if P is finite & $E = \infty$.

$$P = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 \cdot dt \quad \text{watts.}$$

$$P = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N |x(n)|^2$$

$$\text{R.M.S Value of } x(t) = \sqrt{\text{Power}}$$

1. Determine the power of

1) $x(t) = A \sin(\omega t + \phi)$

2) $x(t) = e^{j\omega t} \cos \omega t$

3) $x(t) = 12 \cos(2\omega t + \pi/3) + 16 \sin(3\omega t + \pi/4)$

Ans

$$P = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T A^2 \sin^2(\omega t + \phi) dt$$

$$= \lim_{T \rightarrow \infty} \frac{A^2}{2T} \int_{-T}^T \frac{1 - \cos(2\omega t + 2\phi)}{2} dt$$

$$= \lim_{T \rightarrow \infty} \frac{A^2}{4T} \left\{ \int_{-T}^T 1 dt - \int_{-T}^T \cos(2\omega t + 2\phi) dt \right\}$$

$$= \lim_{T \rightarrow \infty} \frac{A^2}{4T} \int_{-T}^T 1 dt$$

$$= \lim_{T \rightarrow \infty} \frac{A^2}{4T} \cdot 2T$$

$$P = \frac{A^2}{2}$$

$$\text{R.M.S value} = \sqrt{P} = \frac{A}{\sqrt{2}}$$

$$x(t) = e^{j\omega t} \cos \omega t$$

$$= (\cos \omega t + j \sin \omega t) \cdot \cos \omega t$$

$$= \cos^2 \omega t + j \sin \omega t \cdot \cos \omega t$$

i.e. Integration of
Cosine functions.
Full cycle is zero

$$\sin(A+B) = \sin A \cdot \cos B + \cos A \cdot \sin B$$

$$2 \cos A \cdot \cos B = \cos(A+B) + \cos(A-B)$$

$$2 \sin A \cdot \cos B = \sin(A+B) + \sin(A-B)$$

$$\begin{aligned} \therefore P &= \frac{1}{T} \left(\frac{\cos 12t + \cos 8t}{2} \right) + j \left(\frac{\sin 12t + \sin 8t}{2} \right) \\ &= \frac{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \frac{\left(\frac{1}{2}\right)^2}{2} + \frac{\left(\frac{1}{2}\right)^2}{2} = \frac{1}{2} \cdot 4 \cdot \frac{1}{4} \\ &= \underline{\underline{\frac{1}{2}}} \end{aligned}$$

$$\text{R.M.S Value} = \underline{\underline{\frac{1}{\sqrt{2}}}}$$

$$c) x(t) = 12 \cos\left(20t + \frac{\pi}{3}\right) + 16 \sin\left(30t + \frac{\pi}{2}\right)$$

$$P = \frac{12^2}{2} + \frac{16^2}{2} = \frac{144 + 256}{2} = \frac{400}{2} = \underline{\underline{200 \text{ W}}}$$

$$\text{R.M.S value} = \underline{\underline{\sqrt{200}}}$$

2. Determine whether the signal is Energy or power signal.

$$x(t) = \text{rect}\left(\frac{t}{\tau}\right)$$

$$\text{rect}\left(\frac{t}{\tau}\right) = 1$$

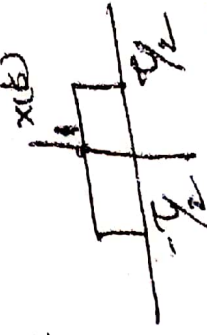
$$= 0$$

$$\text{for } -\tau/2 < t < \tau/2$$

otherwise

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \cdot dt$$

$$= \int_{-\tau/2}^{\tau/2} 1 \cdot dt = \tau/2 + \tau/2 = \underline{\underline{\tau}}$$



Non-periodic signal

$$E = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T/2}^{T/2} 1 dt$$

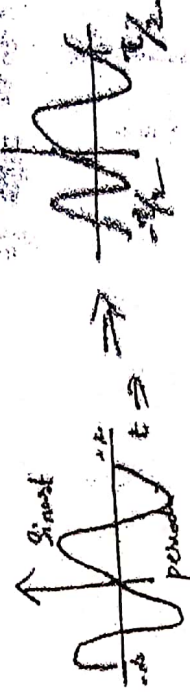
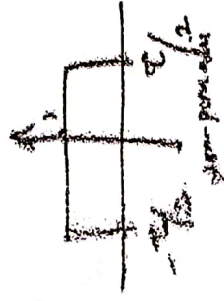
$$= \lim_{T \rightarrow \infty} \frac{t}{2T} = \frac{\infty}{\infty} = 0$$

Because it is Energy signal $\underline{\underline{E = \infty}}$ joules.

Prob:

Non-periodic signal $\Rightarrow$ Energy signal

$$x(t) = \text{rect}(t/a) \cdot \sin \omega t$$



(non-periodic signal)

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{-T/2}^{T/2} \sin^2 \omega t dt$$

$$= \int_{-T/2}^{T/2} \left(\frac{1 - \cos 2\omega t}{2} \right) dt$$

$$= \int_{-T/2}^{T/2} \frac{1}{2} dt = \frac{1}{2} [t]_{-T/2}^{T/2} = \frac{1}{2} T$$

$$E = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt$$

$$\begin{aligned}
 &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \sin^2 \omega t \cdot dt \\
 &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \frac{1 - \cos 2\omega t}{2} \cdot dt \\
 &= \lim_{T \rightarrow \infty} \frac{1}{2T} \cdot \frac{1}{2} \int_{-T}^T 1 \cdot dt \\
 &= \lim_{T \rightarrow \infty} \frac{1}{2T} \cdot \frac{1}{2} \cdot (T/2 + T/2) \\
 &= \lim_{T \rightarrow \infty} \frac{1}{2T} \cdot (T/2) = \frac{1}{4} \times 1 = 0
 \end{aligned}$$

$$P = 0 \quad E = \underline{\underline{T/2}}$$

ie, If the signal is periodic of infinite duration, then it can be power signal. If the signal is periodic over a finite interval & not periodic, then it can be Energy signal.

Qn. Find Energy or power signal or not

$$\begin{aligned}
 a) \quad & \left(\frac{1}{2}\right)^n u(n) \\
 E &= \lim_{N \rightarrow \infty} \sum_{n=-N}^N \left|\left(\frac{1}{2}\right)^n\right|^2 \cdot u(n) \\
 &= \lim_{N \rightarrow \infty} \sum_{n=0}^N \left(\frac{1}{2}\right)^{2n} \cdot u(n)
 \end{aligned}$$

$$= \lim_{N \rightarrow \infty} \sum_{n=0}^N \left| \left(\frac{1}{2} \right)^n \right|^2$$

$$= \lim_{N \rightarrow \infty} \sum_{n=0}^N \left(\frac{1}{4} \right)^n$$

$$= \lim_{N \rightarrow \infty} \frac{1}{1 - \frac{1}{4}}$$

$$E = \underline{\underline{\frac{4}{3} \text{ Joules}}}$$

Note:

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \quad |a| < 1$$

$$\sum_{n=0}^{\infty} a^n = \frac{1-a^{N+1}}{1-a}$$

$$P = \lim_{N \rightarrow \infty} \sum_{n=-N}^N |x[n]|^2$$

$$= \lim_{N \rightarrow \infty} \sum_{n=-N}^N \left(\frac{1}{4} \right)^n$$

$$= \lim_{N \rightarrow \infty} \frac{1}{2N+1} \left[\frac{1 - \left(\frac{1}{4} \right)^{N+1}}{\left(\frac{3}{4} \right)} \right] = 0$$

Energy signal

$$e^{j\left(\frac{\pi}{4}n + \frac{\pi}{2}\right)}$$

$$E = \lim_{N \rightarrow \infty} \sum_{n=-N}^N \left| e^{j\left(\frac{\pi}{4}n + \frac{\pi}{2}\right)} \right|^2$$

$$= \lim_{N \rightarrow \infty} \sum_{n=-N}^N 1 = \lim_{N \rightarrow \infty} (2N+1) = \infty$$

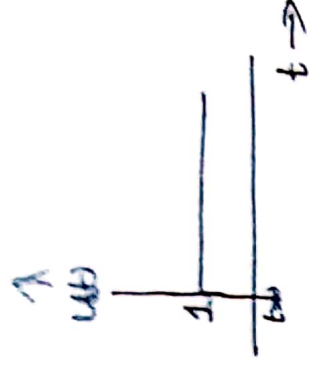
$$P = \lim_{N \rightarrow \infty} \frac{1}{2N+1} = 0$$

Elementary signals

1. Unit step function

$$u(t) = 1 \quad \text{for } t \geq 0$$

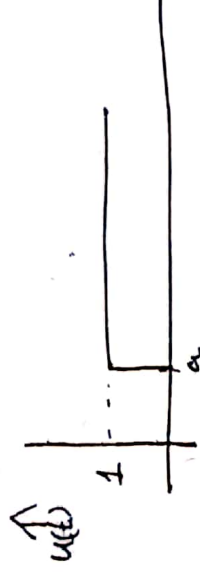
$$= 0 \quad \text{for } t < 0$$



The shifted unit step function $u(t-a)$ defined as

$$u(t-a) = 1 \quad \text{for } t-a \geq 0 \text{ or } t \geq a$$

$$= 0 \quad \text{for } t-a < 0 \text{ or } t < a$$



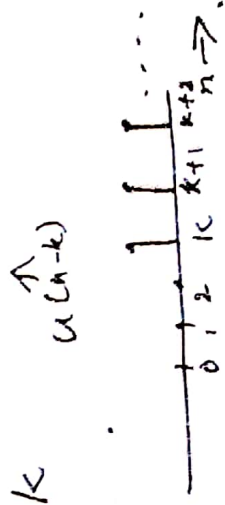
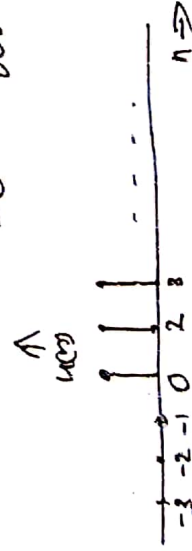
The discrete time unit step function $u(n)$ defined as

$$u(n) = 1 \quad \text{for } n \geq 0$$

$$= 0 \quad \text{for } n < 0$$

$$u(n-k) = 1 \quad \text{for } n \geq k$$

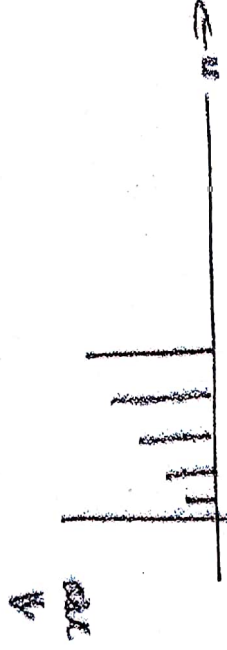
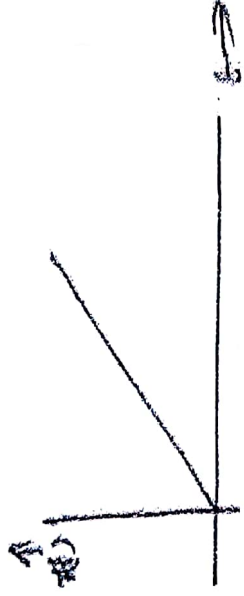
$$= 0 \quad \text{for } n < k$$



3) Ramp Function

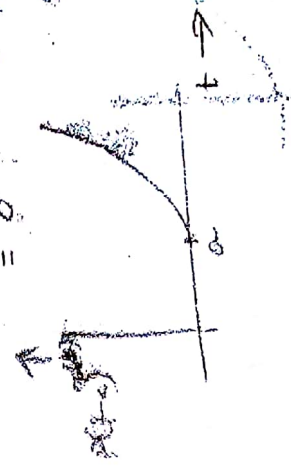
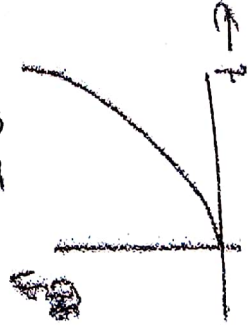
$$r(t) = \begin{cases} t & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

$$r(t) = t \cdot u(t)$$



3) ~~and~~ Parabolic Signal (acceleration signal)

$$p(t) = \begin{cases} \frac{t^2}{2} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$



Discrete:

$$p[n] = \frac{n^2}{2} \text{ for } n \geq 0$$

$$p[n-3] = \frac{(n-3)^2}{2} \text{ for } n \geq 3$$



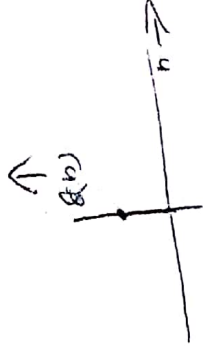
Impulse signal.

$\delta(t)$

$$\int_{-\infty}^{\infty} \delta(t) \cdot dt = 1 \quad \text{at } t=0$$

$$\delta(t) = 0 \quad t \neq 0$$

In discrete time; $\delta(n)$



$$\sum_n \delta(n-3)$$

5. Sinusoidal signal

Sinusoidal signal given

A continuous time

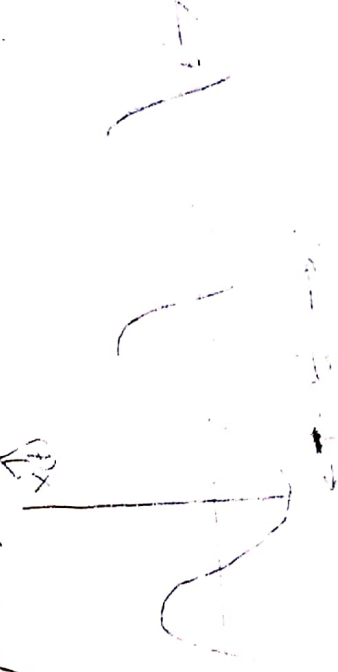
by: $x(t) = A \sin(\omega t + \phi)$

A = Amplitude in radians.

ω = Angular frequency in radians

ϕ = phase angle in radians

Time period, $T = \frac{2\pi}{\omega}$



The discrete time signal is given by

$$x(n) = A \sin(\omega n + \phi)$$

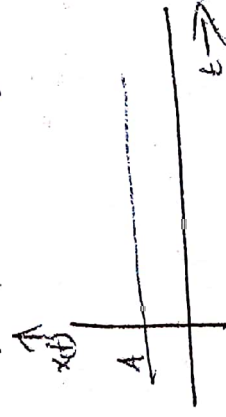


6) Exponential signal

A continuous time real exponential signal is

$$x(t) = A e^{\alpha t}$$

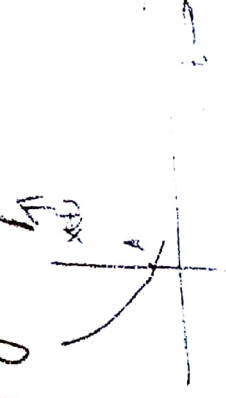
$$\alpha = 0 \quad x(t) = A$$



$\alpha > 0$ $x(t)$ is a growing exponential signal



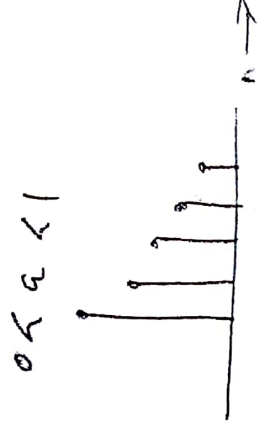
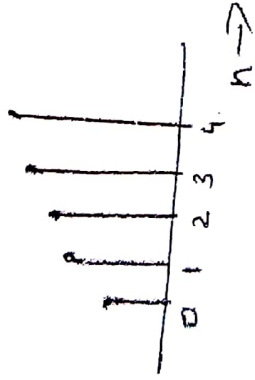
$\alpha < 0$ $x(t)$ is a decaying exponential signal



Discrete time

$$x(n) = a^n$$

$a > 1$ for all n



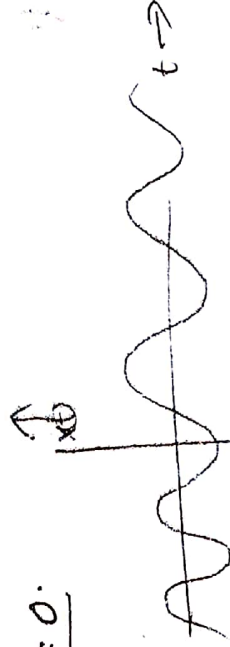
2) Complex Exponential Signal

$$x(t) = A e^{st} \text{ where } s = \sigma + j\omega$$

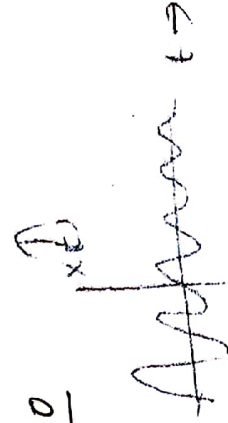
$$= A e^{(\sigma + j\omega)t}$$

$$= A e^{\sigma t} \cdot e^{j\omega t} = A e^{\sigma t} [\cos(\omega t + j \sin(\omega t))]$$

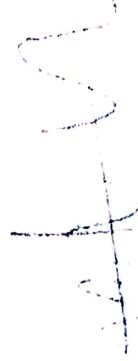
a) $\sigma = 0$



b) $\sigma < 0$



c) $\sigma > 0$



Basic operation on signals

1. Time shifting
2. Time scaling
3. Time Reversal
4. Amplitude Scaling
5. Signal addition
6. Signal Multiplication

1. Time Shifting.

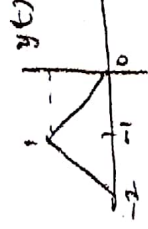
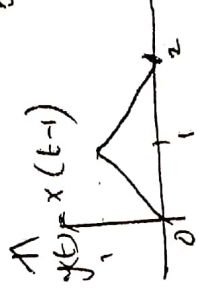
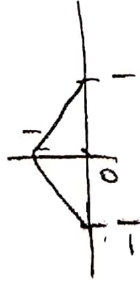
$x(t)$ be a continuous time signal

$$y(t) = x(t+b)$$

$b < 0$; Right shift by ' b ' sec.

$b > 0$; Left shift by ' b ' sec.

$x(t)$

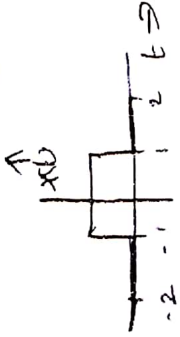


2) Time Scaling

$y(t) = x(at)$ in continuous time

If $a > 1 \rightarrow$ time compression

$a < 1 \rightarrow$ time expansion



$$y(t) = x(t/2)$$



(Expansion by 2)

$$y(t) = x(2t)$$



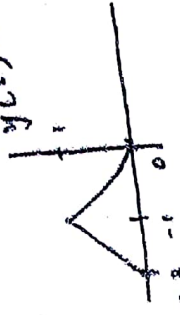
Reversed (Time Reversal) [Reflection]

substituting $y(t) = x(-t)$

→ result in convolution

$y(t) = x(t)$

$y(t) = x(-t)$



In discrete

$y(n) = x(-n)$

$\uparrow$
 $x(n)$



$\uparrow$
 $x(-n)$



Amplitude Scaling

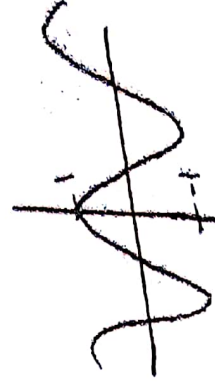
$y(t) = A x(t)$

$A \rightarrow$ constant

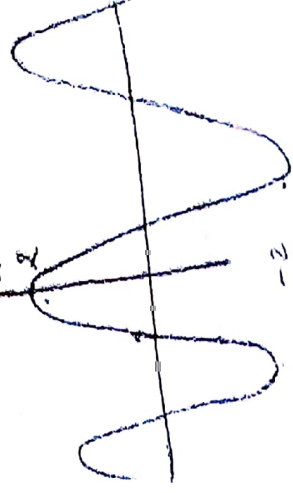
if $A > 1 \rightarrow$ Amplification

$A < 1 \rightarrow$ Attenuation

$\uparrow$
 $x(t)$

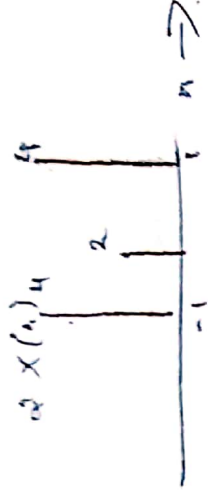
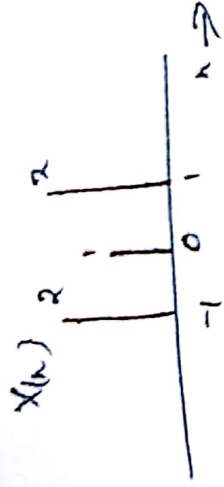


$\uparrow$
 $2x(t)$



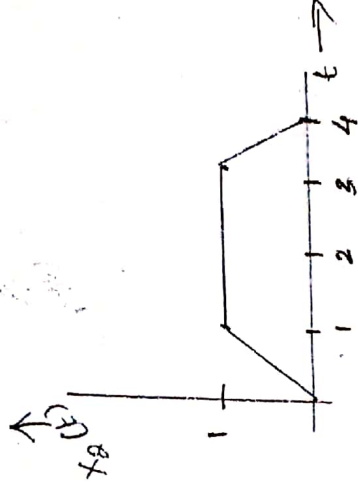
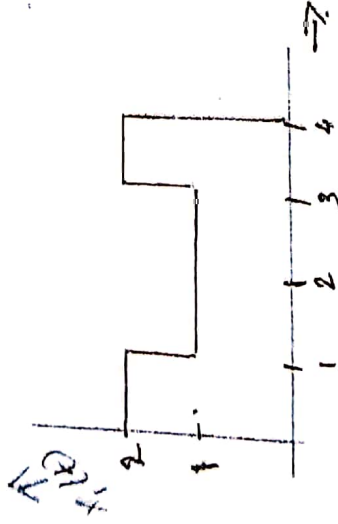
In discrete:

$$y(n) = (x(n))$$

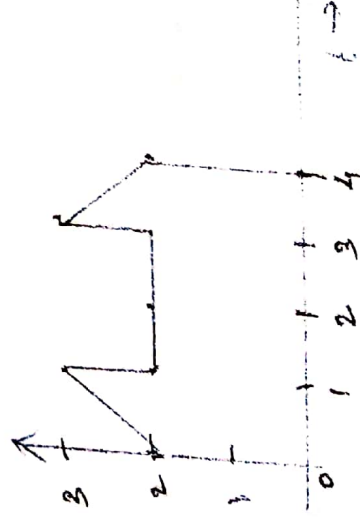


5. Signal addition:

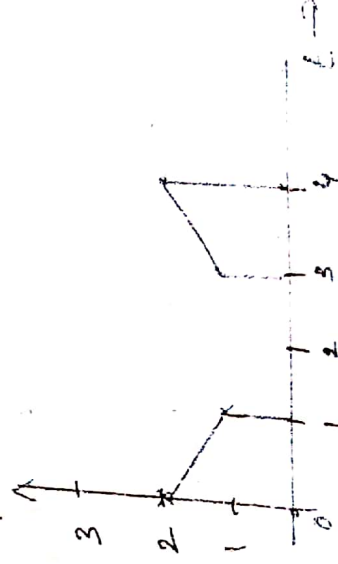
- The sum of two continuous time signal $x_1(t)$ & $x_2(t)$ can be obtained by adding their values at every instant of time. // i.e. the subtraction is done.



$$x_1(t) + x_2(t)$$



$$x_1(t) - x_2(t)$$



In discrete time

$$\text{If } x_1(n) = \{1, 2, 3\}$$

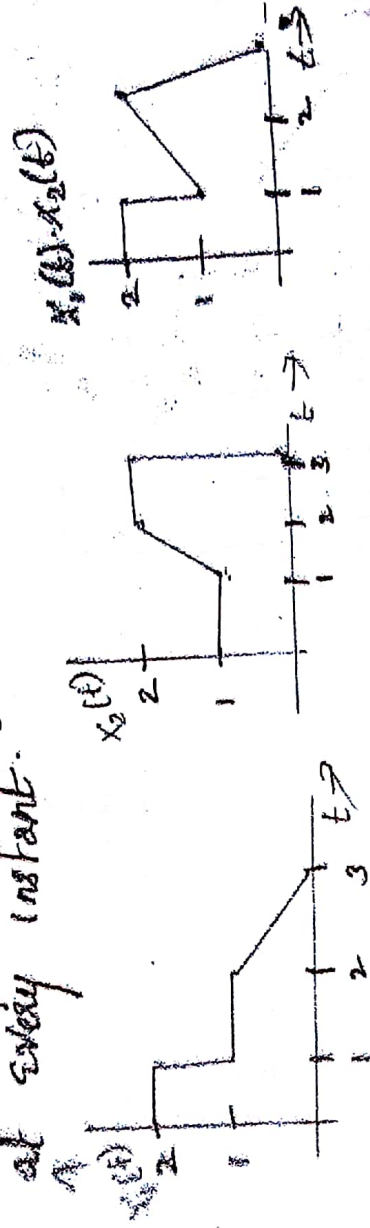
$$x_2(n) = \{2, 3, 4\}$$

$$x_1(n) + x_2(n) = \{3, 5, 7\}$$

$$x_1(n) - x_2(n) = \{-1, -1, -1\}$$

Signal Multiplication:

Multiplication of 2 continuous time signal can be performed by multiplying their values at every instant.



In discrete time;

$$x_1(n) = \{1, -3, 2\} \quad x_2(n) = \{2, -1, 3\}$$

$$x_1(n) \cdot x_2(n) = \{(1 \times 2), (-3 \times -1), (2 \times 3)\}$$

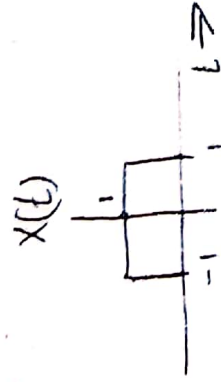
$$= \underline{\underline{\{2, 3, 6\}}}$$

Important Rule for Time Shifting & Time Scaling

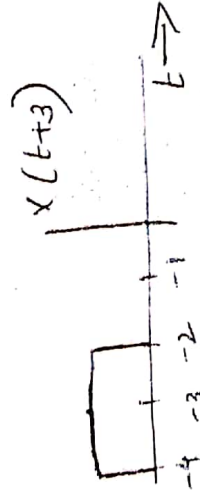
⇒ First, we do shifting operation

⇒ Then, we do the time scaling operation

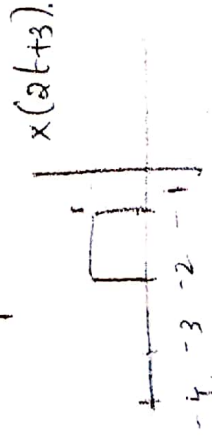
$$1 \quad g(t) = x(-2t+3)$$



(i) Shift $x(t)$ to left by 3 to get $x(t+3)$



(ii) Compress $x(t+3)$ by 2 to get $x(2t+3)$



(iii) $x(2t+3)$ folded in time get $x(-2t+3)$



Systems

Systems

A system may be considered as a set of elements or functional blocks which are connected together and produce an output in response to an input signal.

$$x(t) \rightarrow y(t)$$



$$y(t) = f[x(t)]$$

Eg: filter, amplifier etc.

Properties of Systems:-

Continuous time & Discrete time systems.

- Continuous time system transforms continuous time input signals into continuous time output signals

$$y(t) = T[x(t)]$$

- Discrete time system transforms discrete time input signals into discrete time output signals.

$$y(n) = T[x(n)]$$

2. Static and Dynamic System:

- For a static (Memory less) system, the output at any instant depends only on the input applied at that instant. [present input alone]

Eg: Resistor $i(t) = \frac{v(t)}{R}$

$$\text{Eg: } y(t) = x(t) \quad \text{or} \quad y(n) = x(n)$$

$$y(t) = x^2(t) \quad y(n) = x^2(n)$$

- For a dynamic system (Memory) - the output depends on past or future inputs.

Eg: Inductor

$$i(t) = \frac{1}{L} \int_{-\infty}^t v(t) \cdot dt$$

$$\text{Eg: } y(t) = x(t) + x(t+2)$$

$$y(n) = x(n) + x(n-2)$$

3. Causal & Non-Causal Systems.

- A system is said to be Causal if the present value of output signal depends only on the present or past value or a combination of present & past value of the input signal.

$$y(n) = x(n) + x(n-1)$$

- For causal s/m the o/p does not begin before the input function is applied.

- A system is said to be Non-causal if the output of the system at any time depends on future inputs.

$$y(t) = x(t) + x(t+2)$$

$$y(n) = x(n) + x(2n)$$

4. Linear & Non-Linear Systems:

- A system which obeys the principle of Superposition called linear system.

If i/p $x_1(t)$ produces o/p $y_1(t)$

For linear s/m, $x_2(t)$ " " $y_2(t)$

$$f[a_1x_1(t) + a_2x_2(t)] = a_1y_1(t) + a_2y_2(t)$$

then weighted sum of inputs $x_1(t) + x_2(t)$
 where a_1 & a_2 are constants produce output
 $a_1 y_1(t) + a_2 y_2(t)$ which is the sum of
 weighted outputs.

For discrete time,

$$f[a_1 x_1(n) + a_2 x_2(n)] = a_1 y_1(n) + a_2 y_2(n)$$

5. Time invariant & time variant system.

- A system is said to be time invariant,
 if its input/output characteristic does not
 change with time

$$\text{if } x(t) \rightarrow y(t)$$

$$x(t-T) \rightarrow y(t-T)$$

$$x(t-T) = y(t)$$

if $x(t)$ is the input; $x(t) = x(t-T) \rightarrow y(t-T)$

$$y(t) = f[x(t)]$$

Then $y(t-T)$ is the delayed output

For delayed if $x(t-T)$, $T[x(t-T)] \rightarrow$ output of delayed
 i/p

if $y(t, T) = y(t-T)$; then the system
 is invariant.

$$\text{i.e. } T[x(t-T)] = y(t-T)$$

$$y(t-T) = T[x(t-T)]$$

BIBO Stable & unstable

- A discrete time system is Bounded BIBO output stable (BIBO stable) if every bounded input sequence produces a bounded output sequence.

if $\sup |x(n)| \leq M_x < \infty$ for BIBO stable

then $\sup |y(n)| \leq M_y < \infty$

Invertible and Non invertible

- A system is said to be Invertible if the input to the system may be uniquely determined from the output.

Given 2 inputs $x_1(n)$ & $x_2(n)$ with $x_1(n) \neq x_2(n)$ it must be true that

$y_1(n) \neq y_2(n)$

Check whether the following systems are linear or not.

a) $y(t) = e^{x(t)}$

To check for linearity

$$F[a x_1(t) + b x_2(t)] = a F[x_1(t)] + b F[x_2(t)]$$

$$F[x_1(t)] = e^{x_1(t)} \quad F[x_2(t)] = e^{x_2(t)}$$

$$a F[x_1(t)] + b F[x_2(t)] = a e^{x_1(t)} + b e^{x_2(t)} \quad \text{--- (1) R.H.S}$$

$$F[a x_1(t) + b x_2(t)] = e^{a x_1(t) + b x_2(t)} \quad \text{--- (2) L.H.S}$$

L.H.S $\neq$ R.H.S

Non-linear

$$F[x(n)] = n^2 x(n)$$

b) $y(n) = n^2 x(n)$

$$\text{L.H.S} \quad F[a x_1(n) + b x_2(n)]$$

$$= n^2 [a x_1(n) + b x_2(n)]$$

$$= a n^2 x_1(n) + b n^2 x_2(n)$$

$$F[x_1(n)] = n^2 x_1(n)$$

$$F[x_2(n)] = n^2 x_2(n)$$

$$\text{R.H.S} = a F[x_1(n)] + b F[x_2(n)]$$

$$= a [n^2 x_1(n)] + b [n^2 x_2(n)]$$

$$= a n^2 x_1(n) + b n^2 x_2(n)$$

$$\text{L.H.S} = \text{R.H.S}$$

Linear

$$\frac{1}{2} f(x(n))$$

$$y(n) = x(n)$$

$$L.H.S = [a x_1(n) + b x_2(n)]$$

$$= [a x_1(n^2) + b x_2(n^2)]$$

$$R.H.S = a F[x_1(n)] + b F[x_2(n)]$$

$$= a x_1(n^2) + b x_2(n^2)$$

Linear S/H.

$$y(n) = x(n) + \frac{1}{2} x(n-2) = f[x(n)]$$

$$L.H.S = [a x_1(n) + b x_2(n)]$$

$$= a [x_1(n) + \frac{1}{2} x_1(n-2)] + b$$

$$= a x_1(n) + b x_2(n) + \frac{1}{2} [a x_1(n-2) + b x_2(n-2)]$$

$$R.H.S = a F[x_1(n)] + b F[x_2(n)]$$

$$= a [x_1(n) + \frac{1}{2} x_1(n-2)] + b [x_2(n) + \frac{1}{2} x_2(n-2)]$$

$$L.H.S = R.H.S$$

Check whether the following systems are time invariant or not.

(1) $y(t) = t^2 x(t)$

(2) $y(n) = x(n) + n x(n-2)$

$$\begin{aligned} f(x(t)) &= t^2 x(t) \\ f(x(t-\tau)) &= t^2 x(t-\tau) \end{aligned}$$

Check for time invariance;

$$y(t-\tau) = f[x(t-\tau)]$$

(1) L.H.S; $y(t-\tau) = (t-\tau)^2 x(t-\tau)$

R.H.S $f[x(t-\tau)] = t^2 x(t-\tau)$

L.H.S $\neq$ R.H.S.

Time Variant

(2) $y(n) = x(n) + n x(n-2)$

L.H.S; $y(n-k) = x(n-k) + (n-k)x(n-k-2)$

R.H.S; $f[x(n-k)] = x(n-k) + n x(n-k-2)$

L.H.S $\neq$ R.H.S

Time Variant.

(3) $y(n) = x^2(n-2)$

L.H.S $y(n-k) = x^2(n-k-2)$

R.H.S $f[x(n-k)] = x^2(n-k-2)$

L.H.S = R.H.S

$$y(t) = \frac{d^2}{dt^2} x(t) + x(t)$$

check whether the sys is static or dynamic

differential eqn: $\Rightarrow$ dynamic (Memory)

$\hookrightarrow$ o/p depends on past or future i/p

$$y(n) = x(n) \quad \Rightarrow \text{static (Memoryless)}$$

O/p depends on present i/p only

Determine whether the systems are causal or

Non-causal

$$y(t) = x(-t)$$

$$t = -1 \quad y(-1) = x(1) \quad \Rightarrow \text{future i/p}$$

$$y(0) = x(0)$$

$$y(1) = x(-1)$$

O/p depends on future values of i/p $\Rightarrow$ Non-causal

$$y(t) = x(t) \cdot \cos(t+1)$$

$$t = -1, \quad y(-1) = x(-1) \cdot \cos(0)$$

$$t = 0, \quad y(0) = x(0) \cdot \cos(1)$$

$$t = 1, \quad y(1) = x(1) \cdot \cos(2)$$

O/p depends on present i/p

$\rightarrow$ Causal

$$y(n) = x(n^2)$$

$$t = -1, y(-1) = x(1) \rightarrow \text{future i/p}$$

$$y(0) = x(0)$$

$$y(1) = x(1)$$

O/p depends on future i/p $\Rightarrow$ Non-causal.

$$4) \quad y(n) = x(n) - x(n-1)$$

$$t = -1, y(-1) = x(-1) - x(-2)$$

$$t = 0, y(0) = \underset{\text{present}}{x(0)} - \underset{\text{past}}{x(-1)}$$

O/p depends on present & past i/p

$\Rightarrow$ Causal s/s.

Stable and Unstable Systems

Bounded signal

If $|g(t)| \leq M$ then $g(t)$ is bounded

when M is positive real finite number

A system is called Bounded-input, bounded output (BIBO) stable if and only if every bounded input results in a bounded output. The output of such system does not diverge or does not grow unreasonably large.

Consider a system.

$$y(t) = f(x(t))$$

For the system to be stable

$$|y(t)| < M_y < \infty \text{ for all values of } t.$$

$$|x(t)| \leq M_x < \infty \text{ for all values of } t.$$

where M_x and M_y are some positive finite numbers.

System which do not satisfy the above condition are called unstable system.

Check the stability of S/M obtained by

a) $h(t) = (2 + e^{-3t}) u(t)$

$$\boxed{\int_{-\infty}^{\infty} |h(t)| dt < \infty}$$

$$\begin{aligned} \int_{-\infty}^{\infty} (2 + e^{-3t}) dt &= 2 \int_0^{\infty} 1 dt + \left[\frac{e^{-3t}}{-3} \right]_0^{\infty} \\ &= 2[\infty - 0] + -\frac{1}{3} [e^{-\infty} - e^0] \\ &= 2\infty + -\frac{1}{3} [0 - 1] \\ &= \infty \end{aligned}$$

The S/M is unstable.

b) $y(t) = (t+5) u(t)$

$y(t) = (t+5)$ for $t \geq 0$

$$\int_0^{\infty} |y(t)| dt = \int_0^{\infty} (t+5) dt = \left[\frac{t^2}{2} + 5t \right]_0^{\infty}$$

$= \infty$
So the S/M is unstable.

c) $h(n) = a^n$; $0 < n < \infty$

$$\begin{aligned} \sum_{n=-\infty}^{\infty} |h(n)| &< \infty \\ \sum_{n=0}^{\infty} |a|^n &= \frac{1 - a^{\infty}}{1 - a} < \infty \end{aligned}$$

stable

$$y(n) = a x(n-7)$$

if $x(n-7)$ is bounded

$$x(n-7) \leq |x_2| < \infty$$

$$\text{if } y(n) = a x(n-7) \\ = a |x(n-7)| < \infty$$

So the s/m is stable.

if

$$y(n) = 1(n)$$

$$\text{if } x(n) = \delta(n)$$

$$\Rightarrow 1(n) = a \delta(n-7) \\ = a \delta(n-7)$$

$$\sum_{n=-\infty}^{\infty} |h(n)| = a < \infty$$

∴ so the s/m is stable

Representation of Systems:

Differential Equation representation of continuous time Systems.

Consider an LTI system described by n^{th} order constant coefficient differential equation with a single input $x(t)$

$$a_0 \frac{d^n y(t)}{dt^n} + a_1 \frac{d^{n-1} y(t)}{dt^{n-1}} + \dots + a_{n-1} \frac{dy(t)}{dt} + a_n y(t) = x(t)$$

$$\text{Let } S^n = \frac{d^n}{dt^n}$$

$$[a_0 S^n + a_1 S^{n-1} + \dots + a_{n-1} S + a_n] y(t) = x(t).$$

Natural, forced & Total Response:

The total Response is the sum of forced & natural Response.

The roots of characteristic equation determines the time domain expression of natural Response.

Characteristic eqn: $a_0 S^n + a_1 S^{n-1} + \dots + a_n = 0$

Natural Response: $y_N(t) = k_1 e^{s_1 t} + k_2 e^{s_2 t} + \dots + k_n e^{s_n t}$

where $s_1, s_2, \dots, s_n$ are the roots of characteristic eqn. & $k_1, k_2, \dots, k_n$ are constants.

Natural Response

Form of characteristic equation

Form of Natural Response

1 Real & distinct; γ

$$K e^{\gamma t}$$

2 Complex conjugate
 $a \pm j\omega$

$$e^{at} [K_1 \cos \omega t + K_2 \sin \omega t]$$

3 Real, repeated: γ^{p+1}

$$e^{\gamma t} [K_0 + K_1 t + K_2 t^2 + \dots + K_p t^p]$$

Forced Response

Input

1 C_0

2 $e^{\gamma t}$

3 $\cos(\omega t + \beta)$

4 $e^{\gamma t} \cos(\omega t + \beta)$

5 t

6 t^p

7 $t^p e^{\gamma t}$

Structure of forced response

C_1

$C e^{\gamma t}$

$C_1 \cos \omega t + C_2 \sin \omega t$

$e^{\gamma t} [C_1 \cos \omega t + C_2 \sin \omega t]$

$C_0 + C_1 t$

$C_0 + C_1 t + C_2 t^2 + \dots$

$e^{\gamma t} (C_0 + C_1 t)$

Difference Equation Representation of Discrete Systems

Let us consider an N^{th} order difference equation with a single input:-

$$y(n) + A_1 y(n-1) + A_2 y(n-2) + \dots + A_N y(n-N) = x(n)$$

The solution of above difference equation is the sum of Natural Response & Forced Response.

The characteristic eqn. given by

$$1 + A_1 z^{-1} + A_2 z^{-2} + \dots + A_N z^{-N} = 0$$

$$\Rightarrow z^N + A_1 z^{N-1} + \dots + A_N = 0$$

The natural Response is given by:-

$$Y_N(n) = K_1 z_1^n + K_2 z_2^n + \dots + K_N z_N^n$$

Roots of charach eqn.

Structure of Natural Response.

1. Real & distinct, γ $\Rightarrow \gamma^n$

2. Complex conjugate: γe^{jn}

$$\gamma^n [K_1 \cos(n\omega) + K_2 \sin(n\omega)]$$

3. Real, repeated: γ^{p+1}

$$\gamma^n [K_0 + K_1 n + K_2 n^2 + \dots + K_p n^p]$$

General Solution:-

Ans:-

$$1. C_0$$

$$2. \alpha^n$$

$$3. C_0(n, n+p)$$

$$4. \alpha^n \cos(n\alpha + \beta)$$

$$5. n$$

$$6. n^2$$

$$7. n \alpha^n$$

$$8. n^2 \alpha^n$$

Structure of book

$$C_1$$

$$C_2 \alpha^n$$

$$C_1 \cos n\alpha + C_2 \sin n\alpha$$

$$\alpha^n (C_1 \cos n\alpha + C_2 \sin n\alpha)$$

$$C_0 + C_1 n$$

$$C_0 + C_1 n + C_2 n^2 + \dots + C_p n^p$$

$$\alpha^n [C_0 + C_1 n]$$

$$\alpha^n [C_0 + C_1 n + C_2 n^2 + \dots + C_p n^p]$$

For a system $T[x(n)] = ax(n) + b$

Check for a) Stability b) Causality c) Linearity
d) Time invariant e) Memory

Soln:

$$\text{Let } |x(n)| \leq M_x < \infty$$

$$|y(n)| = |ax(n) + b| = |a||x(n)| + |b| \leq \infty$$

Then the S/M is stable

Causality

$$y(n) = ax(n) + b$$

Causal

Linearity

$$\begin{aligned} T[ax_1(n) + bx_2(n)] &= a[ax_1(n) + bx_2(n)] + b \\ &= a^2x_1(n) + abx_2(n) + b \\ ax_1(n) + b T[x_2(n)] &= a[ax_1(n) + b] + b[ax_2(n) + b] \\ &= a^2x_1(n) + ab + abx_2(n) + b^2 \end{aligned}$$

$LHS \neq RHS$

Not Linear.

Time invariant:

$$y(n-m) = T[x(n-m)]$$

$$\frac{L.H.S}{y(n-m)} = a x(n-m) + b$$

$$\frac{R.H.S}{T[x(n-m)]} = a x(n-m) + b$$

$$L.H.S = R.H.S$$

Time invariant

$$y(n) = a x(n) + b$$

% depends on present i/p
Memory less (static system)

Draw the waveforms
Step functions

represented by following

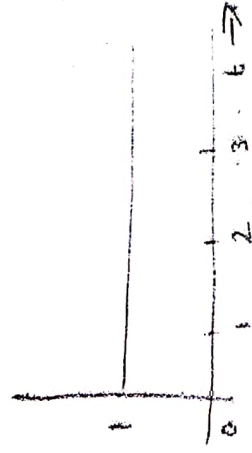
$$1) f_1(t) = 2u(t-1)$$

$$2) f_2(t) = -2u(t-2)$$

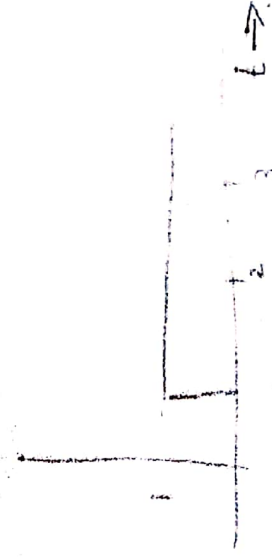
$$3) f(t) = f_1(t) + f_2(t)$$

$$4) f(t) = f_1(t) - f_2(t)$$

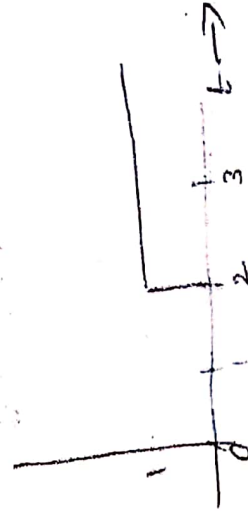
$u(t)$



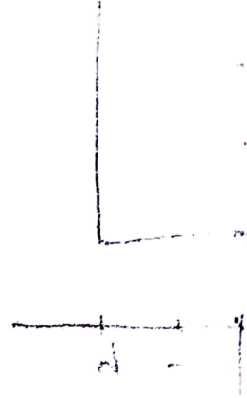
$u(t-1)$



$u(t-2)$



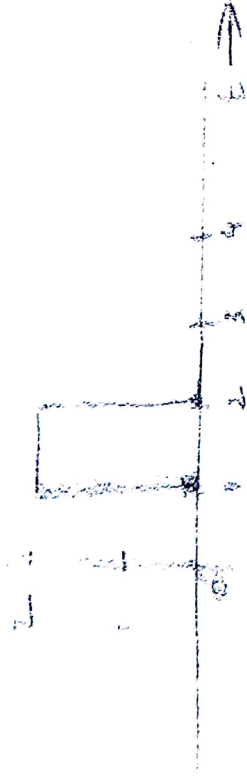
$$f_1(t) = 2u(t-1)$$



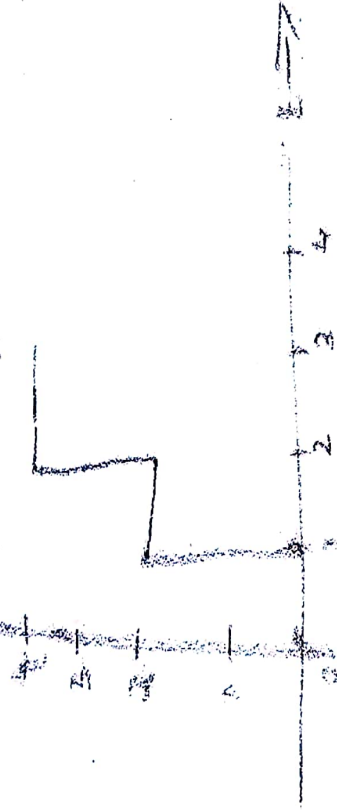
$$f_2(t) = -2u(t-2)$$



$$f(t) = \begin{cases} 1 & 0 \leq t < 2 \\ 0 & t \geq 2 \end{cases}$$

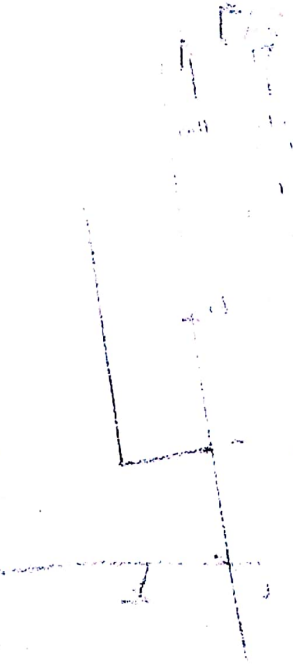


$$f_2(t) = f_1(t) - f_2(t)$$



Q.3. Draw the waveform for the following

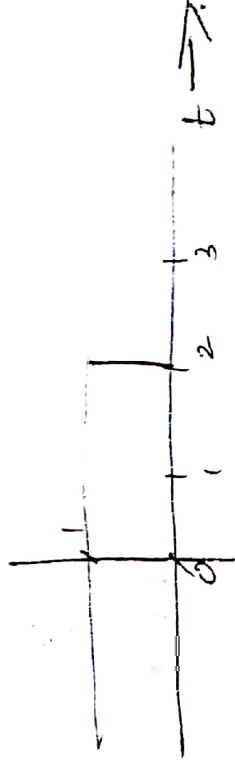
$$f_2(t) = ku(t-1)$$



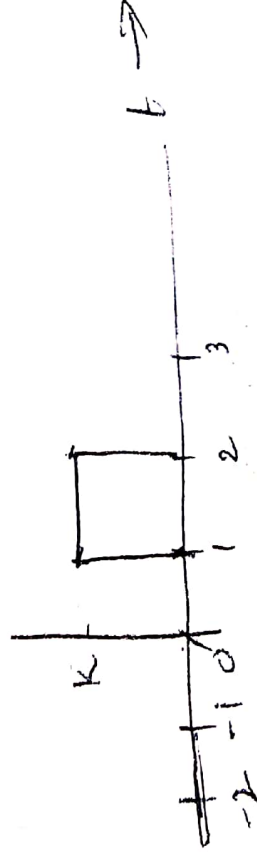
$$f_2(t) = u(t-1)$$



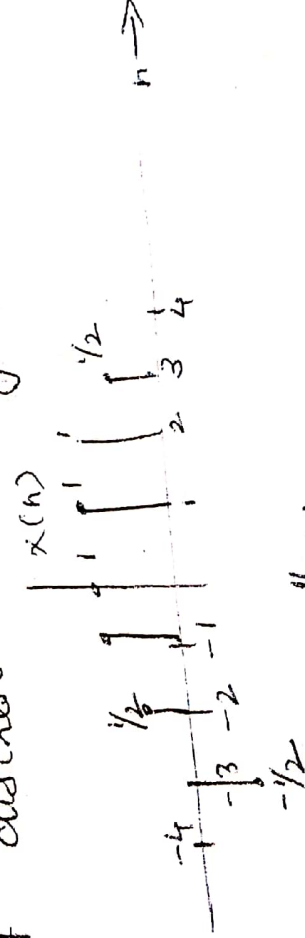
$$f_2(t) = u(-(t-2)) = u(2-t)$$



$$f(t) = f_1(t) \cdot f_2(t)$$

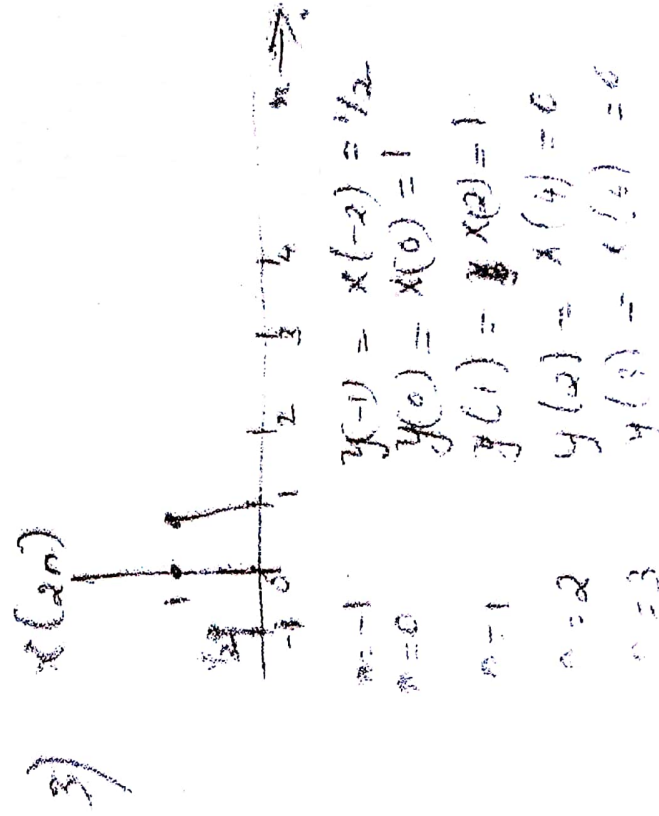
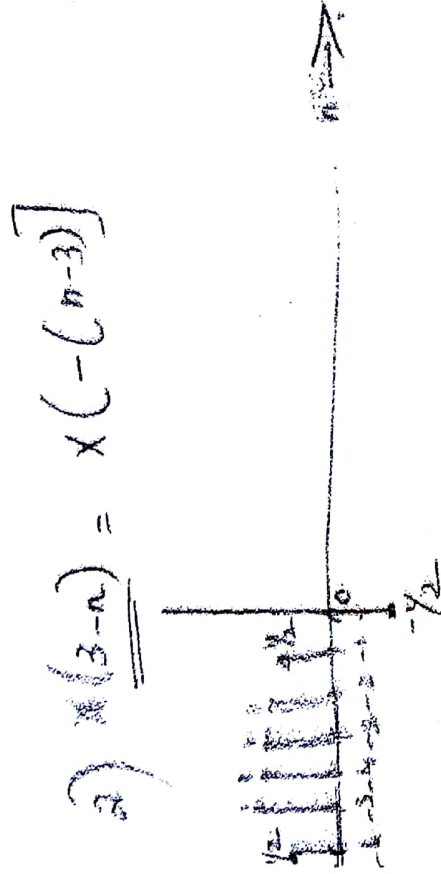
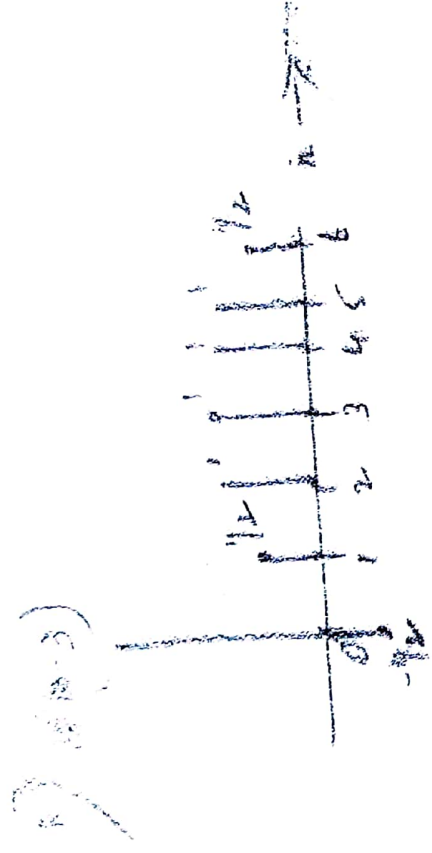


Qn 3. A discrete time signal is as shown as

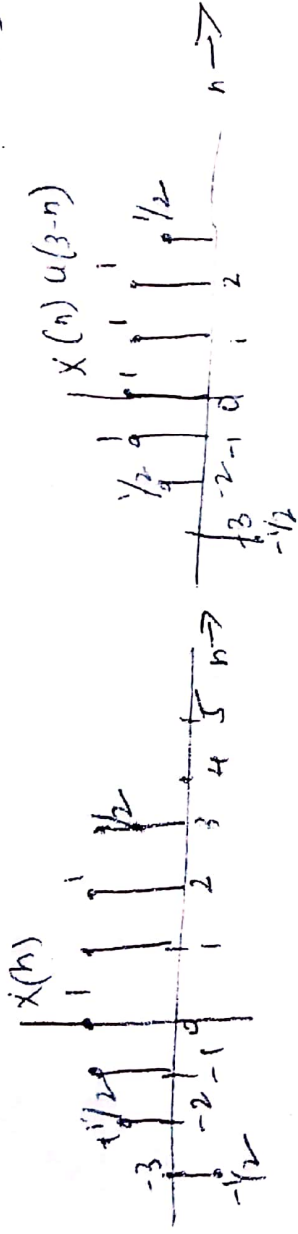
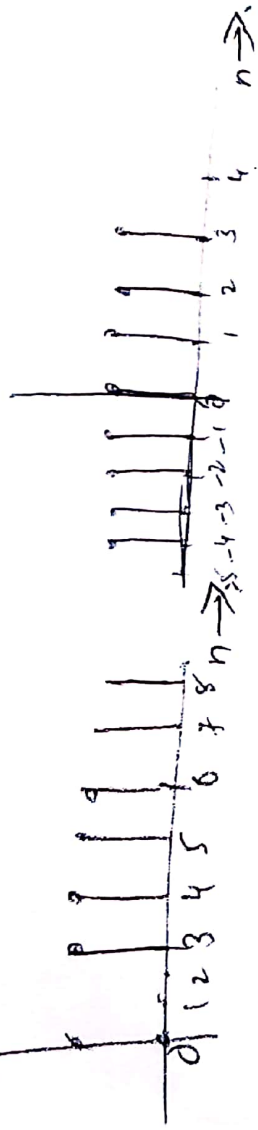


Sketch the following

- 1) $x(n-3)$
- 2) $x(3-n)$
- 3) $x(2n)$
- 4) $x(n) \cdot u(3-n)$
- 5) $[(n-1)^2]$



$$y(n) = x(n) \cdot u(3-n) \quad u(-(n-3)) = u(3-n)$$



$$5) y(n) = x[n(n-1)^2]$$

$$\begin{aligned} n=0 & \quad y(0) = x(1) = 1 \\ n=1 & \quad y(1) = x(0) = 1 \\ n=2 & \quad y(2) = x(1) = 1 \\ n=3 & \quad y(3) = x(4) = 0 \\ n=-1 & \quad y(-1) = x(4) = 0 \\ n=-2 & \quad y(-2) = x(4) = 0 \end{aligned}$$



Module 11

Laplace Transform Analysis of systems - Relation between the Transfer Function & differential equation - Causality & Stability - Inverse system - Determining the time domain & frequency response from poles & zeros.

Laplace Transform:

• LTI system described by differential equations.

The response $y(t)$ is obtained by solving the differential equation. So Laplace Transform is used to solve the differential equations.

It is given by;

$$\mathcal{L}[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt \quad [\text{Bilateral}]$$

where $s = \sigma + j\omega$

$$X(s) = \int_0^{\infty} x(t) \cdot e^{-st} \cdot dt \quad [\text{unilateral}]$$

$$x(t) \xleftrightarrow{\text{LT}} X(s)$$

Relation between Transfer Function & Differential Equation.

Laplace Transform Technique can be used to obtain the transfer function of a constant -

Coefficient differential Equation

$$\frac{d^N}{dt^N} y(t) + a_{N-1} \frac{d^{N-1}}{dt^{N-1}} y(t) + \dots + a_1 \frac{d}{dt} y(t) + a_0 y(t) = b_m \frac{d^m}{dt^m} x(t) + b_{m-1} \frac{d^{m-1}}{dt^{m-1}} x(t) + b_1 \frac{d}{dt} x(t) + b_0 x(t)$$

Taking Laplace Transform on Both sides;

$$\left[s^N + a_{N-1} s^{N-1} + \dots + a_1 s + a_0 \right] Y(s) = \left[b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0 \right] X(s)$$

$$Y(s) A(s) - C(s) = B(s) X(s)$$

$$Y(s) = \frac{B(s) X(s)}{A(s)} + \frac{C(s)}{A(s)}$$

where $Y(s) = Y^F(s) + Y^n(s)$

$$Y^F(s) = \frac{B(s) X(s)}{A(s)} \rightarrow \text{Forced Response}$$

$$Y^n(s) = \frac{C(s)}{A(s)} \rightarrow \text{Natural Response}$$

$C(s) = 0$ if all initial conditions are zero

Note:

$$C(s) = \sum_{k=1}^N \sum_{l=0}^{k-1} a_k s^{k-1-l} \frac{d^l}{dt^l} y(t) \Big|_{t=0^-}$$

Laplace Transform & ROC

($s = \sigma + j\omega$)

The range of σ for which Laplace transform converges is known as Region of Convergence ROC

- ROC is required in computing Laplace Transform & Inverse Laplace Transform.

1. Evaluate the Laplace Transform & ROC of Signal $x(t) = e^{at} u(t)$

$$X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt = \int_{-\infty}^{\infty} e^{at} u(t) \cdot e^{-st} \cdot dt$$

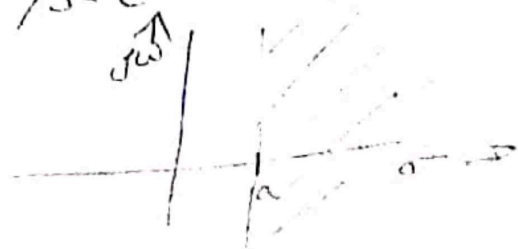
$$= \int_0^{\infty} e^{(a-s)t} \cdot dt$$

$$= \int_0^{\infty} e^{-(s-a)t} \cdot dt = \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^{\infty}$$

$$= 0 + \frac{1}{(s-a)} = \frac{1}{s-a} \quad \text{for } (s-a) > 0$$

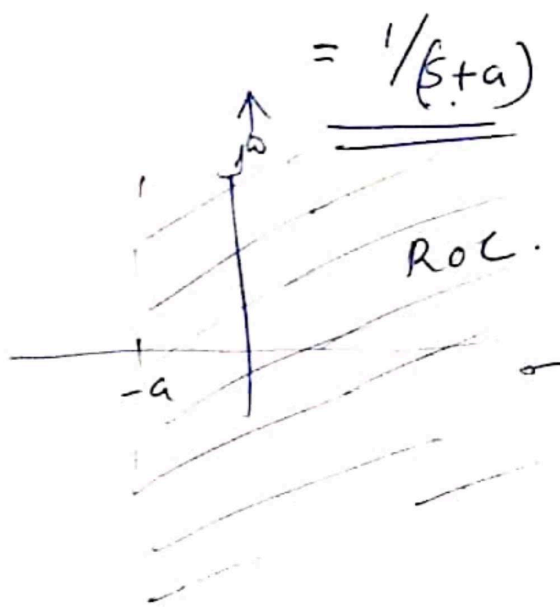
$s > a$

Laplace Transform $e^{at} u(t) = \frac{1}{s-a}$; for ROC $s > a$



2) LT & ROC $e^{-at} u(t)$

$$\begin{aligned}
 X(s) &= \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt \\
 &= \int_0^{\infty} e^{-at} e^{-st} \cdot dt \\
 &= \int_0^{\infty} e^{-(s+a)t} \cdot dt \\
 &= \left[\frac{e^{-(s+a)t}}{-(s+a)} \right]_0^{\infty} = -\frac{1}{(s+a)} \left[0 - \frac{1}{(s+a)} \right]
 \end{aligned}$$



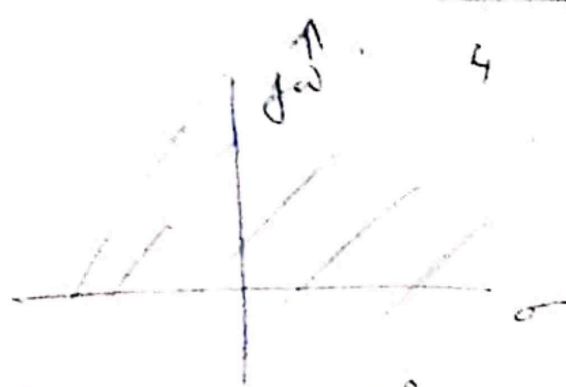
$$= \frac{1}{s+a}$$

$$\begin{aligned}
 & \text{Re}(s+a) > 0 \\
 & \text{Re}(s) > -a \\
 & \sigma > -a
 \end{aligned}$$

Qn. $x(t) = \delta(t)$. Find LT ?

$$\begin{aligned}
 \delta(t) &= 1 && \text{for } t=0 \\
 &= 0 && \text{for } t \neq 0
 \end{aligned}$$

$$L[\delta(t)] = \int_0^{\infty} \delta(t) \cdot e^{-st} \cdot dt = \frac{e^{-st}}{t=0} = 1$$



$$\int u v dx = uv - \int u' v$$

ROC is the entire s-plane.

$$\int u \cdot v \cdot dx = u \int v - \int du \cdot v$$

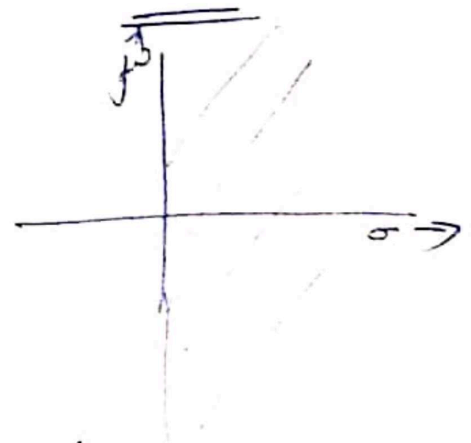
Qn $x(t) = t u(t)$ Find LT.

$$\mathcal{L}[t u(t)] = \int_0^{\infty} t \cdot e^{-st} \cdot dt$$

$$= \left[t \cdot \frac{e^{-st}}{-s} - e^{-st} \cdot \frac{1}{s} \right]_0^{\infty}$$

$$= (\infty > 0) - \left(-\frac{1}{s^2} \right)$$

ROC: $\text{Re}(s^2) > 0$
 $\sigma > 0$

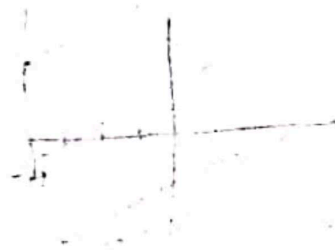
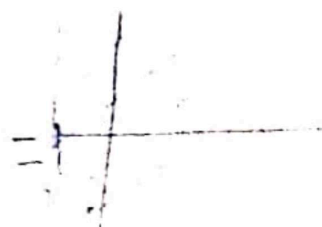


Qn Find the LT of the signal

$$x(t) = e^{-t} u(t) + e^{4t} u(t)$$

$$= \frac{1}{s+1} + \frac{1}{s+4}$$

ROC: $\text{Re}(s) > -1$ $\text{Re}(s) > -4$



Qn. $x(t) = e^{2t} u(t) + e^{-2t} u(-t)$

$$\begin{aligned}
 X(s) &= \int_0^{\infty} e^{2t} e^{-st} dt + \int_{-\infty}^0 e^{-2t} e^{st} dt \\
 &= \int_0^{\infty} e^{-(s-2)t} dt + \int_{-\infty}^0 e^{-(s+2)t} dt \\
 &= \int_0^{\infty} e^{-(s-2)t} dt + \int_0^{\infty} e^{(s+2)t} dt \\
 &= \left[\frac{e^{-(s-2)t}}{-(s-2)} + \frac{e^{(s+2)t}}{s+2} \right]_0^{\infty} \\
 &= \frac{1}{(s-2)} + \frac{1}{-(s+2)}
 \end{aligned}$$

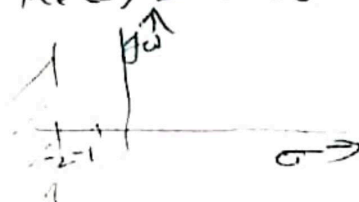
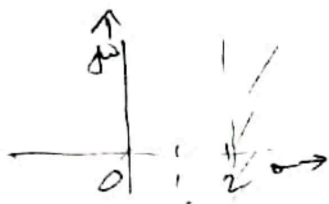
$u(-t) = 1$
 $-t \geq 0$
 $t \leq 0$

ROC: $\text{Re}\{s\} > 2$

$\text{Re}(-s-2) > 0$

$\text{Re}(-s) > 2$

$\text{Re}(s) < -2$



$\rightarrow$ LT does not converge for any real value of 's'
 $\rightarrow$ NO ROC

Properties of Laplace Transform.

1. Linearity Property:

Laplace Transform of a weighted sum of two signals is equal to the weighted sum of individual Laplace Transforms.

$$\text{if } \begin{array}{ccc} x_1(t) & \xleftrightarrow{\text{LT}} & X_1(s) \\ x_2(t) & \xleftrightarrow{\text{LT}} & X_2(s) \end{array}$$

Then

$$a x_1(t) + b x_2(t) \longleftrightarrow a X_1(s) + b X_2(s).$$

Proof:

$$\begin{aligned} \mathcal{L}[a x_1(t) + b x_2(t)] &= \int_0^{\infty} [a x_1(t) + b x_2(t)] e^{-st} dt \\ &= a \int_0^{\infty} x_1(t) \cdot e^{-st} dt + b \int_0^{\infty} x_2(t) \cdot e^{-st} dt \\ &= \underline{a X_1(s) + b X_2(s)} \end{aligned}$$

2. Time shift Property:

$$\text{if } x(t) \xleftrightarrow{\text{LT}} X(s)$$

$$\text{Then } x(t-t_0) \xleftrightarrow{\text{LT}} e^{-st_0} X(s)$$

Proof:

$$\begin{aligned} \mathcal{L}[x(t-t_0)] &= \int_0^{\infty} x(t-t_0) \cdot e^{-st} dt \\ &= \int_0^{\infty} x(p) e^{-s(p+t_0)} dp \\ &= e^{-st_0} \int_0^{\infty} x(p) \cdot e^{-sp} dp \\ &= \underline{e^{-st_0} X(s)} \end{aligned}$$

$$\left| \begin{array}{l} \text{put} \\ t-t_0 = p \\ t = p+t_0 \end{array} \right.$$

3. Time Reversal Property:

$$\text{if } x(t) \xleftrightarrow{LT} X(s)$$

$$\text{Then } x(-t) \xleftrightarrow{LT} X(-s)$$

Proof:

$$\begin{aligned} \mathcal{L}[x(-t)] &= \int_0^{\infty} x(-t) \cdot e^{-st} \cdot dt \\ &= \int_{-\infty}^0 x(p) \cdot e^{sp} \cdot dt \quad \left| \text{put } p = -t \right. \\ &= \int_0^{\infty} x(p) \cdot e^{-(-s)p} \cdot dp \\ &= \underline{\underline{X(-s)}} \end{aligned}$$

4. Frequency shift property:

$$\text{if } x(t) \xleftrightarrow{LT} X(s)$$

$$\text{Then } e^{-at} x(t) \xleftrightarrow{LT} X(s+a)$$

Proof:

$$\begin{aligned} \mathcal{L}[e^{-at} x(t)] &= \int_0^{\infty} e^{-at} x(t) \cdot e^{-st} \cdot dt \\ &= \int_0^{\infty} x(t) \cdot e^{-(s+a)t} \cdot dt \\ &= \underline{\underline{X(s+a)}} \end{aligned}$$

5. Time convolution property:

$$\text{if } \begin{matrix} x_1(t) & \xleftrightarrow{LT} & X_1(s) \\ x_2(t) & \xleftrightarrow{LT} & X_2(s) \end{matrix} \quad \& \quad \text{Then} \quad x_1(t) * x_2(t) = X_1(s) \cdot X_2(s)$$

$$\text{Then } x_1(t) * x_2(t) = \int_0^{\infty} x_1(\tau) x_2(t-\tau) d\tau$$

$$= \int_0^{\infty} x_1(t-\tau) \cdot x_2(\tau) d\tau$$

$$\Rightarrow \mathcal{L}[x_1(t) * x_2(t)] = \int_0^{\infty} \left[\int_0^{\infty} x_1(t-\tau) \cdot x_2(\tau) d\tau \right] e^{-st} dt$$

$$= \int_0^{\infty} x_2(\tau) \cdot \int_0^{\infty} x_1(p) \cdot e^{-(t+p)s} dp d\tau \quad \left| \begin{array}{l} \text{put } t-\tau=p \\ dt = dp \end{array} \right.$$

$$= \int_0^{\infty} x_2(\tau) \cdot e^{-s\tau} d\tau \cdot \int_0^{\infty} x_1(p) \cdot e^{-sp} dp$$

$$= \underline{X_2(s) \cdot X_1(s)}$$

$$= \underline{X_1(s) \cdot X_2(s)}$$

6. Transform of derivatives property:

$$\text{if } x(t) \xleftrightarrow{LT} X(s)$$

$$\text{Then } \frac{d}{dt} x(t) \xleftrightarrow{LT} sX(s) - x(0)$$

Initial value Theorem

$$\text{if } x(t) \xleftrightarrow{LT} X(s)$$

$$\text{Then } \lim_{t \rightarrow 0} x(t) = x(0) = \lim_{s \rightarrow \infty} s X(s)$$

Proof:

$$\mathcal{L} \left[\frac{d}{dt} x(t) \right] = S X(S) - X(0)$$

$$\int_0^{\infty} \frac{d}{dt} x(t) \cdot e^{-st} \cdot dt = S X(S) - X(0)$$

Take $S \rightarrow \infty$ on both sides.

$$\lim_{S \rightarrow \infty} \left[\int_0^{\infty} \frac{d}{dt} x(t) \cdot e^{-st} \cdot dt \right] = \lim_{S \rightarrow \infty} [S X(S) - X(0)]$$

$$0 = \lim_{S \rightarrow \infty} [S X(S) - X(0)]$$

$$\underline{\underline{X(0) = \lim_{S \rightarrow \infty} S X(S)}}$$

8. Final value Theorem

$$\text{if } x(t) \xleftrightarrow{LT} X(S)$$

$$\text{Then } \lim_{t \rightarrow \infty} x(t) = x(\infty) = \lim_{S \rightarrow 0} S X(S)$$

Proof:

$$\mathcal{L} \left[\frac{d}{dt} x(t) \right] = S X(S) - X(0)$$

$$\lim_{S \rightarrow 0} \int_0^{\infty} \frac{d}{dt} x(t) \cdot e^{-st} \cdot dt = \lim_{S \rightarrow 0} [S X(S) - X(0)]$$

$$\lim_{s \rightarrow 0} \left[\int_0^{\infty} \frac{d}{dt} x(t) \cdot dt \right] = \lim_{s \rightarrow 0} s X(s) - x(0)$$

$$\left[x(t) \right]_0^{\infty} = \lim_{s \rightarrow 0} s X(s) - x(0)$$

$$x(\infty) - x(0) = \lim_{s \rightarrow 0} s X(s) - x(0)$$

$$\underline{\underline{x(\infty) = \lim_{s \rightarrow 0} s X(s)}}$$

Commonly used Laplace Transform Pairs

SL. NO.	$x(t)$	$X(s)$	ROC
1.	$\delta(t)$	1	Entire s-plane
2.	$u(t)$	$1/s$	$\text{Re}(s) > 0$
3.	$x(t) = t$	$1/s^2$	$\text{Re}(s) > 0$
4.	$e^{at} u(t)$	$1/s-a$	$\text{Re}(s) > a$
5.	$e^{-at} u(-t)$	$1/s-a$	$\text{Re}(s) < a$
6.	$\sin \omega_0 t \cdot u(t)$	$\frac{\omega_0}{s^2 + \omega_0^2}$	$\text{Re}(s) > 0$
7.	$\cos \omega_0 t \cdot u(t)$	$\frac{s}{s^2 + \omega_0^2}$	$\text{Re}(s) > 0$
8.	$e^{-at} \sin \omega_0 t$	$\frac{\omega_0}{(s+a)^2 + \omega_0^2}$	$\text{Re}(s) > -a$
9.	$e^{-at} \cos \omega_0 t$	$\frac{s+a}{(s+a)^2 + \omega_0^2}$	$\text{Re}(s) > -a$
10.	$\frac{t^{n-1}}{(n-1)!} u(t)$	$1/s^n$	$\text{Re}(s) > 0$

- | | | | |
|-----|---|-------------------------------|-----------------------------|
| 11. | $-\frac{t^{n-1}}{(n-1)!} u(-t)$ | $\frac{1}{s^n}$ | $\operatorname{Re}(s) > 0$ |
| 12. | $\frac{t^{n-1}}{(n-1)!} e^{-at} u(t)$ | $\frac{1}{(s+a)^n}$ | $\operatorname{Re}(s) > -a$ |
| 13. | $-\frac{t^{n-1}}{(n-1)!} e^{-at} u(-t)$ | $\frac{1}{(s+a)^n}$ | $\operatorname{Re}(s) < -a$ |
| 14. | $\cosh at$ | $\frac{s}{s^2 - a^2}$ | |
| 15. | $\sinh at$ | $\frac{a}{s^2 - a^2}$ | |
| 16. | $e^{-at} \cosh bt$ | $\frac{(s+a)}{(s+a)^2 - b^2}$ | |
| 17. | $e^{-at} \sinh bt$ | $\frac{b}{(s+a)^2 - b^2}$ | |

Inverse Laplace Transform:

9

$$x(t) = \frac{1}{2\pi j} \int_{C-j\infty}^{C+j\infty} X(s) \cdot e^{st} \cdot ds$$

Partial Fraction Expansion can be used, or convolution Integration.

(Case: a

Review of Partial Fraction Expansion:-

Case 1: Function with separate poles

$$T(s) = \frac{k}{s(s+p_1)(s+p_2)} = \frac{A}{s} + \frac{B}{s+p_1} + \frac{C}{s+p_2}$$

Residues A, B, C

$$A = T(s) \cdot s \bigg|_{s=0}$$

$$B = T(s) (s+p_1) \bigg|_{s=-p_1}$$

$$C = T(s) (s+p_2) \bigg|_{s=-p_2}$$

For eg: $T(s) = \frac{2}{s(s+1)(s+2)}$

$$\frac{2}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$A = \frac{2}{s(s+1)(s+2)} \cdot s \bigg|_{s=0} = \frac{1}{(s+1)(s+2)} \bigg|_{s=0}$$

$$B = \frac{-2}{(s)(s+2)} \bigg|_{s=-1}$$

$$C = \frac{1}{(s)(s+1)} \bigg|_{s=-2}$$

$$A(s+1)(s+2) + B(s)(s+2) + C(s)(s+1)$$

To find A

$$A = \frac{2}{s(s+1)(s+2)} \times s \bigg|_{s=0} = \frac{2}{(-1)(-2)} = 1$$

$$= \frac{1}{s} - \frac{2}{s+1} + \frac{1}{s+2}$$

$$\mathcal{L}^{-1} [T(s)] = \underline{\underline{u(t) - 2e^{-t} + e^{-2t}}}$$

Case 2: Function with Multiple poles.

$$T(s) = \frac{k}{s(s+p_1)(s+p_2)^2}$$

$$T(s) = \frac{A}{s} + \frac{B}{s+p_1} + \frac{C}{(s+p_2)^2} + \frac{D}{(s+p_2)}$$

The residues

$$A = T(s) \cdot s \Big|_{s=0}$$

$$B = T(s) \cdot (s+p_1) \Big|_{s=-p_1}$$

$$C = T(s) \cdot (s+p_2)^2 \Big|_{s=-p_2}$$

$$D = \frac{d}{ds} [T(s) \cdot (s+p_2)^2] \Big|_{s=-p_2}$$

$$= \frac{d}{ds} \left[\frac{2}{s(s+1)} \right] \\ = \frac{d}{ds} \left(\frac{2}{s^2+s} \right)$$

for eg: $T(s) = \frac{2}{s(s+1)(s+2)^2}$

$$= \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+2)^2} + \frac{D}{(s+2)}$$

$$A = \frac{1}{2}$$

$$B = -2$$

$$C = 1$$

$$D = 1.5$$

$$= \frac{(\frac{1}{2})}{s} - \frac{2}{(s+1)} + \frac{1}{(s+2)^2} + \frac{1.5}{(s+2)}$$

$$\mathcal{L}^{-1}[T(s)] = \frac{1}{2} - 2e^{-t} + t \cdot e^{-2t} + 1.5e^{-2t}$$

Case 3:

Function with complex conjugate poles.

$$T(s) = \frac{k}{(s+p_1)(s^2+bs+c)}$$

$$= \frac{A}{s+p_1} + \frac{Bs+C}{s^2+Bs+C}$$

$$A = T(s) \cdot (s+p_1) / s = -p_1$$

For eg:

$$T(s) = \frac{1}{(s+2)(s^2+s+1)}$$

$$= \frac{A}{s+2} + \frac{Bs+C}{s^2+s+1}$$

$$A = \underline{\underline{1/3}}$$

$$T(s) = \frac{1}{(s+2)(s^2+s+1)}$$

$$\frac{1}{(s+2)(s^2+s+1)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+s+1}$$

Taking LCM in RHS

$$\frac{A(s^2+s+1) + Bs+C(s+2)}{s^2+s+1}$$

equating numerator

$$1 = A(s^2+s+1) + Bs+C(s+2)$$

$$1 = A(s^2 + s + 1) + (Bs + C)(s + 2)$$

Comp. coeff. s^2

Compute coefficient of 's'
 $0 = A + B$ i.e.; $B = -\frac{1}{3}$

Comp. coeff. s

$$0 = A + 2B + C$$

Compute coefficient of 's'
 $= \frac{1}{3} + \frac{-2}{3} + C$
 $= -\frac{1}{3} + C$

$$C = \frac{1}{3}$$

$$T(s) = \frac{(\frac{1}{3})}{(s+2)} + \frac{-\frac{1}{3}s + \frac{1}{3}}{(s+2)^2}$$

$$= \frac{1}{3} \frac{1}{(s+2)} - \frac{1}{3} \frac{s}{(s+0.5)^2 + 0.75} + \frac{1}{3} \frac{1}{(s+0.5)^2 + 0.75}$$

$$\mathcal{L}^{-1}[T(s)] = \frac{1}{3} e^{-2t} - \frac{1}{3} \left\{ \frac{s+0.5-0.5}{(s+0.5)^2 + 0.75} \right\} + \frac{1}{3} \left[\frac{1}{(s+0.5)^2 + 0.75} \right]$$

Case: b

if Degree of N_s = Degree of D_s & $N_s > D_s$
Then divide Numerator by Denominator.

$$X(s) = \frac{N(s)}{D(s)}$$

$$\text{Let } F(s) = \frac{s^3 + 3s^2 + 3s + 2}{s^2 + 2s + 2}$$

$$= (s+1) - \left\{ \frac{s}{s^2 + 2s + 2} \right\}$$

$$\begin{array}{r} s+1 \\ \hline s^3 + 3s^2 + 3s + 2 \\ s^3 + 2s^2 + 2s \\ \hline s^2 + s + 2 \\ s^2 + 2s + 2 \\ \hline -s \end{array}$$

For eg:

$$1) \quad X(s) = \frac{s^3 + 1}{s(s+1)(s+2)} = \frac{s^3 + 1}{s^3 + 3s^2 + 2s}$$

$$X(s) = 1 - \frac{(3s^2 + 2s + 1)}{s(s+1)(s+2)}$$

$$\begin{array}{r} 1 \\ s^3 + 3s^2 + 2s \overline{) s^3 + 1} \\ \hline -3s^2 - 2s - 1 \end{array}$$

$$\text{Take } \frac{3s^2 + 2s + 1}{s(s+1)(s+2)}$$

$$= \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$A = -\frac{1}{2}$$

$$B = \frac{3 - 2 + 1}{(-1)(1)} = 0$$

$$C = \frac{(2 \times 1) - 1 + 1}{(-2)(-1)} = \frac{2}{2} = 1$$

$$X(s) = 1 - \left[\frac{-\frac{1}{2}}{s} + \frac{0}{s+1} + \frac{3.5}{s+2} \right]$$

$$\mathcal{L}^{-1}[X(s)] = x(t) = \underline{\underline{\delta(t) + \frac{1}{2}u(t) - 3.5e^{-2t}u(t)}}$$

$$2) \quad X(s) = \frac{(s-1)}{(s+1)(s^2+2s+5)}$$

$$\frac{(s-1)}{(s+1)(s^2+2s+5)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+2s+5}$$

$$A = \frac{-2}{1-2+5} = \frac{-2}{4} = \underline{\underline{-\frac{1}{2}}}$$

$$s-1 = A(s^2+2s+5) + (Bs+C)(s+1)$$

$$\text{Comp. } s^2 \quad 0 = A + B$$

$$B = \underline{\underline{\frac{1}{2}}}$$

$$\text{Comp. } s \quad 1 = 2A + B + C$$

$$1 = -1 + \frac{1}{2} + C$$

$$\text{Comp. } \text{const} \quad -1 = 5A + C$$

$$= -\frac{5}{2} + C$$

$$C = -1 + \frac{5}{2} = \underline{\underline{\frac{3}{2}}}$$

$$\begin{aligned}
&= \frac{\left(-\frac{1}{2}\right)}{s+1} + \frac{\frac{1}{2}s + \frac{3}{2}}{s^2+2s+5} \quad 12 \\
&= \frac{\left(-\frac{1}{2}\right)}{s+1} + \frac{\frac{1}{2}s}{(s+1)^2+2^2} + \frac{3/2}{(s+1)^2+2^2} \\
&= \left(-\frac{1}{2}\right) \frac{1}{(s+1)} + \frac{1}{2} \left[\frac{s+1}{(s+1)^2+2^2} - \frac{1}{(s+1)^2+2^2} \right] \\
&\quad + \frac{3/2}{2} \left[\frac{2}{(s+1)^2+2^2} \right] \\
&= \left(-\frac{1}{2}\right) \cdot \frac{1}{(s+1)} + \frac{1}{2} \cdot \frac{(s+1)}{(s+1)^2+2^2} - \frac{\left(\frac{1}{2}\right)}{(s+1)^2+2^2} \\
&\quad + \frac{3}{4} \cdot \frac{2}{(s+1)^2+2^2} \\
&= \left(-\frac{1}{2}\right) \frac{1}{(s+1)} + \frac{1}{2} \cdot \frac{(s+1)}{(s+1)^2+2^2} + \frac{3}{2} \cdot \frac{1}{(s+1)^2+2^2} \\
&\quad e^{-t} \cos 2t + \frac{1}{2} e^{-t} \sin 2t \\
\mathcal{L}^{-1} [X(s)] &= -\frac{1}{2} e^{-t} + \frac{1}{2} e^{-t} \cos 2t + \frac{1}{2} e^{-t} \sin 2t
\end{aligned}$$

$$\begin{aligned}
&\underline{Qn. 1} \quad \mathcal{L}^{-1} \text{ of } \frac{1}{(s+2)^2} = \underline{t \cdot e^{-2t} u(t)} \\
&\mathcal{L}^{-1} \text{ of } \frac{s}{(s+2)^2+1} = \mathcal{L}^{-1} \left[\frac{s+2-2}{(s+2)^2+1^2} \right] \\
&= \mathcal{L}^{-1} \left[\frac{s+2}{(s+2)^2+1^2} \right] - 2 \mathcal{L}^{-1} \left[\frac{1}{(s+2)^2+1^2} \right]
\end{aligned}$$

$$= \underline{e^{-2t} \cos t} - 2 \cdot e^{-2t} \sin t$$

$$\text{Qn. } \mathcal{L}^{-1} \left[\frac{s}{s^2-8} \right] = \mathcal{L}^{-1} \left[\frac{s}{2(s^2-4)} \right] = \frac{1}{2} \mathcal{L}^{-1} \left[\frac{s}{s^2-2^2} \right]$$

$$= \underline{\underline{\frac{1}{2} \cdot \cosh 2t}}$$

Qn. Obtain the Transfer function of the system
 if $y(t) = u(t) + e^{-t}u(t) - e^{-3t}u(t) - e^{-4t}u(t)$
 & if $x(t) = e^{-2t}u(t)$

$$Y(s) = \frac{1}{s} + \frac{1}{(s+1)} - \frac{1}{(s+3)} - \frac{1}{(s+4)}$$

$$= \frac{(s+1)(s+3)(s+4) + s(s+3)(s+4) - s(s+1)(s+4) - s(s+1)(s+3)}{s(s+1)(s+3)(s+4)}$$

$$= \frac{(s^2+7s+12)(s+1) + (s^2+7s+12)s - s(s^2+5s+4) - s(s^2+4s+3)}{s(s+1)(s+3)(s+4)}$$

$$= \frac{(s^3+s^2+7s^2+7s+12s+12) + s^3+7s^2+12s - s^3-5s^2-4s - s^3-4s^2-3s}{s(s+1)(s+3)(s+4)}$$

$$G(s) = \frac{6s^2 + 24s + 12}{s(s+1)(s+3)(s+4)}$$

$$= \frac{6(s^2 + 4s + 2)}{s(s+1)(s+3)(s+4)}$$

$$X(s) = \frac{1}{s+2}$$

$$\therefore T.F = \frac{Y(s)}{X(s)} = \frac{6(s^2 + 4s + 2)}{\cancel{s(s+1)(s+3)(s+4)} \cdot \left(\frac{1}{s+2}\right)}$$

$$= \frac{6(s^2 + 4s + 2)(s+2)}{s(s+1)(s+3)(s+4)}$$

Qn: Find the T.F & impulse Response of the S/M.

$$\frac{d^2 y(t)}{dt^2} + 7 \frac{dy}{dt} + 6y(t) = 7x(t) + 11 \frac{dx(t)}{dt}$$

$$s^2 Y(s) + 7sY(s) + 6Y(s) = 7X(s) + 11sX(s)$$

$$[s^2 + 7s + 6] Y(s) = [7 + 11s] X(s)$$

$$T.F = \frac{Y(s)}{X(s)} = \frac{11s + 7}{s^2 + 7s + 6}$$

impulse Response

$$x(t) = \delta(t)$$

$$X(s) = 1$$

$$\frac{Y(s)}{1} = \frac{11s+7}{s^2+7s+6} = \frac{11s+7}{(s+6)(s+1)}$$

$$= \frac{A}{s+1} + \frac{B}{s+6}$$

$$A = \underline{\underline{\frac{-4}{5}}} \quad B = \underline{\underline{\frac{+59}{5}}}$$

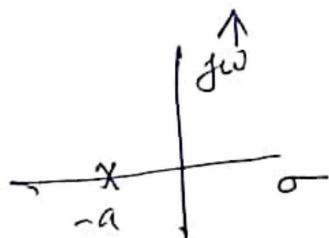
$$Y(s) = \frac{(-4/5)}{s+1} + \frac{(59/5)}{s+6}$$

$$\underline{\underline{y(t) = -4/5 \cdot e^{-t} + 59/5 \cdot e^{-6t}}}$$

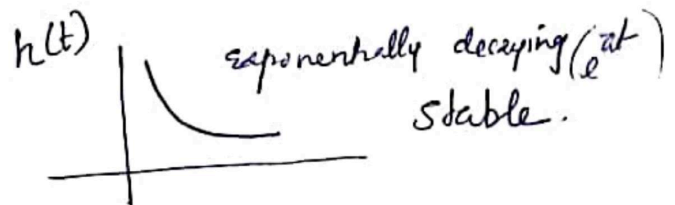
Determination of Time domain Response

Time domain response obtained from pole zero plot.

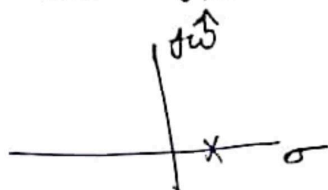
1) when the pole are at left side of s-domain



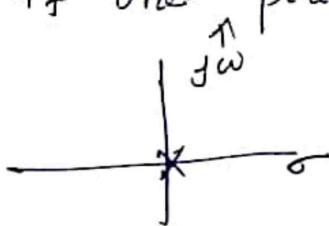
$$T_{1/2} = \frac{1}{\sigma + a}$$



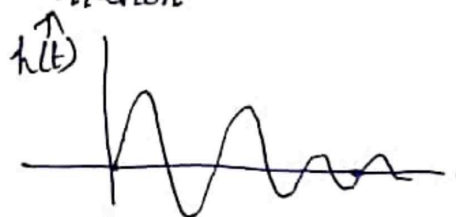
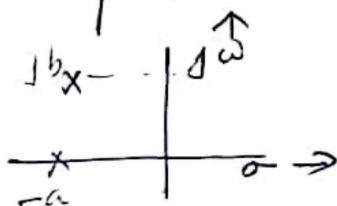
2) when the pole at Right side of s-plane
Then the response is exponentially increasing



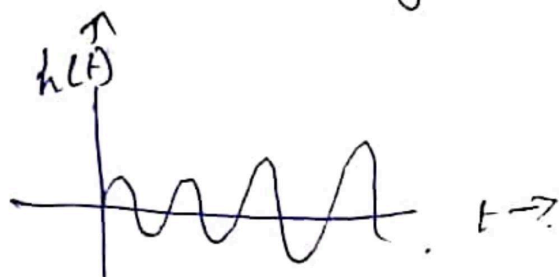
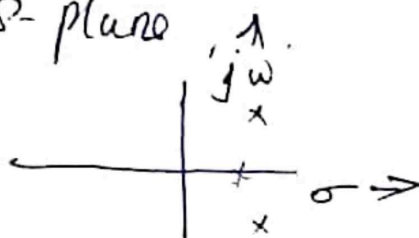
3) If the pole are at origin, Response is constant



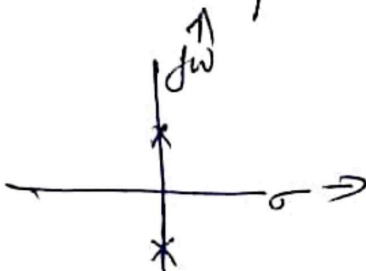
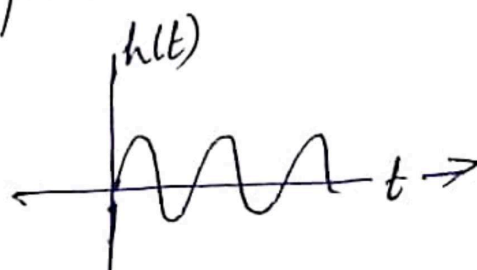
4) If the pair of poles at the left side of s-plane
Response is Sustained oscillation



5) If the pair of poles at the Right side of s-plane

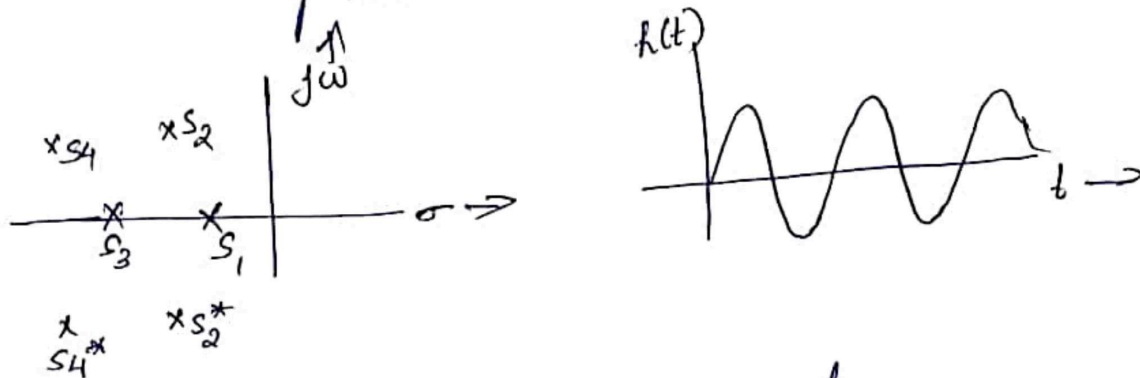


6) If the pair of poles at the Imaginary axis

marginaly stable or unstable.

7) If all the poles are at the left side of s-plane, The Response is stable



s_1 & s_3 are the Real poles
 s_2 & s_4 are the Conjugate poles.

For Real poles: $h(t) = k_1 e^{-s_1 t} + k_3 e^{-s_3 t}$

Quadratic Eqn: $s^2 + 2\zeta\omega_n s + \omega_n^2$ for $\zeta > 1$

$$s_1, s_3 \rightarrow -\zeta\omega_n \pm \omega_n \sqrt{\zeta^2 - 1}$$

For Conjugate poles

$$h(t) = k_2 e^{-s_2 t} + k_4 e^{-s_4 t}$$

The Quadratic Eqn: $s^2 + 2\zeta\omega_n s + \omega_n^2$ for $\zeta < 1$

$$s_2, s_2^*, s_4, s_4^* \rightarrow -\zeta\omega_n \pm j\omega_n \sqrt{1 - \zeta^2}$$

Time Domain Response from pole zero plot.

$$\text{Let } \bar{L}(s) = H \frac{(s-s_a)(s-s_b) \dots (s-s_n)}{(s-s_1)(s-s_2) \dots (s-s_m)}$$

H = Scale factor (gain)

$$\bar{L}(s) = \frac{k_1}{s-s_1} + \frac{k_2}{s-s_2} + \dots + \frac{k_m}{s-s_m}$$

$$i(t) = \bar{L}^{-1} \left[\frac{k_1}{s-s_1} + \frac{k_2}{s-s_2} + \dots + \frac{k_m}{s-s_m} \right]$$

To find

$$k_1 = H \frac{M_{1a} M_{1b} \dots M_{1n}}{M_{11} M_{12} \dots M_{1m}} \times e^{j(\phi_{1a} + \phi_{1b} + \dots + \phi_{1n}) - (\phi_{11} + \phi_{12} + \dots + \phi_{1m})}$$

Similarly, all coefficients $k_1, k_2 \dots k_m$ are obtained which constitute Magnitude & phase angle.

The residues obtained by pole-zero plot

1. Obtain the pole zero plot for the given N/w function
2. Measure the distances $M_{1a}, M_{1b} \dots M_{1n}$ of a given pole from each of the other zeros
3. Measure the distances $M_{11}, M_{12}, \dots M_{1m}$ of a given pole from each of the other poles.

4. Measure the angle $\phi_{1a}, \phi_{1b}, \dots, \phi_{1n}$ of the line joining that pole to each of other zeros
5. Measure the angle $\phi_{l1}, \phi_{l2}, \dots, \phi_{lm}$ of the line joining that pole to each of other poles.
6. Substitute these values in required residue equation.

Qn: Draw the pole zero diagram for the given N/w function $\bar{L}(s)$ & hence obtain $i(t)$

$$\bar{L}(s) = \frac{20s}{(s+5)(s+2)}$$

N/w function $L(s) = \frac{P(s)}{Q(s)}$

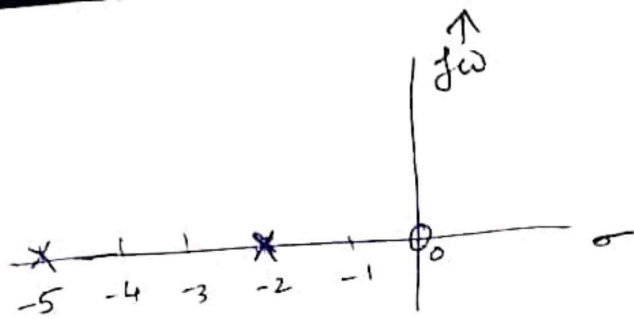
$$P(s) = 20s$$

$$Q(s) = (s+5)(s+2)$$

$$\bar{L}(s) = \frac{k_1}{s+2} + \frac{k_2}{s+5}$$

$$i(t) = k_1 e^{-2t} + k_2 e^{-5t}$$

k_1, k_2 determined by pole-zero plot



K_1 for $C(s) = -2$ i.e.; pole at -2 .

$$K_1 = H \frac{M_{02}}{M_{52}} \frac{e^{j\phi_{02}}}{e^{j\phi_{52}}} = 20 \times \frac{2}{3} \times e^{j180} = \underline{\underline{-13.33}}$$

$$H = 20$$

$$M_{02} = \underline{\underline{2}}$$

$$M_{52} = 3$$

$$\phi_{02} = 180^\circ$$

$$\phi_{52} = 0$$

$$K_2 = H \frac{M_{05}}{M_{25}} \frac{e^{j\phi_{05}}}{e^{j\phi_{25}}} = 20 \times \frac{5}{3} \times e^{j(180-180)} =$$

$$M_{05} = 5$$

$$M_{25} = 3$$

$$\phi_{05} = 180^\circ$$

$$\phi_{25} = 180^\circ$$

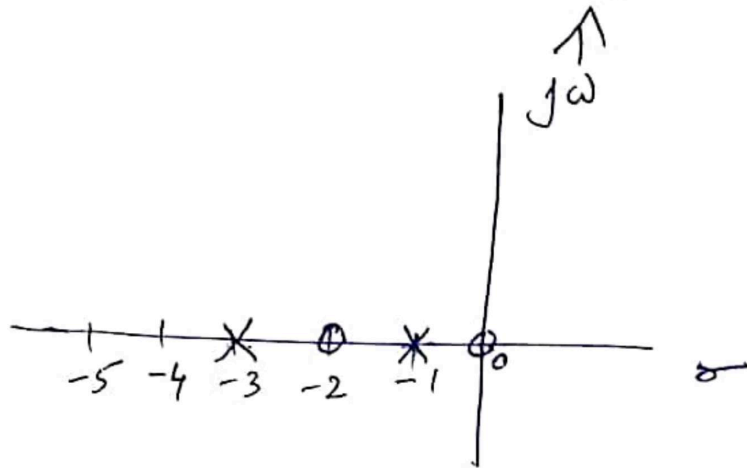
$$I(s) = \frac{-13.33}{s+2} + \frac{33.3}{s+5}$$

$$i(t) = \underline{\underline{-13.33 e^{-2t} + 33.3 e^{-5t}}}$$

Qn. Draw the pole zero diagram for the given N/w fn. & obtain $v(t)$

$$V(s) = \frac{4(s+2)s}{(s+1)(s+3)}$$

$$= \frac{K_1}{(s+1)} + \frac{K_2}{(s+3)}$$



$$K_1 \text{ at } s = -1$$

$$= H \cdot \frac{M_{01} M_{21} e^{j(\phi_{01} + \phi_{21})}}{M_{31} e^{j(\phi_{31})}}$$

$$H = 4$$

$$M_{01} = 1$$

$$\phi_{01} = 180^\circ$$

$$M_{21} = 1$$

$$\phi_{21} = 0^\circ$$

$$M_{31} = 2$$

$$\phi_{31} = 0^\circ$$

$$K_1 = 4 \times \frac{1}{2} \cdot e^{j180^\circ}$$

$$= \underline{\underline{-2}}$$

$$K_2 = H \cdot \frac{M_{03} M_{23} e^{j(\phi_{03} + \phi_{23})}}{M_{13} e^{j(\phi_{13})}} = \frac{2}{4} \times \frac{3 \times 1}{2} \cdot \frac{e^{j360^\circ}}{e^{j180^\circ}}$$

$$M_{03} = 3$$

$$\phi_{03} = 180^\circ$$

$$M_{23} = 1$$

$$\phi_{23} = 180^\circ$$

$$M_{13} = 2$$

$$\phi_{13} = 180^\circ$$

$$= 6 e^{j180^\circ} e^{j180^\circ}$$

$$= \underline{\underline{-6}}$$

$$V(s) = \frac{-2}{s+1} + \frac{-6}{s+3}$$

$$v(t) = \underline{\underline{-2e^{-t} - 6e^{-3t}}}$$

Q.3. For the given N/w function, draw the pole-zero diagram & hence obtain $i(t)$.

$$\hat{I}(s) = \frac{5s}{(s+1)(s^2+4s+8)}$$

$$\hat{I}(s) = \frac{K_1}{s+1} + \frac{K_2}{s+2+j2} + \frac{K_3}{s+2-j2}$$

$$i(t) = K_1 e^{-t} + K_2 e^{-(2+j2)t} + K_3 e^{-(2-j2)t}$$

$$K_1 = H \frac{M_{01} e^{j\phi_{01}}}{M_{p11} M_{p21} e^{j(\phi_{p11} + \phi_{p21})}}$$

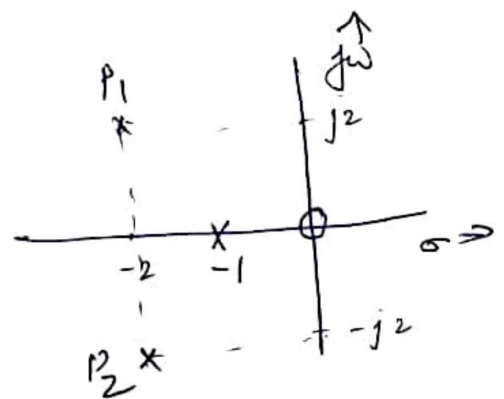
$$H = 5$$

$$M_{01} = 1 \quad \phi_{01} = 180^\circ$$

$$M_{p11} = \sqrt{5} \quad \phi_{p11} = \underline{\underline{-63.44^\circ}}$$

$$M_{p21} = \sqrt{5} \quad \phi_{p21} = \underline{\underline{63.44^\circ}}$$

$$K_1 = \frac{5 \times 1 \times e^{j180}}{\sqrt{5} \times \sqrt{5} e^{j0}} = \underline{\underline{-1}}$$



$$\tan^{-1} \frac{1}{2}$$

$$K_2 = H \frac{M_{op1}}{e^{j\phi_{op1}}}$$

Module - IIT

Fourier Representations of Continuous time signal

Fourier Series - Harmonic Analysis of Common signals

Fourier Transform - Existence Properties of FT

Energy Spectral density - Power Spectral density

Frequency Response of LTI systems.

Fourier Series

- Fourier series represents a periodic waveform in the form of sum of infinite number of Sine and Cosine terms.
- It is a tool used to analyze any periodic signal.
- Fourier series is applicable only for periodic signals. It cannot be applied to Non-periodic signals.

Existence of Fourier Series:

- The conditions under which a periodic signal can be represented by a Fourier series are known as Dirichlet conditions.

- (i) The function $x(t)$ must be a single valued function
- (ii) The function $x(t)$ has only a finite number of maxima and minima.
- (iii) The function $x(t)$ has a finite number of discontinuities
- (iv) The function $x(t)$ is absolutely integrable over one period
$$\int_0^{T_0} |x(t)| \cdot dt < \infty$$

Fourier Series Representation of Periodic Signal

- A periodic signal is one which repeats itself periodically over $-\infty < t < \infty$.

It satisfies $x(t+T) = x(t)$.

- Any periodic signal $x(t)$ can be written in terms of cosine & sine terms.

$$x(t) = a_0 + \sum_{n=1}^k [a_n \cos n\omega_0 t + b_n \sin n\omega_0 t] \quad \dots (1)$$

$$x(t+T) = a_0 + \sum_{n=1}^k [a_n \cos n\omega_0(t+T) + b_n \sin n\omega_0(t+T)]$$

$$T = \frac{2\pi}{\omega_0}$$

$$\begin{aligned} x(t+T) &= a_0 + \sum_{n=1}^k \left[a_n \cos n\omega_0 \left(t + \frac{2\pi}{\omega_0} \right) + b_n \sin n\omega_0 \left(t + \frac{2\pi}{\omega_0} \right) \right] \\ &= a_0 + \sum_{n=1}^k [a_n \cos [n\omega_0 t + 2\pi n] + b_n \sin (n\omega_0 t + 2\pi n)] \end{aligned}$$

$$\text{Let } \cos (n\omega_0 t + 2\pi n) = \cos n\omega_0 t$$

$$\sin (n\omega_0 t + 2\pi n) = \sin n\omega_0 t$$

$$x(t+T) = a_0 + \sum_{n=1}^k [a_n \cos n\omega_0 t + b_n \sin n\omega_0 t] \quad \dots (2)$$

$$\therefore \underline{x(t) = x(t+T)}$$

Fourier Series representation of periodic signal;

$$x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos n\omega_0 t + b_n \sin n\omega_0 t]$$

→ Trigonometric representation of Fourier series

a_0 = dc component

$\omega_0 = 2\pi/T$; Fundamental frequency.

$n\omega_0$ = harmonic frequencies.

$a_0, a_n, b_n \rightarrow$ Fourier coefficients.

$$a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) \cdot dt$$

$$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \cdot \cos n\omega_0 t \cdot dt$$

$$b_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \cdot \sin n\omega_0 t \cdot dt$$

3 types of Fourier series

1. Trigonometric
2. Polar
3. Exponential

Symmetry Conditions:

Even Symmetry:

$$a_0 = \frac{2}{T} \int_{t_0}^{t_0+T/2} x(t) \cdot dt$$

$$a_n = \frac{4}{T} \int_{t_0}^{t_0+T/2} x(t) \cdot \cos n\omega_0 t \cdot dt$$

$$b_n = 0$$

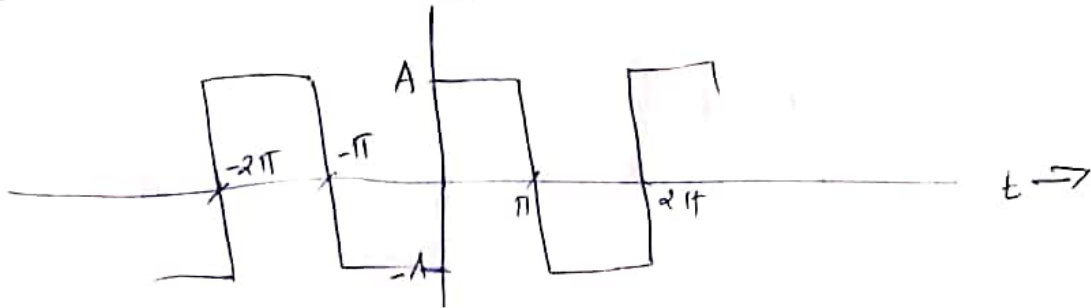
2. odd symmetry:

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{4}{T} \int_0^{T/2} x(t) \cdot \sin n\omega t \cdot dt$$

Qn. Obtain the trigonometric Fourier series for the given periodic signal $x(t)$



$$x(t) = A \quad ; \quad 0 < t < \pi$$

$$= -A \quad ; \quad \pi < t < 2\pi$$

$$T = 2\pi$$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

$$a_0 = \frac{1}{T} \left[\int_0^{\pi} A \cdot dt + \int_{\pi}^{2\pi} -A \cdot dt \right]$$

$$= \frac{1}{T} [A \cdot \pi - A \pi] = \underline{\underline{0}}$$

$$a_n = \frac{2}{T} \left[\int_0^{\pi} A \cdot \cos n\omega_0 t \cdot dt + \int_{\pi}^{2\pi} -A \cdot \cos n\omega_0 t \cdot dt \right]$$

$$= \frac{2}{2\pi} \left[A \left[\frac{\sin n\omega_0 t}{n\omega_0} \right]_0^{\pi} - A \left[\frac{\sin n\omega_0 t}{n\omega_0} \right]_{\pi}^{2\pi} \right]$$

$$= \frac{1}{\pi} A \left[-(\frac{\sin n\omega_0 \pi}{n\omega_0} - \frac{\sin 0}{n\omega_0}) + A \left[\frac{\sin n\omega_0 2\pi}{n\omega_0} - \frac{\sin n\omega_0 \pi}{n\omega_0} \right] \right]$$

$$= \underline{\underline{0}}$$

$$b_0 = 0$$

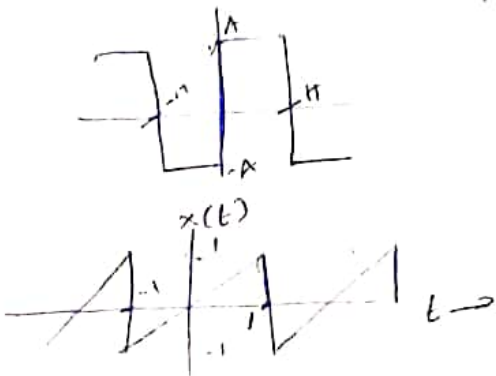
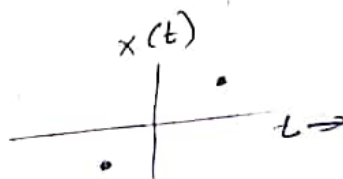
$$T/2 = \frac{2\pi}{2} = \pi$$

$$\omega_0 = \frac{2\pi}{T} = 1$$

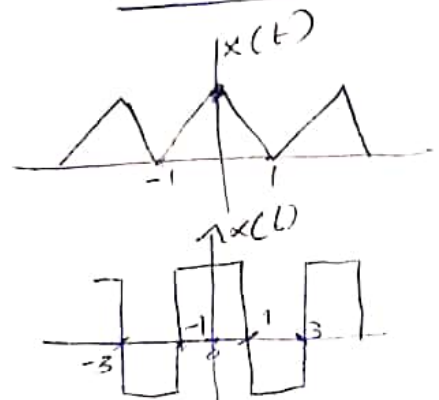
$$\begin{aligned} b_n &= \frac{4}{2\pi} \left[\int_0^{\pi} A \cdot \sin n\omega_0 t \cdot dt \right] \\ &= \frac{2A}{\pi} \left(-\frac{\cos n\omega_0 t}{n\omega_0} \right) \Big|_0^{\pi} \\ &= \frac{-2A}{\pi} (\cos n\pi - \cos n/0) \\ &= \underline{\underline{\frac{-2A}{\pi} [(-1)^n - 1]}} \end{aligned}$$

$$\begin{aligned} x(t) &= a_0 + \sum_{n=1}^{\infty} [a_n \cos n\omega_0 t + b_n \sin n\omega_0 t] \\ &= \sum_{n=1}^{\infty} \frac{-2A}{\pi} [(-1)^n - 1] \sin n\omega_0 t \end{aligned}$$

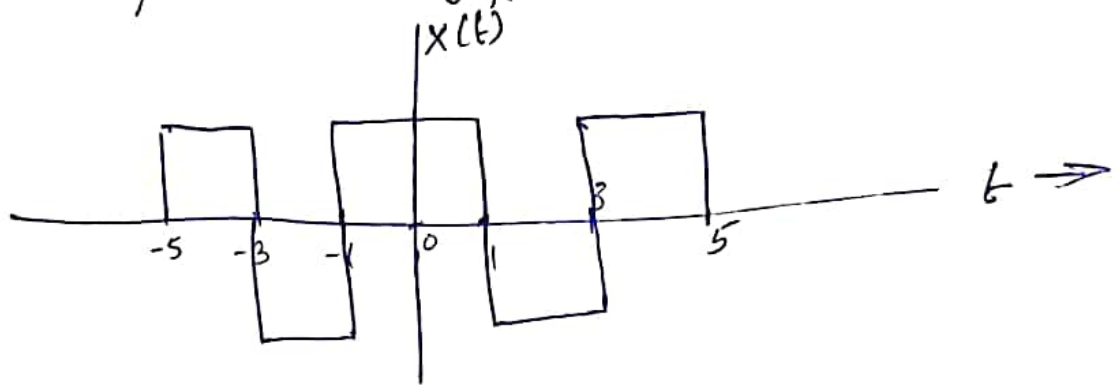
odd



Even



Qn. 2 Find the Trigonometric Fourier Series for periodic signal



$$T = (3 - (-1)) = \underline{\underline{4}}$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{4} = \underline{\underline{\pi/2}}$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos n\omega t + b_n \sin n\omega t]$$

$$a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) \cdot dt$$

$$= \frac{1}{4} \left[\int_{-1}^3 x(t) \cdot dt \right]$$

$$= \frac{1}{4} \left[\int_{-1}^1 1 \cdot dt + \int_1^3 (-1) \cdot dt \right]$$

$$= \frac{1}{4} \left[(t)_{-1}^1 - (t)_{1}^3 \right]$$

$$= \frac{1}{4} [2 - 2] = \underline{\underline{0}}$$

$$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \cdot \cos n\omega t \cdot dt$$

$$= \frac{2}{T} \int_{-1}^3 x(t) \cdot \cos(n \frac{\pi}{2} t) \cdot dt$$

$$= \frac{2}{4} \left[\int_{-1}^3 x(t) \cdot \cos\left(\frac{n\pi}{2}t\right) \cdot dt \right]$$

$$= \frac{1}{2} \left[\int_{-1}^1 \cos\left(\frac{n\pi}{2}t\right) \cdot dt + \int_1^3 (-1) \cdot \cos\left(\frac{n\pi}{2}t\right) \cdot dt \right]$$

$$= \frac{1}{2} \left[\left[\frac{\sin\left(\frac{n\pi}{2}t\right)}{\left(\frac{n\pi}{2}\right)} \right]_{-1}^1 - \left[\frac{\sin\left(\frac{n\pi}{2}t\right)}{\left(\frac{n\pi}{2}\right)} \right]_1^3 \right]$$

$$= \frac{1}{2} \left[\frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) - \frac{2}{n\pi} \left(-\sin\left(\frac{3n\pi}{2}\right) \right) \right]$$

$$= \frac{1}{2} \left[\frac{8}{n\pi} \sin\left(\frac{n\pi}{2}\right) \right]$$

$$= \frac{4}{n\pi} \sin\left(\frac{n\pi}{2}\right)$$

$$b_n = \frac{2}{T} \left[\int_{t_0}^{t_0+T} x(t) \cdot \sin n\omega t \cdot dt \right]$$

$$= \frac{2}{4} \left[\int_{-1}^1 \sin\left(\frac{n\pi}{2}t\right) \cdot dt + \int_1^3 -\sin\left(\frac{n\pi}{2}t\right) \cdot dt \right]$$

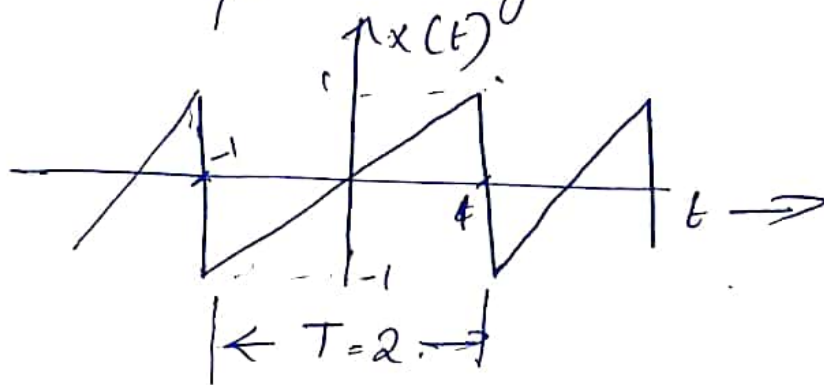
$$= \frac{1}{2} \left\{ -\left[\frac{\cos\left(\frac{n\pi}{2}t\right)}{\left(\frac{n\pi}{2}\right)} \right]_{-1}^1 + \left[\frac{\cos\left(\frac{n\pi}{2}t\right)}{\left(\frac{n\pi}{2}\right)} \right]_1^3 \right\}$$

$$= \underline{\underline{0}}$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi}{2}t + b_n \sin \frac{n\pi}{2}t \right]$$

$$= \sum_{n=1}^{\infty} \frac{4}{n\pi} \sin\left(\frac{n\pi}{2}t\right) \cdot \cos\left(\frac{n\pi}{2}t\right)$$

2. Find the Trigonometric Fourier series for the periodic signal $x(t)$



Odd signal.

$$a_0 = a_n = 0$$

$$\omega = \frac{2\pi}{T}$$

$$= \underline{\underline{\pi}}$$

$$b_n = \frac{4}{T} \int_0^{T/2} x(t) \cdot \sin n\omega t \cdot dt$$

$$= 2 \int_0^1 t \cdot \sin n\pi t \cdot dt$$

$$= 2 \left[t \cdot \int_0^1 \sin n\pi t \cdot dt - \int_0^1 1 \cdot \int \sin n\pi t \cdot dt \right]$$

$$= 2 \left[t \cdot \left[\frac{\cos n\pi t}{n\pi} \right]_0^1 - \int_0^1 -\frac{\cos n\pi t}{n\pi} \cdot dt \right]$$

$$= 2 \left[-t \cdot \left(\frac{\cos n\pi t}{n\pi} \right)_0^1 + \frac{1}{n\pi} \cdot \left[\frac{\sin n\pi t}{n\pi} \right]_0^1 \right]$$

$$= 2 \left[-\frac{1}{n\pi} [\cos n\pi - 0] + \frac{1}{(n\pi)^2} [\sin n\pi - \sin 0] \right]$$

$$= 2 \times \frac{-1}{n\pi} \cos n\pi = \underline{\underline{\frac{-2}{n\pi} \cos n\pi}}$$

$$x(t) = \sum_{n=1}^{\infty} \underline{\underline{\frac{-2}{n\pi} \cos n\pi \cdot \sin n\pi t}}$$

Cosine Representation (Polar Form or Wueman form)

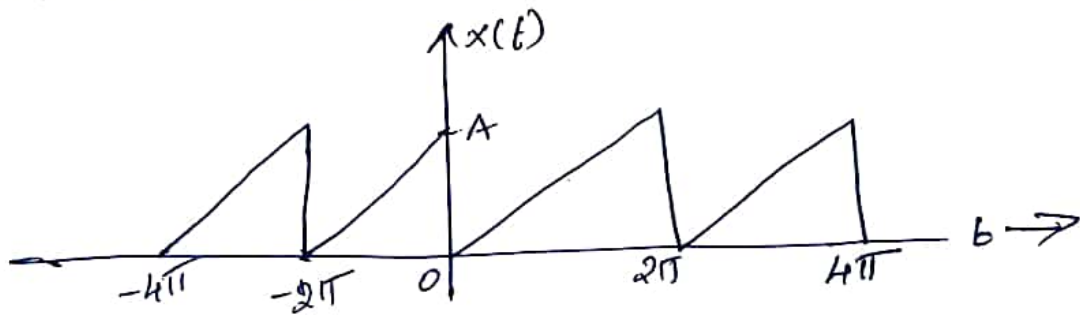
$$x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + Q_n)$$

where $A_0 = a_0$

$$A_n = \sqrt{a_n^2 + b_n^2}$$

$$Q_n = -\tan^{-1}\left(\frac{b_n}{a_n}\right)$$

Qw. Find the cosine Fourier series for the waveform



$$x(t) = \frac{A}{2\pi} \cdot t$$

$$T = 2\pi$$

$$\omega = \frac{2\pi}{T} = 1$$

$$a_0 = \frac{1}{T} \int_0^T x(t) \cdot dt = \frac{1}{2\pi} \int_0^{2\pi} \frac{A}{2\pi} \cdot t \cdot dt$$

$$= \frac{A}{2}$$

$$a_n = \frac{2}{T} \int_0^T x(t) \cdot \cos n\omega t \cdot dt = \frac{2}{2\pi} \int_0^{2\pi} \frac{A}{2\pi} \cdot t \cdot \cos nt \cdot dt$$

$$= 0$$

$$b_n = \frac{2}{T} \int_0^T x(t) \cdot \sin n\omega t \cdot dt = \frac{2}{2\pi} \int_0^{2\pi} \frac{A}{2\pi} \cdot t \cdot \sin nt \cdot dt$$

$$= \left(\frac{2A}{2\pi}\right) \left\{ \left[t \cdot \frac{-\cos nt}{n} \right]_0^{2\pi} - \int_0^{2\pi} \frac{-\cos nt}{n} \cdot dt \right\}$$

$$= \frac{2A}{(2\pi)^2} \left[\int_0^{2\pi} \left[t \cdot \left(-\frac{\cos nt}{n} \right) \right]_0^{2\pi} + \frac{1}{n} \left(\frac{\sin nt}{n} \right)_0^{2\pi} \right]$$

$$= \frac{\cancel{2A}}{\cancel{(2\pi)^2}} - \frac{A}{\pi n}$$

$$A_n = \sqrt{a_n^2 + b_n^2} = \sqrt{\left(\frac{-A}{\pi n} \right)^2} = \frac{A}{n\pi}$$

$$\begin{aligned} \phi_n &= -\tan^{-1} \left(\frac{b_n}{a_n} \right) = -\tan^{-1} \left(\frac{-A/n\pi}{0} \right) \\ &= \underline{\underline{-\pi/2}} \end{aligned}$$

Cosine form:

$$x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \phi_n)$$

$$= \frac{A}{2} + \sum_{n=1}^{\infty} \frac{A}{n\pi} \cos\left(nt - \pi/2\right)$$

Exponential Fourier Series:

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$\omega_0 = \frac{2\pi}{T}$$

$$C_n = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) \cdot e^{-jn\omega_0 t} \cdot dt$$

$C_n \rightarrow$ Fourier series coefficients of Exponential Fourier series.

Proof:

Let Cosine Fourier series:

$$A_n \cos(n\omega_0 t + \phi_n) = A_n \left[\frac{e^{j(n\omega_0 t + \phi_n)} + e^{-j(n\omega_0 t + \phi_n)}}{2} \right]$$

$$x(t) = A_0 + \sum_{n=1}^{\infty} \frac{A_n}{2} \left[e^{j(n\omega_0 t + \phi_n)} + e^{-j(n\omega_0 t + \phi_n)} \right]$$

$$= A_0 + \sum_{n=1}^{\infty} \frac{A_n}{2} \left[e^{jn\omega_0 t} \cdot e^{j\phi_n} \right] + \sum_{n=1}^{\infty} \frac{A_n}{2} e^{j(-\phi_n)} \cdot e^{-jn\omega_0 t}$$

$$= A_0 + \sum_{n=1}^{\infty} \left(\frac{A_n}{2} \cdot e^{j\phi_n} \right) e^{jn\omega_0 t} + \sum_{n=1}^{\infty} \left(\frac{A_n}{2} \cdot e^{j\phi_n} \right) e^{-jn\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} = C_0 + \sum_{n=-\infty}^{-1} C_n e^{jn\omega_0 t} + \sum_{n=1}^{\infty} C_n e^{jn\omega_0 t}$$

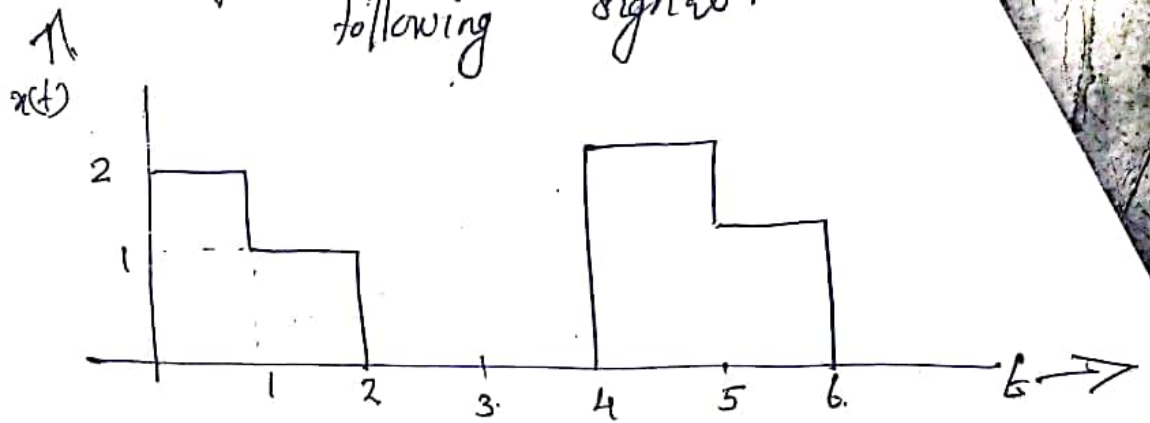
$$C_n = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) \cdot e^{-jn\omega_0 t} \cdot dt$$

$$C_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) \cdot dt$$

if C_n is complex number;

$$|C_n| = |C_{-n}|$$

Compute the Exponential Fourier series of following signal.



$$T = \underline{\underline{4}}$$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{4} = \pi/2$$

$$C_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) \cdot dt = \frac{1}{4} \int_0^4 x(t) \cdot dt$$

$$= \frac{1}{4} \left[\int_0^1 2 \cdot dt + \int_1^2 1 \cdot dt \right]$$

$$= \frac{1}{4} \left[2(t)_0^1 + (t)_1^2 \right] = \frac{1}{4} [2+1] = \underline{\underline{\frac{3}{4}}}$$

$$C_n = \frac{1}{T} \left[\int_0^T x(t) \cdot e^{-jn\omega_0 t} \cdot dt \right]$$

$$= \frac{1}{4} \left[\int_0^1 2 \cdot e^{-jn\pi/2 t} \cdot dt + \int_1^2 1 \cdot e^{-jn\pi/2 t} \cdot dt \right]$$

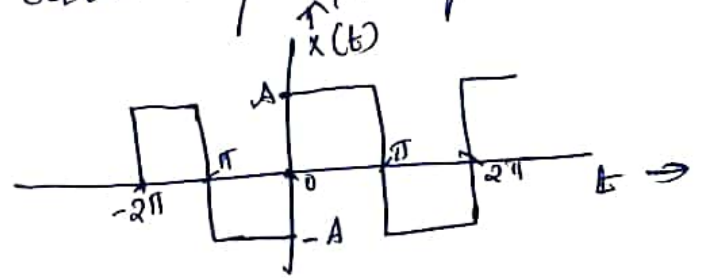
$$= \frac{1}{4} \left[2 \cdot \left(\frac{e^{-jn\pi/2 t}}{-jn\pi/2} \right)_0^1 + \left(\frac{e^{-jn\pi/2 t}}{-jn\pi/2} \right)_1^2 \right]$$

$$= \frac{1}{2(-jn\pi/2)} \left[e^{-jn\pi/2} - 1 \right] + \frac{1}{2(-jn\pi/2)} \left[e^{-jn\pi} - e^{-jn\pi/2} \right]$$

$$= -\frac{1}{jn\pi} \left[e^{-jn\pi/2} - 1 \right] - \frac{1}{2jn\pi} \left[e^{-jn\pi} - e^{-jn\pi/2} \right]$$

$$\begin{aligned}
&= \frac{-1}{jn\pi} e^{-jn\pi/2} + \frac{1}{jn\pi} - \frac{1}{2jn\pi} e^{-jn\pi} + \frac{1}{2jn\pi} e^{-jn\pi/2} \\
&= \frac{1}{jn\pi} \left[1 - e^{-jn\pi/2} - \frac{1}{2} e^{-jn\pi} + \frac{1}{2} e^{-jn\pi/2} \right] \\
&= \frac{1}{jn\pi} \left[1 - \frac{1}{2} e^{-jn\pi/2} - \frac{1}{2} e^{jn\pi} \right] \\
&= \frac{1}{jn\pi} \left[1 - \frac{1}{2} e^{-jn\pi/2} - \frac{1}{2} [\cos n\pi - j \sin n\pi] \right] \\
&= \frac{1}{jn\pi} \left[1 - \frac{1}{2} e^{-jn\pi/2} - \frac{1}{2} (-1)^n \right]
\end{aligned}$$

Qn. Obtain the Exponential Fourier Series for the waveform. Also draw Frequency Spectrum.



$$\begin{aligned}
x(t) &= A \quad 0 < t < \pi \\
&= -A \quad \pi < t < 2\pi
\end{aligned}$$

$$t_0 + T = 2\pi$$

$$\phi = 0$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

$$x(t) = C_0 + \sum_{n=-\infty}^{\infty} C_n e^{jn\omega t}$$

$$C_0 = \frac{1}{T} \int_0^T x(t) \cdot dt$$

$$= \frac{1}{2\pi} \left[\int_0^{\pi} A \cdot dt + \int_{\pi}^{2\pi} -A \cdot dt \right]$$

$$= \frac{1}{2\pi} \left[A(t)_{\pi} - A(t)_{\pi} \right]$$

$$= \frac{1}{2\pi} [A\pi - A(\pi - \pi)] = \underline{\underline{0}}$$

$$C_n = \frac{1}{T} \int_0^T x(t) \cdot e^{-jn\omega t} \cdot dt$$

$$= \frac{1}{2\pi} \left[\int_0^{\pi} A \cdot e^{-jn\omega t} \cdot dt + \int_{\pi}^{2\pi} -A \cdot e^{-jn\omega t} \cdot dt \right]$$

$$= \frac{A}{2\pi} \left[\left(\frac{e^{-jn\omega t}}{-jn\omega} \right)_{\pi} - \left(\frac{e^{-jn\omega t}}{-jn\omega} \right)_{\pi} \right]$$

$$= \frac{A}{-2\pi j n \omega} \left[(e^{-jn\omega \pi})_{\pi} - (e^{-jn\omega \pi})_{\pi} \right]$$

$$= \frac{-A}{j 2\pi n \omega} \left[(e^{-jn\omega \pi} - e^0) - (e^{-jn\omega \pi} - e^{-jn\omega \pi}) \right]$$

$$= \frac{-A}{j 2\pi n \omega} \left[(\cos n\pi - \sin n\pi) - 1 \right]$$

$$= \frac{-A}{j 2\pi n \omega} \left[[(-1)^n - 1] - [1 - (-1)^n] \right]$$

$$= \frac{-A}{j 2\pi n \omega} [2(-1)^n - 2]$$

$$= \frac{-2A}{j2n\pi} [(-1)^n - 1]$$

$$= j \frac{A}{n\pi} [(-1)^n - 1]$$

$$C_n = -j \frac{2A}{n\pi} \quad ; \text{ for odd } n$$

$$= 0 \quad ; \text{ for even } n$$

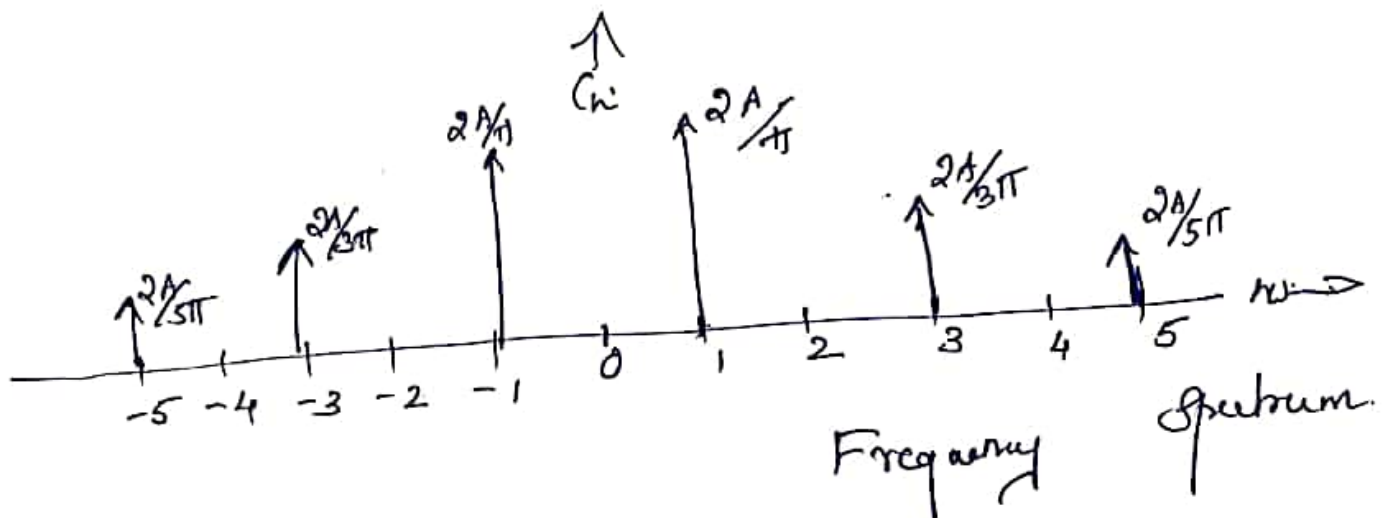
$$x(t) = C_0 + \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{-j2A}{n\pi} \right) e^{jn\omega_0 t} \quad ; \text{ for odd } n$$

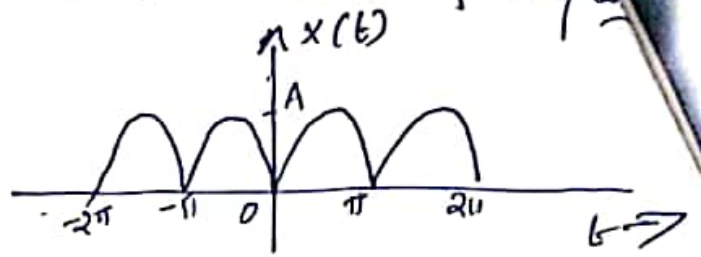
$$C_0 = 0 \quad |C_1| = |C_{-1}| = \frac{2A}{\pi}$$

$$C_3 = C_{-3} = \frac{2A}{3\pi}$$

$$C_5 = C_{-5} = \frac{2A}{5\pi}$$



Q.2. Find the Exponential Fourier Series & Spectrum.



$$x(t) = A \sin t$$

$$\omega = 2\pi/T = \frac{2\pi}{2\pi} = 2$$

Exponential Fourier Series;

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega t}$$

$$C_n = \frac{1}{T} \int_0^T A \sin t \cdot e^{-jn\omega t} \cdot dt$$

$$= \frac{A}{T} \int_0^T \left[\frac{e^{jt} - e^{-jt}}{2j} \right] \cdot e^{-jn\omega t} \cdot dt$$

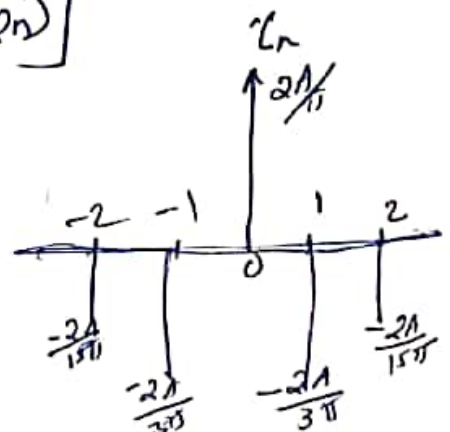
$$= \frac{A}{2jT} \left[\int_0^T \left[e^{j(1-2n)t} - e^{-j(1+2n)t} \right] \cdot dt \right]$$

$$= \frac{A}{2jT} \left[\frac{e^{j(1-2n)t}}{j(1-2n)} - \frac{e^{-j(1+2n)t}}{-j(1+2n)} \right]_0^T$$

$$= \frac{A}{2jT} \left[\frac{(-1-1)}{j(1-2n)} + \frac{(-1-1)}{j(1+2n)} \right]$$

$$= \frac{2A}{T(1-4n^2)}$$

$$C_0 = \frac{2A}{T}$$



Properties of Continuous time Fourier Series

1. Linearity property:

$$x_1(t) \xleftrightarrow{Fs} C_n$$

$$x_2(t) \xleftrightarrow{Fs} D_n$$

$$\text{Then } Ax_1(t) + Bx_2(t) \xleftrightarrow{Fs} AC_n + BD_n$$

Proof:

$$\begin{aligned} Fs [Ax_1(t) + Bx_2(t)] &= \frac{1}{T} \int_{t_0}^{t_0+T} [Ax_1(t) + Bx_2(t)] \cdot e^{-jn\omega t} dt \\ &= \frac{A}{T} \int_{t_0}^{t_0+T} x_1(t) \cdot e^{-jn\omega t} dt + \frac{B}{T} \int_{t_0}^{t_0+T} x_2(t) \cdot e^{-jn\omega t} dt \\ &= AC_n + BD_n \end{aligned}$$

2. Time Shifting property:

$$\text{if } x(t) \xleftrightarrow{Fs} C_n$$

$$x(t-t_0) \xleftrightarrow{Fs} e^{-jn\omega t_0} C_n$$

Proof:

$$x(t) = \sum_{n=-\infty}^{\infty} C_n \cdot e^{jn\omega t}$$

$$\begin{aligned} x(t-t_0) &= \sum_{n=-\infty}^{\infty} C_n e^{jn\omega(t-t_0)} \\ &= \sum_{n=-\infty}^{\infty} C_n \cdot e^{jn\omega t} \cdot e^{-jn\omega t_0} \\ &= e^{-jn\omega t_0} C_n \end{aligned}$$

3. Time Reversal Property:

$$x(t) \xleftrightarrow{FS} C_n$$

$$x(-t) \xleftrightarrow{FS} C_{-n}$$

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty} C_{-p} e^{j\omega_0 p t}$$

$$= \sum_{p=-\infty}^{\infty} C_{-p} e^{j\omega_0 p t}$$

$$-n = p$$

$$\underline{\underline{x(t) = C_{-n}}}$$

4. Time Scaling Property:

$$\text{if } x(t) \xleftrightarrow{FS} C_n$$

$$x(\alpha t) \xleftrightarrow{FS} C_n$$

Proof:

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$x(\alpha t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 \alpha t}$$

$$= \underline{\underline{C_n}}$$

5. Time Convolution Theorem

$$\text{if } x_1(t) \xleftrightarrow{FS} C_n$$

$$x_2(t) \xleftrightarrow{FS} D_n$$

Then $x_1(t) * x_2(t) \xleftrightarrow{FS} T C_n D_n$.

Proof:

$$FS[x_1(t) * x_2(t)] = \frac{1}{T} \int_0^{t+T} [x_1(t) * x_2(t)] e^{-jn\omega_0 t} dt$$

$$x_1(t) * x_2(t) = \int_0^T x_1(\tau) \cdot x_2(t-\tau) \cdot d\tau$$

$$= \frac{1}{T} \int_0^T \int_0^T x_1(\tau) \cdot x_2(t-\tau) \cdot e^{-jn\omega_0 t} dt \cdot d\tau$$

$$= \frac{1}{T} \int_0^T x_1(\tau) \cdot d\tau \int_0^T x_2(t-\tau) \cdot e^{-jn\omega_0 t} dt$$

put $t - \tau = m$
 $t = m + \tau$
 $dt = dm$

$$= \frac{1}{T} \int_0^T x_1(\tau) \cdot d\tau \int_0^T x_2(m) \cdot e^{-jn\omega_0(m+\tau)} \cdot dm$$

$$= F \cdot \left[\frac{1}{T} \int_0^T x_1(\tau) \cdot e^{-jn\omega_0 \tau} \cdot d\tau \int_0^T x_2(m) \cdot e^{-jn\omega_0 m} \cdot dm \right]$$

$$= T \cdot C_n \cdot D_n$$

6. Time Differentiation Property:

$$\text{if } x(t) \xleftrightarrow{FS} C_n \\ \frac{d}{dt} x(t) \xleftrightarrow{FS} j n \omega_0 \cdot C_n$$

Proof:

$$x(t) = \sum_{n=-\infty}^{\infty} C_n \cdot e^{j n \omega_0 t}$$

$$\frac{d}{dt} x(t) = \sum_{n=-\infty}^{\infty} C_n \frac{d}{dt} (e^{j n \omega_0 t})$$

$$= j n \omega_0 \sum_{n=-\infty}^{\infty} C_n \cdot e^{j n \omega_0 t}$$

$$= j n \omega_0 \sum_{n=-\infty}^{\infty} C_n \cdot e^{j n \omega_0 t} = \underline{\underline{j n \omega_0 \cdot C_n}}$$

7. Time Integration property:

$$\text{if } x(t) \xleftrightarrow{FS} C_n$$

$$\int_{-\infty}^{\infty} x(t) \cdot dt \longleftrightarrow \frac{C_n}{j n \omega_0}$$

8. Parseval's Theorem or Rayleigh's Theorem

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \cdot dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \cdot d\omega$$

$$\text{if } x(t) \xleftrightarrow{FT} X(\omega)$$

6. Time Differentiation Property:

$$\text{if } x(t) \xleftrightarrow{FS} C_n$$

$$\frac{d}{dt} x(t) \xleftrightarrow{FS} j n \omega_0 \cdot C_n$$

Proof:

$$x(t) = \sum_{n=-\infty}^{\infty} C_n \cdot e^{j n \omega_0 t}$$

$$\frac{dx(t)}{dt} = \sum_{n=-\infty}^{\infty} C_n \frac{d}{dt} (e^{j n \omega_0 t})$$

$$= j n \omega_0 \sum_{n=-\infty}^{\infty} C_n \cdot e^{j n \omega_0 t}$$

$$= j n \omega_0 \sum_{n=-\infty}^{\infty} C_n \cdot e^{j n \omega_0 t} = \underline{\underline{j n \omega_0 \cdot C_n}}$$

7. Time Integration property:

$$\text{if } x(t) \xleftrightarrow{FS} C_n$$

$$\int_{-\infty}^{\infty} x(t) \cdot dt \longleftrightarrow \frac{C_n}{j n \omega_0}$$

8. Parseval's Theorem or Rayleigh's Theorem

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \cdot dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |x(\omega)|^2 \cdot d\omega$$

$$\text{if } x(t) \xleftrightarrow{FT} X(\omega)$$

Fourier Transform.

- It is a transformation technique which transforms signal from continuous time domain to the corresponding frequency domain
- It can be applied to periodic & non-periodic signals

$$x(t) = \frac{1}{2\pi} \int$$

FT of $x(t)$ is $X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot dt$

$X(\omega)$ = Frequency spectrum of $x(t)$
(Spectral density Function)

$$x(t) \xleftrightarrow{FT} X(\omega) \quad F[x(t)] = X(\omega)$$

Inverse Fourier Transform, $F^{-1}[X(\omega)] = x(t)$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t} \cdot d\omega$$

Magnitude ; $|x(\omega)| = \sqrt{X_R(\omega)^2 + X_I(\omega)^2}$

Phase ; $\angle X(\omega) = \tan^{-1} \frac{X_I(\omega)}{X_R(\omega)}$

Existence of FT. (Dirichlet condition)

1. $x(t)$ is absolutely Integrable over the interval $-\infty$ to ∞ ; $\int_{-\infty}^{\infty} |x(t)| \cdot dt < \infty$

2. $x(t)$ has a finite number of maxima and minima in every finite time interval.

2. $x(t)$ has a finite no. of discontinuities in every finite time interval. Each of discontinuities must be finite.

Qn. Find FT of $x(t) = \delta(t)$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot dt = \int_{-\infty}^{\infty} \delta(t) \cdot e^{-j\omega t} \cdot dt \Big|_{t=0} \\ = \underline{\underline{1}}$$

Qn. $x(t) = e^{-at} u(t)$.

$$X(\omega) = \int_{-\infty}^{\infty} e^{-at} u(t) \cdot e^{-j\omega t} \cdot dt$$

$$= \int_0^{\infty} e^{-at} \cdot e^{-j\omega t} \cdot dt$$

$$= \int_0^{\infty} e^{-(a+j\omega)t} \cdot dt$$

$$= \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right]_0^{\infty}$$

$$= 0 - \left[-\frac{1}{a+j\omega} \right] = \underline{\underline{\frac{1}{a+j\omega}}}$$

Q.3) $x(t) = e^{-a|t|}$

Ans:

Also plot Magnitude response & phase response.

$$x(t) = e^{at} ; t \leq 0$$

$$= e^{-at} ; t \geq 0$$

for $t \leq 0, x(t) = e^{-a \cdot t}$
 $= e^{at}$

for $t > 0, x(t) = e^{-at}$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot dt$$

$$= \int_{-\infty}^0 e^{at} \cdot e^{-j\omega t} \cdot dt + \int_0^{\infty} e^{-at} \cdot e^{-j\omega t} \cdot dt$$

$$= \int_{-\infty}^0 e^{(a-j\omega)t} \cdot dt + \int_0^{\infty} e^{-(a+j\omega)t} \cdot dt$$

$$= \int_0^{\infty} e^{-(a-j\omega)t} \cdot dt + \int_0^{\infty} e^{-(a+j\omega)t} \cdot dt$$

$$= \left[\frac{e^{-(a-j\omega)t}}{-(a-j\omega)} \right]_0^{\infty} + \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right]_0^{\infty}$$

$$= \frac{(e^{-\infty} - e^0)}{-(a-j\omega)} + \frac{(e^{-\infty} - e^0)}{-(a+j\omega)}$$

$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{2a}{a^2 + \omega^2}$$



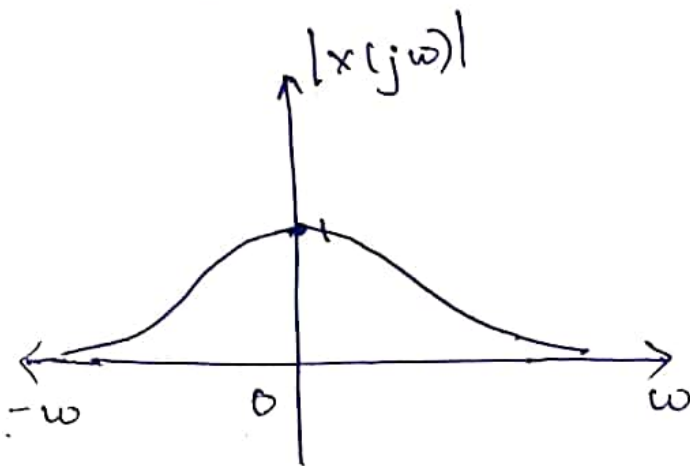
$$x(j\omega) = \frac{2a}{a^2 + \omega^2}$$

Magnitude $|x(j\omega)| = \frac{2a}{a^2 + \omega^2}$

$$\angle x(j\omega) = -\tan^{-1} \frac{0}{\left(\frac{2a}{a^2 + \omega^2}\right)} = -\tan^{-1} 0 = \underline{\underline{0}}$$

$$\begin{aligned} x(j\omega) &= a + jb \\ |x(j\omega)| &= \sqrt{a^2 + b^2} \\ \angle x(j\omega) &= \tan^{-1} \frac{b}{a} \end{aligned}$$

Mag. spectrum



Phase spectrum

$\angle x(j\omega) = 0$ for all values of ω .

At $\omega = 0$

$$|x(j\omega)| = \frac{2a}{a^2} = \frac{2}{a}$$

Let $a = 2$

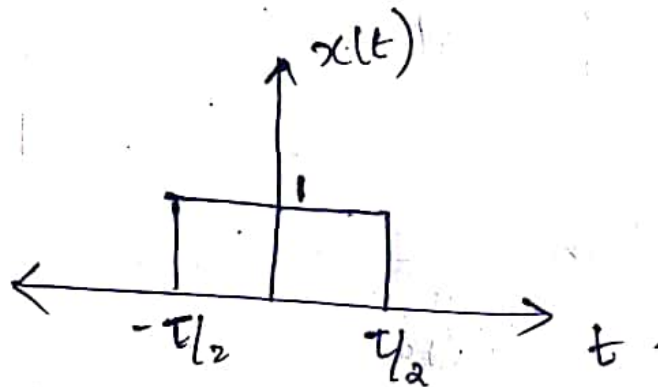
$$\frac{2}{a} = 1$$

$$\omega = \infty, |x(j\omega)| = \frac{2a}{a^2 + \infty^2} = 0$$

$$\omega = -\infty, |x(j\omega)| = \frac{1}{\infty} = 0$$

Q.4. $x(t) = \text{rect} \cdot (t/\tau) = 1 \quad -\tau/2 < t < \tau/2$
 $= 0$ otherwise.

Ans:



$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$\Rightarrow \int_{-\tau/2}^{\tau/2} 1 \cdot e^{-j\omega t} dt = \left[\frac{e^{-j\omega t}}{-j\omega} \right]_{-\tau/2}^{\tau/2}$$

$$= \frac{-1}{j\omega} \left[e^{-j\omega \tau/2} - e^{j\omega \tau/2} \right]$$

$$= \frac{e^{+j\omega \tau/2} - e^{-j\omega \tau/2}}{j\omega}$$

$$= \frac{2}{\omega} \left[\frac{e^{j\omega \tau/2} - e^{-j\omega \tau/2}}{2j} \right]$$

$$= \frac{2}{\omega} \sin \omega \tau/2$$

$$\left[\sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \right]$$

$$= \frac{2}{\omega} \sin \omega T/2$$

$$\left[\frac{\sin \theta}{\theta} = \text{sinc } \theta \right]$$

$$= \frac{\sin \omega T/2}{\frac{\omega}{2}} \times \frac{T}{T}$$

$$= T \cdot \frac{\sin \omega T/2}{\frac{\omega T}{2}}$$

$$= T \cdot \underline{\underline{\text{sinc } \omega T/2}}$$

Note

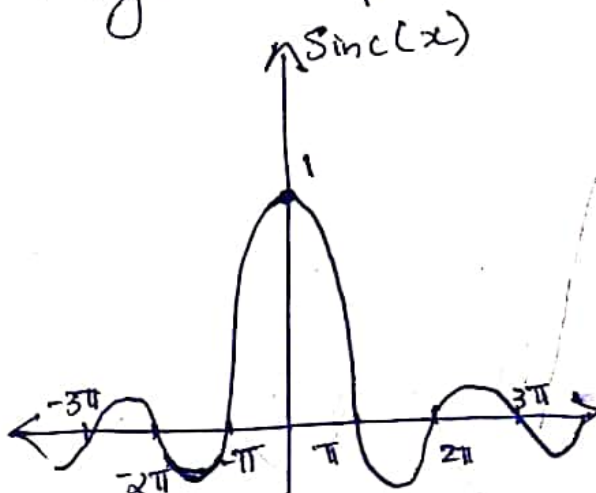
Sampling function or interpolating function or sinc function

$$\text{sinc}(x) = \frac{\sin x}{x} = \text{Sa}(x)$$

$$\text{sinc}(0) = 1$$

$$\text{sinc}(x) = 0 \text{ for } x = \pm\pi, \pm 2\pi, \pm 3\pi, \dots$$

Magnitude spectrum

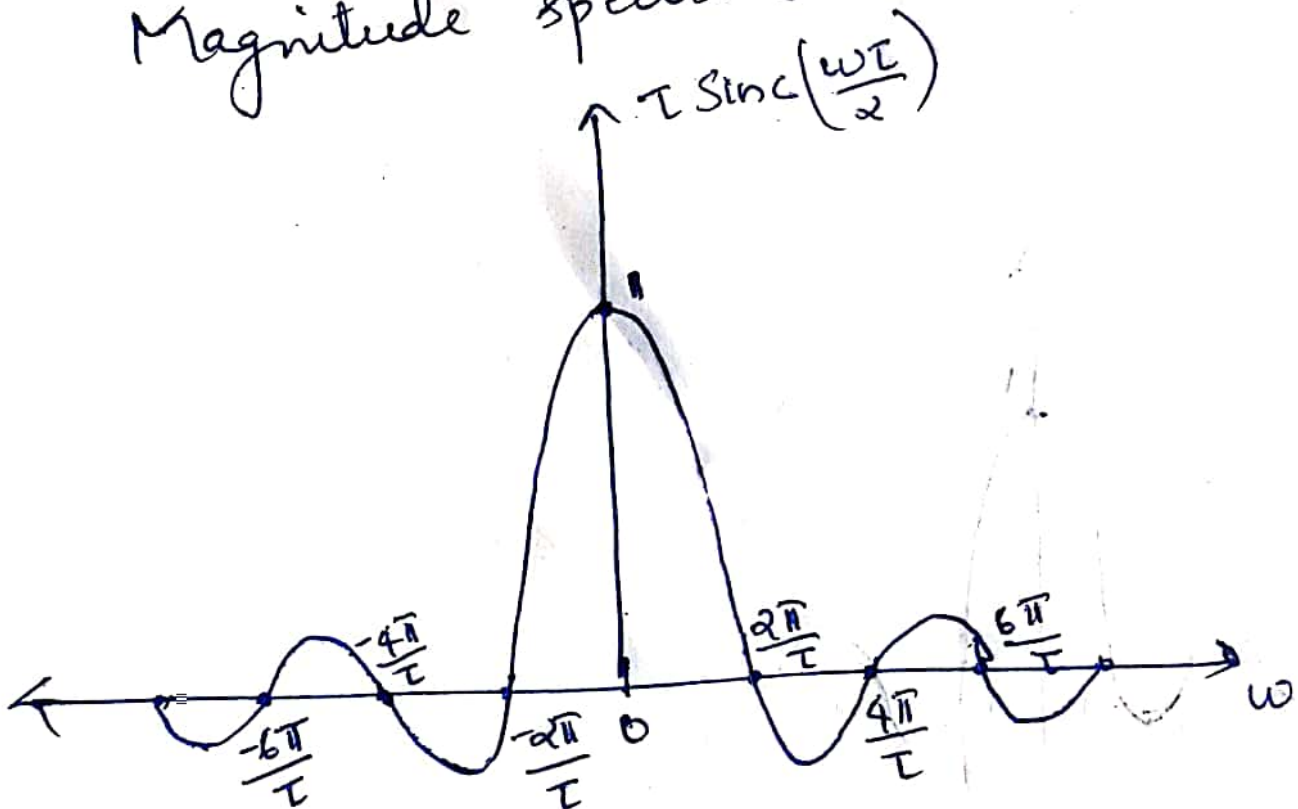


$$X(j\omega) = T \cdot \text{Sinc}\left(\frac{\omega T}{2}\right)$$

$$\text{Sinc}\left(\frac{\omega T}{2}\right) = 0 \quad \text{when} \quad \frac{\omega T}{2} = \pm n\pi$$

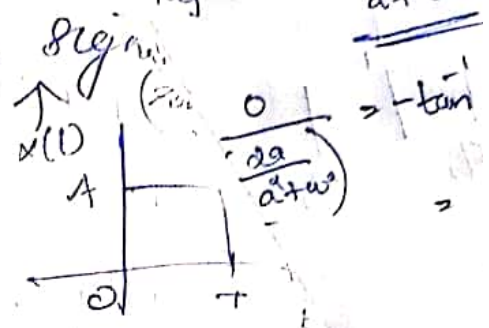
$$\omega = \pm \frac{2n\pi}{T} \quad \text{for } n = 1, 2, 3, \dots$$

Magnitude spectrum



Qn5) Find the FT of the given signal

$x(t) = \begin{cases} A & 0 < t < T \\ 0 & \text{other wise} \end{cases}$



$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot dt$$

$$= \int_0^T A \cdot e^{-j\omega t} \cdot dt = A \left[\frac{e^{-j\omega t}}{-j\omega} \right]_0^T$$

$$= A \left[\frac{e^{-j\omega T} - 1}{-j\omega} \right] = A \left[\frac{1 - e^{-j\omega T}}{j\omega} \right]$$

$$= \frac{2A}{\omega} \left[\frac{e^{j\omega T/2} \cdot e^{-j\omega T/2} - e^{-j\omega T/2} \cdot e^{j\omega T/2}}{2j} \right]$$

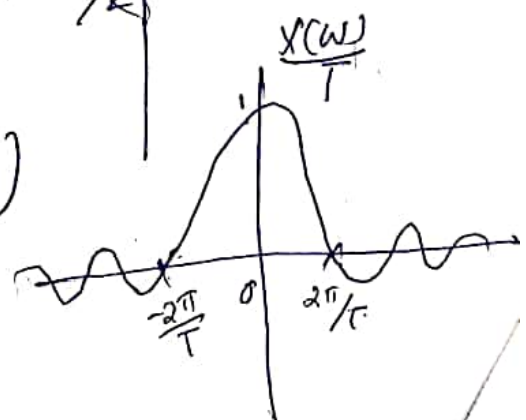
$$= \frac{2A}{\omega} e^{-j\omega T/2} \left[\frac{e^{j\omega T/2} - e^{-j\omega T/2}}{2j} \right]$$

$$= \frac{2A}{\omega} \cdot e^{-j\omega T/2} \cdot \sin(\omega T/2)$$

$$= \frac{2A}{\omega} \cdot e^{-j\omega T/2} \cdot \text{sinc}(\omega T/2) \cdot \omega T/2$$

$$= A T e^{-j\omega T/2} \cdot \text{sinc}(\omega T/2)$$

Filtering or
Interpolating fn.
or sampling or sinc =

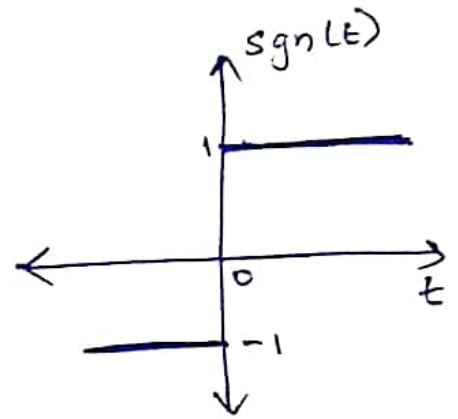
$$\text{sinc}(\omega T/2) = \frac{\sin(\omega T/2)}{(\omega T/2)}$$


note: ... = 1

Note:

Signum function:

$$\begin{aligned} \text{sgn}(t) &= -1 \text{ for } t < 0 \\ &= 0 \text{ for } t = 0 \\ &= 1 \text{ for } t > 0 \end{aligned}$$



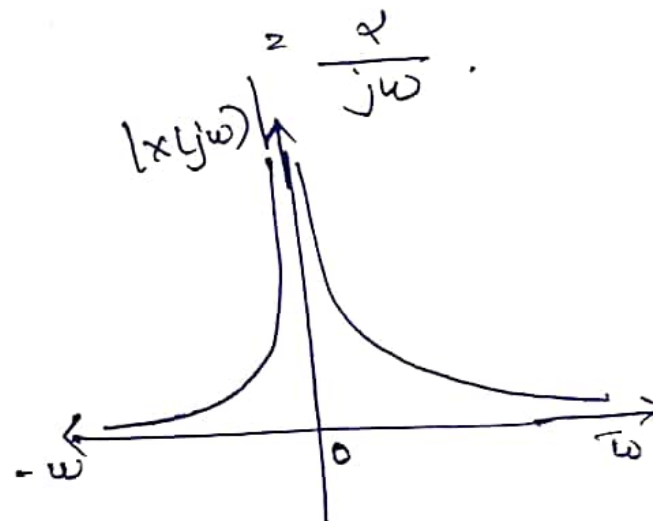
Q.6. $x(t) = \text{sgn}(t)$.

Ans:
$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \\ &= \int_{-\infty}^0 -1 e^{-j\omega t} dt + \int_0^{\infty} 1 \cdot e^{-j\omega t} dt \\ &= - \left[\frac{e^{-j\omega t}}{-j\omega} \right]_{-\infty}^0 + \left[\frac{e^{-j\omega t}}{-j\omega} \right]_0^{\infty} \\ &= \frac{1}{j\omega} + \frac{-1}{j\omega} (0 - 1) = \frac{1}{j\omega} + \frac{1}{j\omega} \end{aligned}$$

Magnitude spectrum.

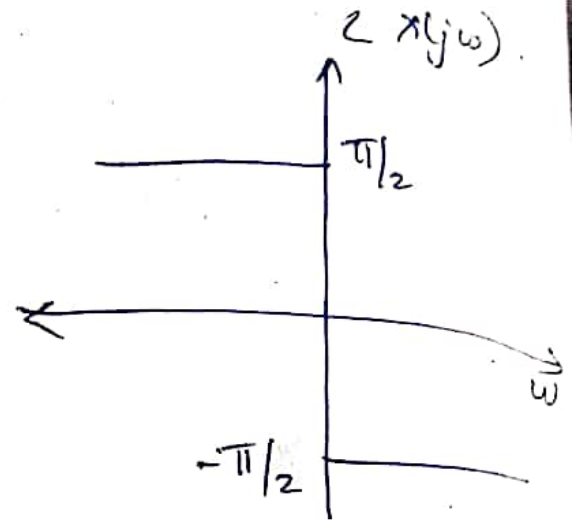
$$|X(j\omega)| = \frac{2}{j\omega}$$

At $\omega = 0$ $|X(j\omega)| = \infty$
 $\omega = \infty$ $|X(j\omega)| = 0$
 $\omega = -\infty$ $|X(j\omega)| = 0$



Phase spectrum -

$$\angle x(j\omega) = -\tan^{-1} \left(\frac{\text{imag.}}{\text{real}} \right)$$



Note:

$$F'(\delta\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi}$$

$$\delta(\omega) = 1$$

at $\omega = 0$.

ie $F'(\delta\omega) = \frac{1}{2\pi}$

$$\Rightarrow F\left(\frac{1}{2\pi}\right) = \delta\omega$$

$$\frac{1}{2\pi} \xrightarrow{\text{FT}} \delta(\omega)$$

$$1 \rightarrow 2\pi \delta(\omega)$$

ie $f(t) \rightarrow 2\pi \delta(\omega)$

Q.7) Find FT of $x(t) = e^{-A|t|} \text{sgn}(t)$

$$x(t) = -e^{At} \quad t < 0$$

$$= 0 \quad t = 0$$

$$= e^{-At} \quad t > 0$$

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot dt \\ &= \int_{-\infty}^0 -e^{At} \cdot e^{-j\omega t} \cdot dt + \int_0^{\infty} e^{-At} \cdot e^{-j\omega t} \cdot dt \\ &= \int_{-\infty}^0 -e^{(A-j\omega)t} \cdot dt + \int_0^{\infty} e^{-(A+j\omega)t} \cdot dt \\ &= - \left[\frac{e^{(A-j\omega)t}}{(A-j\omega)} \right]_{-\infty}^0 + \left[\frac{e^{-(A+j\omega)t}}{-(A+j\omega)} \right]_0^{\infty} \\ &= -\frac{1}{(A-j\omega)} + \frac{1}{A+j\omega} = \underline{\underline{\frac{-2j\omega}{A^2 + \omega^2}}} \end{aligned}$$

Properties of Fourier Transform:

1. Linearity Property:

$$\text{If } F[x_1(t)] = X_1(\omega)$$

$$F[x_2(t)] = X_2(\omega)$$

$$\text{Then } F[a x_1(t) + b x_2(t)] = a X_1(\omega) + b X_2(\omega).$$

Proof:

$$\begin{aligned} F[a x_1(t) + b x_2(t)] &= \int_{-\infty}^{\infty} [a x_1(t) + b x_2(t)] \cdot e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} a x_1(t) \cdot e^{-j\omega t} dt + \int_{-\infty}^{\infty} b x_2(t) \cdot e^{-j\omega t} dt \\ &= \underline{\underline{a X_1(\omega) + b X_2(\omega)}} \end{aligned}$$

2. Time Shifting Property:

$$\text{If } F[x(t)] = X(\omega)$$

$$\text{Then } F[x(t-t_0)] = e^{-j\omega t_0} X(\omega)$$

3. Frequency Shifting Property:

$$\text{If } F[x(t)] = X(\omega)$$

$$\text{Then } F[e^{j\omega_0 t} x(t)] = X(\omega - \omega_0)$$

4. Time Reversal Property:

$$\text{If } F[x(t)] = X(\omega)$$

$$\text{Then } F[x(-t)] = X(-\omega)$$

5. Convolution Theorem:

$$\text{If } F[x_1(t)] = X_1(\omega)$$

$$F[x_2(t)] = X_2(\omega)$$

$$F[x_1(t) * x_2(t)] = X_1(\omega) \cdot X_2(\omega)$$

6. Time Scaling Property:

$$\text{If } F[x(t)] = X(\omega)$$

$$\text{Then } F[x(at)] = \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

7. Differentiation in time domain:

$$\text{If } F[x(t)] = X(\omega)$$

$$\text{Then } F\left[\frac{d}{dt} x(t)\right] = j\omega X(\omega)$$

8. Differentiation in frequency domain:

$$\text{If } F[x(t)] = X(\omega)$$

$$\text{Then } F[t x(t)] = j \frac{d}{d\omega} X(\omega)$$

Modulation property.

$$\underline{\underline{F[x(t) \cdot \cos \omega_0 t] = \frac{1}{2} [X(\omega - \omega_0) + X(\omega + \omega_0)]}}$$

Fourier Transform of Standard Signals.

1. $F[\delta(t)] = 1$

2. $F[\delta(t - t_0)] = e^{-j\omega t_0}$

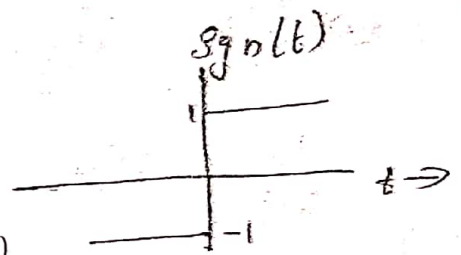
3. $F[e^{j\omega_0 t}] = 2\pi \delta(\omega - \omega_0)$

4. ~~$F[u(t)] = \dots$~~ since $F[1] = 2\pi \delta(\omega)$

4. $F[u(t)] = \pi \delta(\omega) + \frac{1}{j\omega}$

5. $F[\text{sgn}(t)] = \frac{2}{j\omega}$

$$\text{sgn}(t) = \begin{cases} -1 & t < 0 \\ 0 & t = 0 \\ 1 & t > 0 \end{cases} \quad \left. \begin{matrix} u(t) - u(-t) \end{matrix} \right\}$$



Qn. Use the properties; find FT of $\sin \omega_0 t$

$$x(t) = \sin \omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

$$\begin{aligned} F[x(t)] &= \frac{1}{2j} [F(e^{j\omega_0 t}) - F(e^{-j\omega_0 t})] \\ &= \frac{1}{2j} [2\pi \delta(\omega - \omega_0) - 2\pi \delta(\omega + \omega_0)] \\ &= \underline{\underline{-j\pi [\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]}} \end{aligned}$$

Inverse Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t} d\omega$$

Q₁ $X(\omega) = \delta(\omega)$

$$\begin{aligned} x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) \cdot e^{j\omega t} d\omega \\ &= \frac{1}{2\pi} \left[e^{j\omega t} \right]_{\omega=0} = \underline{\underline{\frac{1}{2\pi}}} \end{aligned}$$

Q₂ $X(\omega) = \delta(\omega - \omega_0)$

$$\begin{aligned} x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega \\ &= \frac{1}{2\pi} \left[e^{j\omega t} \right]_{\omega=\omega_0} = \underline{\underline{\frac{e^{j\omega_0 t}}{2\pi}}} \end{aligned}$$

Q₃ $F^{-1}[\delta(\omega)] = \frac{1}{2\pi}$

$$F^{-1}[2\pi \delta(\omega)] = 1$$

$$F^{-1}[\delta(\omega - \omega_0)] = \frac{e^{j\omega_0 t}}{2\pi}$$

$$F^{-1}[2\pi \delta(\omega - \omega_0)] = e^{j\omega_0 t}$$

Find $\mathcal{F}_T [\cos \omega_0 t]$

$$\mathcal{F}_T \left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} \right] = \frac{1}{2} [2\pi \delta(\omega - \omega_0) + 2\pi \delta(\omega + \omega_0)]$$
$$= \underline{\underline{\pi [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]}}$$

$$\mathcal{F}^{-1} [\delta(\omega)] = \frac{1}{2\pi} e^{j\omega_0 t}$$

$$\mathcal{F}^{-1} [\delta(\omega - \omega_0)] = \frac{e^{j\omega_0 t}}{2\pi}$$

$$\mathcal{F}(1) = 2\pi \delta(\omega)$$

$$\mathcal{F}(e^{j\omega_0 t}) = 2\pi \delta(\omega - \omega_0)$$

The input and output of a causal LTI system are related by differential eqn:

$$\frac{d^2 y(t)}{dt^2} + 6 \frac{dy}{dt} + 8 y(t) = 2x(t).$$

What is the Response of this system if $x(t) = t e^{-2t}$ assume initial condition is zero & use FT Method.

Take FT on both sides:

$$(j\omega)^2 Y(\omega) + 6 j\omega Y(\omega) + 8 Y(\omega) = 2 X(\omega)$$

$$\frac{Y(\omega)}{X(\omega)} = H(\omega) = \frac{2}{(j\omega)^2 + 6j\omega + 8} = \frac{2}{(j\omega + 2)(j\omega + 4)}$$

$$x(t) = t e^{-2t} u(t); X(\omega) = \frac{1}{(j\omega + 2)^2}$$

$$Y(\omega) = \frac{2}{(j\omega + 2)(j\omega + 4)} \cdot X(\omega) = \frac{2}{(j\omega + 2)^3 (j\omega + 4)}$$

$$Y(\omega) = \frac{A}{j\omega + 4} + \frac{B}{j\omega + 2} + \frac{C}{(j\omega + 2)^2} + \frac{D}{(j\omega + 2)^3}$$

$$A = \frac{2}{(-2)^3} = -\frac{2}{8} = -\frac{1}{4}$$

$$2 = A(j\omega + 2)^3 + B(j\omega + 4)(j\omega + 2)^2 + C(j\omega + 2)(j\omega + 4) + D(j\omega + 4)$$

$$2 = A[(j\omega)^3 + 6(j\omega)^2 + 12j\omega + 8] + B[(j\omega + 4)(j\omega^2 + 4j\omega + 4)] + C[(j\omega)^2 + 6j\omega + 8] + D(j\omega + 4)$$

$$\begin{aligned} \text{const.} \\ \frac{d^2}{dt^2} 2 &= 8A + 16B + 8C + 4D \\ \frac{d}{dt} 0 &= A + B \\ \frac{d^2}{dt^2} 0 &= 6A + 8B \end{aligned}$$

$$6A - 8B$$

$$A =$$

$$A = \frac{1}{4} \quad B = -\frac{1}{2} \quad D = \frac{1}{4}$$

$$Y(\omega) = \frac{1/4}{(j\omega+4)} + \frac{(-1/2)}{j\omega+2} + \frac{1}{(j\omega+2)^2} + \frac{(-1/4)}{(j\omega+2)^3}$$

$$y(t) = \frac{1}{4} e^{-4t} + \frac{(-1/2)}{1} e^{-t} + t \cdot e^{-2t} + \frac{1}{4} t^2 e^{-2t}$$

Q: Find the frequency Response of an LTI S/M described by diff. eqn.

$$\frac{d^2 y(t)}{dt^2} + 5 \frac{dy(t)}{dt} + 6 y(t) = 2x(t)$$

$$(j\omega)^2 Y(\omega) + 5(j\omega) Y(\omega) + 6 Y(\omega) = 2X(\omega)$$

$$Y(\omega) [(j\omega)^2 + 5j\omega + 6] = 2X(\omega)$$

$$\frac{Y(\omega)}{X(\omega)} = H(\omega) = \frac{2}{(j\omega+3)(j\omega+2)}$$

$$= \frac{A}{(j\omega+2)} + \frac{B}{(j\omega+3)}$$

$$A /_{j\omega=-2} = \frac{2}{(j\omega+3)(j\omega+2)} /_{j\omega=-2}$$

$$= \frac{2}{1} = \underline{2}$$

$$B /_{j\omega=-3} = \frac{-2}{(j\omega+2)} /_{j\omega=-3} = \underline{-2}$$

$$H(j\omega) = \frac{2}{(j\omega+2)} - \frac{2}{(j\omega+3)} = \underline{2e^{-2t} - 2e^{-3t}}$$

Relation b/w Fourier & Laplace Transforms:

$$\underline{\text{FT}} \quad X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot dt \quad \text{--- (1)}$$

$$\underline{\text{Inverse FT}} \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t} \cdot d\omega$$

put $j\omega = s$ in eq (1)

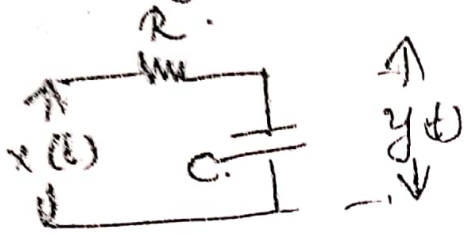
$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-(\sigma + j\omega)t} \cdot dt$$

$$\sigma + j\omega = s$$

$$X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt \quad \text{--- (2)}$$

$$\underline{X(\omega) = X(s) /_{s=j\omega}}$$

Find the Frequency Response of RC circuit
plot Magnitude & phase Response for $RC=1$



Differential eqn:

$$i(t) R + \frac{1}{C} \int i(t) dt = x(t) \quad \text{--- (1)}$$

$$\frac{1}{C} \int i(t) dt = y(t) \quad \text{--- (2)}$$

FT of eq (1) & (2).

$$I(\omega) R + \frac{1}{C j\omega} I(\omega) = X(j\omega)$$

$$\frac{1}{C j\omega} I(\omega) = Y(j\omega)$$

$$I(\omega) \left[R + \frac{1}{j\omega C} \right] = X(\omega)$$

$$I(\omega) \frac{1}{j\omega C} = Y(\omega)$$

$$\frac{Y(\omega)}{X(\omega)} = H(\omega) = \frac{(1/j\omega C)}{\left[R + \frac{1}{j\omega C} \right]} = \frac{1}{R j\omega C + 1}$$

$$H(\omega) = \frac{1}{1 + j\omega RC}$$

$$|H(\omega)| = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}}$$

$$\angle = -\tan^{-1}(\omega RC)$$

put $RC = 1$.

$$|H(\omega)| = \frac{1}{\sqrt{1 + \omega^2}}$$

$$\angle = -\tan^{-1}(\omega)$$

	1	2	3	5	10	20	50	∞
$ H $	1	0.707	0.447	0.316	.196	0.099	.0499	0
$\angle$ (deg)	0	-78.5	-1.107	-1.249	-1.373	-1.47	-1.52	-1.59



Q. Find FT of $x(t-b) = e^{-a(t-b)}$

$$F[x(t-b)] = e^{-j\omega b} X(j\omega)$$

$$F[e^{-at}] = \frac{1}{j\omega + 1}$$

$$F[e^{-a(t-b)}] = e^{-j\omega b} \frac{1}{1+j\omega}$$

Q. Find the FT

$$x(t) = [u(t+1) - u(t-1)] \cos 2\pi t$$

$$F[\cos 2\pi t] = \pi \delta(\omega - 2\pi) + \pi \delta(\omega + 2\pi)$$

$$F[u(t+1) - u(t-1)] = \int_{-1}^1 1 \cdot e^{-j\omega t} dt$$

$$= \left[\frac{e^{-j\omega t}}{-j\omega} \right]_{-1}^1$$

$$= \frac{-1}{j\omega} [e^{-j\omega} - e^{j\omega}]$$

$$= \frac{e^{j\omega} - e^{-j\omega}}{j\omega} = \frac{2}{\omega} \sin \omega$$

Using Convolution

$$F[u(t+1) - u(t-1)] = \frac{2}{\omega} \sin \omega$$

$$F[\cos 2\pi t] = \pi \delta(\omega - 2\pi) + \pi \delta(\omega + 2\pi)$$

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau) \cdot d\tau$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2 \sin \tau}{\tau} \cdot \pi [\delta(\omega - 2\pi - \tau) + \pi \delta(\omega + 2\pi - \tau)] d\tau$$

$$= \int_{-\infty}^{\infty} \frac{\sin \tau}{\tau} \cdot \delta(\omega - 2\pi - \tau) \cdot d\tau + \int_{-\infty}^{\infty} \frac{\sin \tau}{\tau} \cdot \delta(\omega + 2\pi - \tau) \cdot d\tau$$

$$= \frac{\sin(\omega - 2\pi)}{\omega - 2\pi} + \frac{\sin(\omega + 2\pi)}{\omega + 2\pi}$$

Spectral Density function:

- The Spectral density of a signal defines the distribution of energy or power per unit Bandwidth as a function of frequency.
- The Spectral density of Energy signals is called Energy Spectral Density (ESD)
- The Spectral density of power signals is called Power Spectral Density (PSD)

Energy Spectral Density (ESD)

- According to Rayleigh's Energy theorem, total Energy of signal given by;

$$E = \int_{-\infty}^{\infty} |x(f)|^2 df$$

Definition of ESD: squared amplitude spectrum $|X(f)|^2$

$$ESD = \psi(f) = |x(f)|^2$$

$$\psi(\omega) = |X(\omega)|^2$$

Energy Spectral Density (ESD) : Properties

Property: 1

ESD of Energy signal $x(t)$ is a Non-negative real valued function of frequency.

$$\Psi(\omega) \geq 0 \text{ for all values of } \omega$$

Property: 2.

ESD of a real valued energy signal $x(t)$ is an Even (symmetric) function of frequency.

$$\Psi(\omega) = \Psi(-\omega)$$

Property: 3.

Total Area under the curve of ESD of an energy signal $x(t)$ is equal to signal energy.

$$E = \int_{-\infty}^{\infty} \Psi(\omega) \cdot d\omega$$

Property: 4.

When an Energy signal $x(t)$ is passed through LTI S/M, ESD of o/p signal is equal to the product of ESD of input and squared amplitude response of S/M.

$$\Psi_o(\omega) = |H(\omega)|^2 \Psi_i(\omega)$$

$$\Psi_o(\omega) = |H(\omega)|^2 \Psi_i(\omega)$$

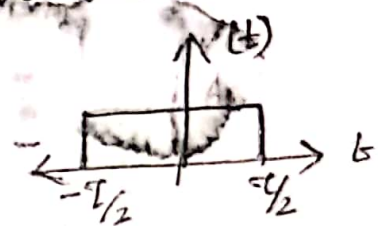
Property 5 :

For an energy signal $x(t)$, the auto-correlation function and Energy spectral density form a Fourier transform pair.

$$FT [R(t)] = \Psi(\omega)$$
$$\text{or } F^{-1}[\Psi(\omega)] = R(t)$$

Qn. Determine the ESD of a gate function

$$x(t) = A \quad -\tau/2 \leq t \leq \tau/2$$



$$ESD = |X(j\omega)|^2$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt$$

$$= \int_{-\tau/2}^{\tau/2} A \cdot e^{-j\omega t} dt = A \left(\frac{e^{-j\omega t}}{-j\omega} \right)_{-\tau/2}^{\tau/2}$$

$$= \frac{A}{-j\omega} \left[e^{-j\omega \tau/2} - e^{j\omega \tau/2} \right]$$

$$= \frac{A}{j\omega} \left[e^{j\omega \tau/2} - e^{-j\omega \tau/2} \right] = \frac{A \cdot 2j \sin \omega \tau/2}{j\omega}$$

$$= \underline{\underline{\frac{2A}{\omega} \sin \omega \tau/2}}$$

$$\Psi(\omega) ESD = |X(j\omega)|^2 = \frac{4A^2}{\omega^2} \sin^2(\omega \tau/2)$$

Q. Determine ESD of a sinc pulse.

$$x(t) = A \operatorname{sinc}(\omega t) \text{ \& Find total energy}$$

Ans:- $|x(f)|^2$

$$A \operatorname{sinc}(\omega t) \longleftrightarrow \frac{A}{\omega} \operatorname{rect}\left(\frac{f}{\omega}\right)$$

$$X(f) = \frac{A}{\omega} \operatorname{rect}\left[\frac{f}{\omega}\right]$$

$$\text{ESD} = |X(f)|^2 = \frac{A^2}{\omega^2} \operatorname{rect}\left[\frac{f}{\omega}\right]$$

$$\text{Energy} = \int_{-\infty}^{\infty} |x(f)|^2 df = \int_{-\omega}^{\omega} \frac{A^2}{\omega^2} df$$

$$= \frac{A^2}{\omega^2} \left(f \right)_{-\omega}^{\omega} = \frac{A^2}{\omega^2}$$

Q. Determine ESD of a sinc pulse

$x(t) = A \text{sinc}(2\omega t)$ & Find total energy

Ans:- $|x(f)|^2$

$$A \text{sinc}(2\omega t) \longleftrightarrow \frac{A}{2\omega} \text{rect}\left(\frac{f}{2\omega}\right)$$

$$X(f) = \frac{A}{2\omega} \text{rect}\left[\frac{f}{2\omega}\right]$$

$$\text{ESD} = |X(f)|^2 = \frac{A^2}{4\omega^2} \text{rect}\left[\frac{f}{2\omega}\right]$$

$$\text{Energy} = \int_{-\infty}^{\infty} |x(f)|^2 df = \int_{-\omega}^{\omega} \frac{A^2}{4\omega^2} df$$

$$= \frac{A^2}{4\omega^2} \left(f \right)_{-\omega}^{\omega} = \frac{A^2}{2\omega}$$

Power Spectral Density (PSD)

Let $x(t)$ be real valued power signal.

$$P = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t) \cdot dt.$$

According to Parseval's power theorem, average signal power given by

$$P = \sum_{n=-\infty}^{\infty} |c_n|^2$$

$$c_n = \frac{1}{T_0} x(nf_0)$$

$$\therefore P = \sum_{n=-\infty}^{\infty} \frac{1}{T_0^2} |x(nf_0)|^2 = \frac{1}{T_0^2} \sum_{n=-\infty}^{\infty} |x(nf_0)|^2$$

Define power spectral density (PSD) of a periodic signal, $x(t)$ as under :-

$$\text{PSD, } S(f) = \frac{1}{T_0^2} \sum_{n=-\infty}^{\infty} |x(nf_0)|^2 \delta(f - nf_0).$$

$$S(f) = \frac{1}{T_0^2} \sum_{n=-\infty}^{\infty} |x(nf_0)|^2$$

$\therefore$ Area under $\text{PSD} = \text{Average power of signal}$

$$\underline{\underline{P = \int_{-\infty}^{\infty} S(f) \cdot df}}$$

PSD: Properties

Property 1:

PSD of a power signal $x(t)$ is a Non-negative real valued function of frequency.

$$S(f) \geq 0 \quad \text{for all values of } f$$

Property 2:

PSD of a real valued power signal $x(t)$ is an even function of frequency

$$S(f) = S(-f)$$

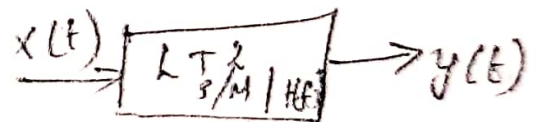
Property 3:

Area under the curve of PSD = average signal power

$$P = \int_{-\infty}^{\infty} S(f) \cdot df$$

Property 4:

$$S_o(f) = |H(f)|^2 S_i(f)$$



$S_o(f)$ = PSD of output of an LTI S/M

$S_i(f)$ = PSD of input

$|H(f)|$ = Amplitude response of S/M

Property 5:

Auto correlation function $R(\tau)$ & power spectral function form a Fourier Transform Pair.

PSD: Properties

Property 1:

PSD of a power signal $x(t)$ is a Non-negative real valued function of frequency.

$$S(f) \geq 0 \text{ for all values of } f$$

Property 2:

PSD of a real valued power signal $x(t)$ is an even function of frequency

$$S(f) = S(-f)$$

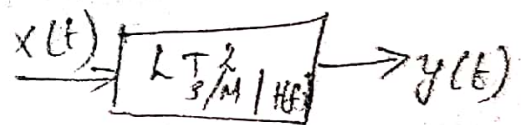
Property 3:

Area under the curve of PSD = average signal power

$$P = \int_{-\infty}^{\infty} S(f) \cdot df$$

Property 4:

$$S_o(f) = |H(f)|^2 S_i(f)$$



$S_o(f)$ = PSD of output of an LTI S/M

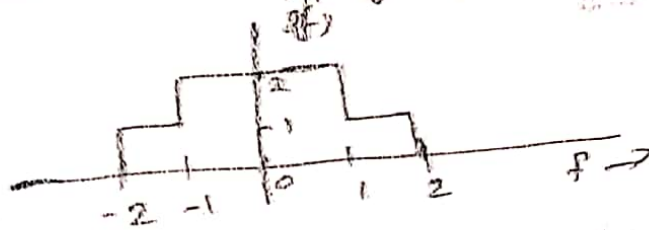
$S_i(f)$ = PSD of input

$|H(f)|$ = Amplitude response of S/M

Property 5:

Auto correlation function $R(\tau)$ & power Spectral function form a Fourier Transform Pair

Q. Evaluate the average power of the signal



$$P = \int_{-\infty}^{\infty} s(f) \cdot df = \int_{-\infty}^{-1} 1 \cdot df + \int_{-1}^1 2 \cdot df + \int_1^{\infty} 1 \cdot df$$

$$= (f)_{-\infty}^{-1} + 2(f)_{-1}^1 + (f)_{1}^{\infty}$$

$$= 1 + 4 + 1 = \underline{\underline{6}}$$

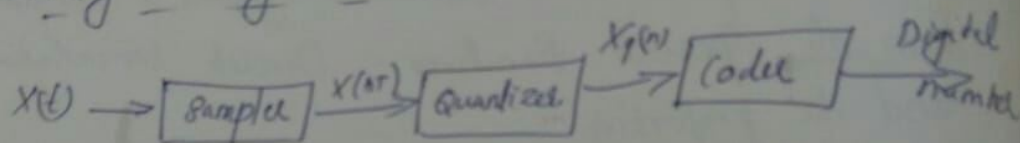
Module IV.

Sampled data systems - Sampling Process -
Sampling Theorem - Signal Re Construction - Zero
order & First order hold circuits - Difference
Equation representation of LTI Systems - Discrete
form of special functions - Discrete Convolution
and its properties

Sampling

- It is a process of converting a continuous time signal into discrete time signal.

Analog to Digital Conversion:



- Sampler, samples the input signal with a sampling interval T_s , produces an output $X(nT_s)$.

$X(nT_s)$ is discrete in time but continuous in amplitude.

- The output of Quantizer, which converts into discrete time, discrete amplitude signal.

Sampling Theorem:

- A continuous-time signal may be completely represented in its samples and recovered back if the sampling frequency is $f_s \geq 2f_m$.

f_s is the sampling frequency

f_m maximum frequency present in the signal

Nyquist Rate & Nyquist Interval

Nyquist rate:-

When sampling rate = $2f_m$ Samples/second
→ Minimum sampling rate.

$$f_s = 2f_m$$

Nyquist Interval:

→ Maximum sampling interval

$$T_s = \frac{1}{2f_m} \text{ seconds.}$$

Prob 1) $x(t) = 3 \cos 50\pi t + 10 \sin 300\pi t - \cos 100\pi t$

$$f_1 = \frac{50\pi}{2\pi} = \underline{\underline{25 \text{ Hz}}}$$

$$f_2 = \underline{\underline{150 \text{ Hz}}}$$

$$f_3 = \underline{\underline{50 \text{ Hz}}}$$

$$f_m = \underline{\underline{150 \text{ Hz}}}$$

$$f_s = 2f_m = \underline{\underline{300 \text{ Hz}}}$$

2) $x(t) = \frac{1}{2\pi} (\cos(4000\pi t) \cdot \cos(10000\pi t))$
 $= \frac{1}{4\pi} [\cos 5000\pi t + \cos 3000\pi t]$

$$f_1 = 2500 \text{ Hz}$$

$$f_m = 2500 \text{ Hz}$$

$$f_2 = 1500 \text{ Hz}$$

$$f_s = 2f_m = \underline{\underline{5000 \text{ Hz}}}$$

$$T_s = \frac{1}{f_s} = \frac{1}{5000} = 2 \mu\text{sec}$$

Qn. $\sin^2(200\pi t)$

$$t \cdot \frac{T}{4} = 200\pi t$$

~~$$T = 800\pi$$~~

$$\omega_m = 2 \cdot \frac{T}{2} = \underline{\underline{400\pi}}$$

$$f_m = \underline{\underline{200 \text{ Hz}}}$$

$$f_s = 2f_m = \underline{\underline{400 \text{ Hz}}}$$

$$\begin{aligned} \frac{T}{2} \cdot \frac{1}{2} &= 200\pi \\ \frac{T}{2} &= 400\pi \\ \omega_m &= 2f_m = \frac{400\pi}{2} = 200\pi \\ f_m &= \underline{\underline{200}} \end{aligned}$$

Qn. $\sin^2(200\pi t) + 3\sin^2(120\pi t)$

$$\omega_{m1} = \frac{T}{2} = \underline{\underline{200\pi}}$$

$$f_{m1} = \underline{\underline{100 \text{ Hz}}}$$

$$f_s = 2 \times f_m = 2 \times 120 = \underline{\underline{240 \text{ Hz}}}$$

~~$$\omega_{m2} = \frac{T}{4} = 120\pi$$~~

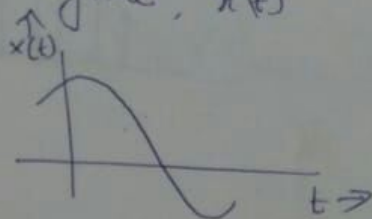
~~$$T = \frac{T}{2} = 240\pi$$~~

$$\omega_{m2} = \underline{\underline{240\pi}}$$

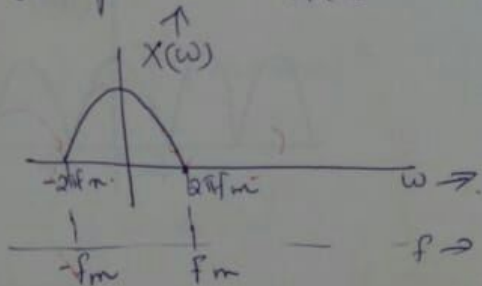
$$f_{m2} = \underline{\underline{120 \text{ Hz}}}$$

Proof of Sampling Theorem:

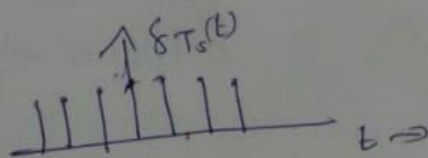
Consider a continuous-time signal, $x(t)$



Fourier Transform or Frequency spectrum $X(\omega)$

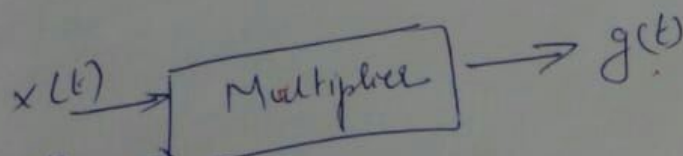


Sampling of $x(t)$ at a rate of ' f_s ' Hz achieved by multiplying $x(t)$ by an impulse train $\delta_{T_s}(t)$.

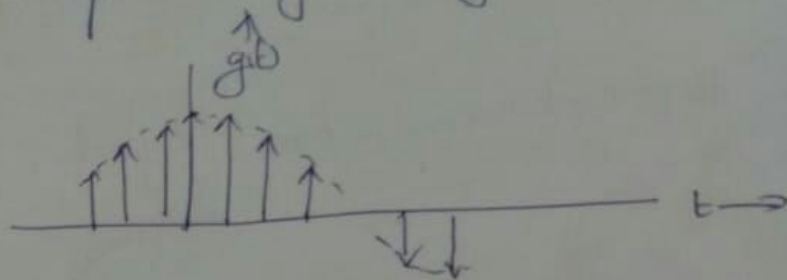


Impulse train $\delta_{T_s}(t)$ consist of unit impulses repeating periodically every ' T_s ' seconds

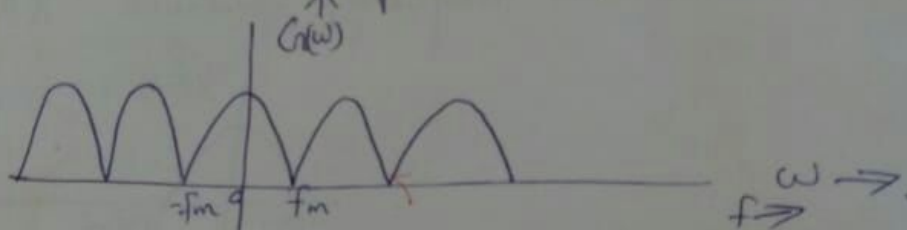
$$T_s = 1/f_s$$



The sampled signal $g(t)$



Spectrum of sampled signal $G(\omega)$



Let the sampled signal, $g(t) = x(t) \cdot \delta_{T_s}(t)$

$$\delta_{T_s}(t) = \frac{1}{T_s} (1 + 2\cos \omega_s t + 2\cos 2\omega_s t + 2\cos 3\omega_s t + \dots)$$

$$\omega_s = \frac{2\pi}{T_s} = 2\pi f_s$$

$$g(t) = \frac{1}{T_s} [x(t) + 2x(t) \cdot \cos \omega_s t + 2x(t) \cdot \cos 2\omega_s t + 2x(t) \cdot \cos 3\omega_s t + \dots]$$

$$\text{FT of } x(t) = X(\omega)$$

$$2x(t) \cos \omega_s t = X(\omega - \omega_s) + X(\omega + \omega_s)$$

$$2x(t) \cos 2\omega_s t = [X(\omega - 2\omega_s) + X(\omega + 2\omega_s)]$$

$$\text{FT of } g(t) =$$

$$\therefore G(\omega) = \frac{1}{T_s} [X(\omega) + X(\omega - \omega_s) + X(\omega + \omega_s) + X(\omega - 2\omega_s) + X(\omega + 2\omega_s) + \dots]$$

$$\underline{G(\omega)} = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$

ie; Spectrum $G(\omega)$ consists of $X(\omega)$ repeating periodically with period $\omega_s = 2\pi f_s$.

In order to reconstruct $x(t)$ from $g(t)$, we must be able to recover $X(\omega)$ from $G(\omega)$. This is possible if there is no overlap between successive cycles of $G(\omega)$. It requires

$$\underline{f_s > 2f_m}$$

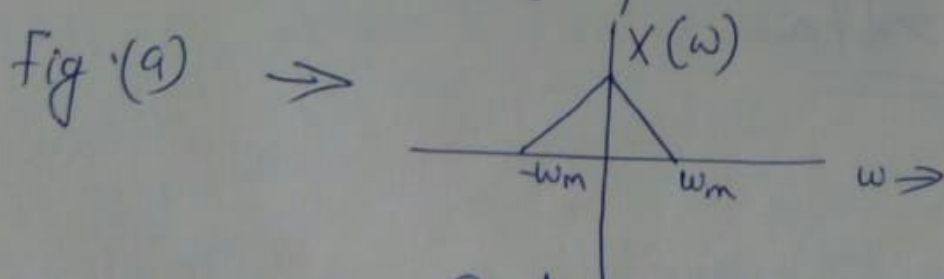
Reconstruction of Sampled Signal.

- According to Sampling theorem, discrete time signal can be perfectly reconstructed from its samples only if it sample at a rate of $f_s \geq 2f_m$. If the sampling rate is less than $2f_m$, the original signal cannot be recovered from the samples.

There is an overlap between lower and higher frequency ~~comp~~ components in the spectrum of sampled signals. This effect is called "Aliasing".

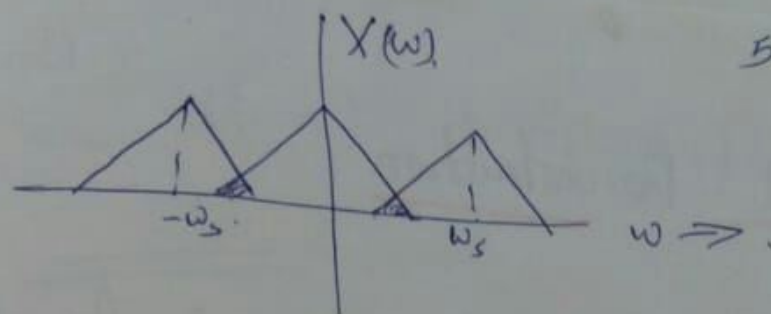
To avoid this aliasing effect, an anti aliasing filter is provided before the interpolation.

If sampling rate greater than Nyquist rate, No aliasing phenomenon occurs.



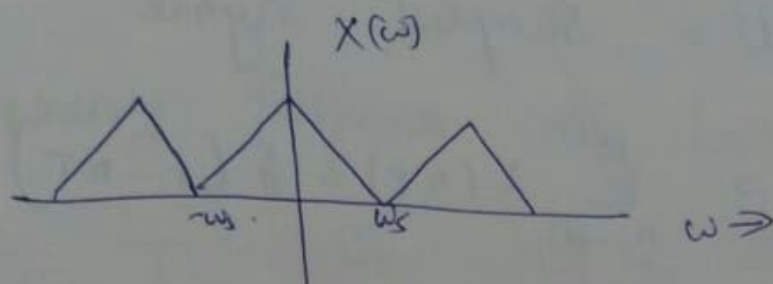
Spectrum of original signal $x(t)$.

Fig(b)



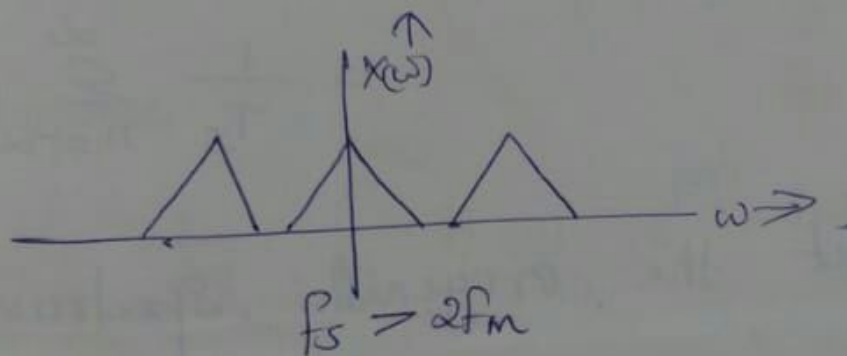
Spectrum of sampled signal. for $F_s < 2f_m$.

Fig(c)



Spectrum of sampled signal for $F_s = 2f_m$.

Fig.d.



Anti aliasing Filter:

A Filter is used to reject high frequency signals before it is sampled to reduce the aliasing is called anti aliasing Filter.

Signal Reconstruction:

$x(t)$ = Band limited signal

$x_s(t)$ = Sampled signal.

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT) \cdot \delta(t - nT)$$

$$\begin{aligned} \text{FT of } x_s(t) ; \quad X_s(e^{j\omega}) &= X_s(\omega) \\ &= \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s) \end{aligned}$$

To get the original spectrum, $x_s(t)$ is to be passed through a reconstruction (low-pass) filter with impulse response

$$x_s(t) \rightarrow \boxed{h(t)} \rightarrow x(t)$$

$$\begin{aligned} x(t) &= x_s(t) * h(t) = \left[\sum_{n=-\infty}^{\infty} x(nT) \cdot \delta(t - nT) \right] * h(t) \\ &= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(nT) \cdot \delta(\lambda - nT) \cdot h(t - \lambda) d\lambda \\ &= \sum_{n=-\infty}^{\infty} x(nT) \cdot \int_{-\infty}^{\infty} \delta(\lambda - nT) \cdot h(t - \lambda) d\lambda \end{aligned}$$

$$\delta(\lambda - nT) = 1 \text{ for } \lambda = nT$$

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT) \cdot h(t-nT)$$

$$h(t-nT) = ?$$

Let Frequency Response of low pass filter

$$H(\omega) = T \quad \text{for } |\omega| \leq \frac{\omega_s}{2}$$

$$= 0 \quad \text{for } |\omega| > \frac{\omega_s}{2}$$

Impulse Response

$$h(t) = \frac{1}{2\pi} \int_{-\frac{\omega_s}{2}}^{\frac{\omega_s}{2}} H(\omega) \cdot e^{j\omega t} \cdot d\omega$$

$$= \frac{T}{2\pi} \int_{-\frac{\omega_s}{2}}^{\frac{\omega_s}{2}} e^{j\omega t} \cdot d\omega$$

$$= \frac{T}{\pi t} \cdot \sin \frac{\omega_s t}{2}$$

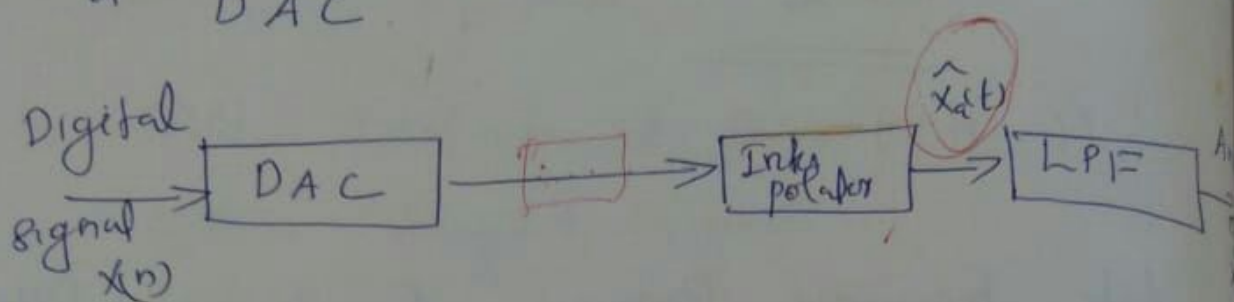
$$= \frac{T}{\pi} \cdot \frac{\sin \left(\frac{\omega_s t}{2} \right)}{t}$$

$$= \frac{T}{\pi} \cdot \frac{\sin \left(\frac{\pi t}{T} \right)}{t}$$

$$h(t-nT) = \frac{T}{\pi} \cdot \frac{\sin \frac{\pi}{T}(t-nT)}{(t-nT)} = \text{sinc} \frac{\pi(t-nT)}{T}$$

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT) \cdot \text{sinc} \frac{\pi}{T}(t-nT)$$

To Reconstruct the signal, best approach is DAC.

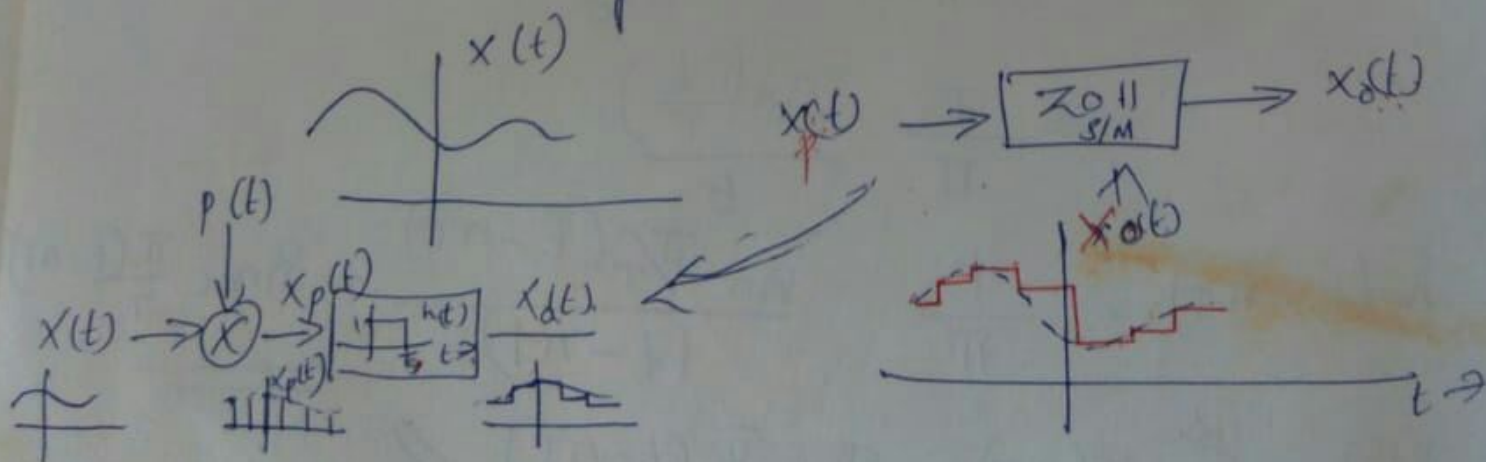


→ DAC produces output voltage proportional to the value of binary word.

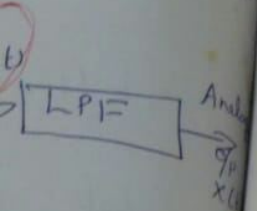
→ The output of DAC applied to interpolator which converts output samples of DAC into analog signal. Several Interpolators

1) Zero order hold (ZOH)

→ ZOH system samples a continuous time signal $x(t)$ at a given instant and holds that value until the next instant at which another sample is taken.



best approach



proportional

interpolator

of DAC

polynomial

no time
hold this
which

(1)

2 →

Frequency Response:

put $s = j\omega$

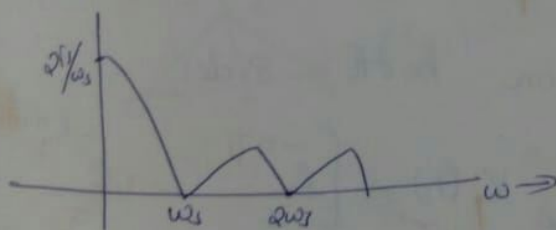
$$G_0(j\omega) = \frac{1 - e^{-Tj\omega}}{j\omega}$$

$$= \frac{e^{-jT\omega/2} \cdot e^{jT\omega/2} - e^{-Tj\omega}}{j\omega}$$

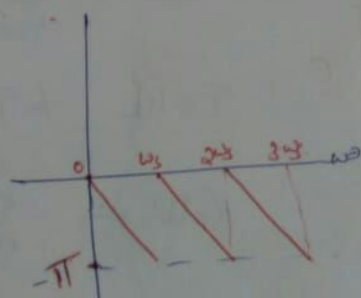
$$= e^{-jT\omega/2} \left[\frac{e^{jT\omega/2} - e^{-jT\omega/2}}{j\omega} \right]$$

$$= 2 \frac{e^{-jT\omega/2}}{\omega} \left(\sin \omega T/2 \right)$$

=====



Magnitude Response



Phase Response

→ ZOH reduces the effect the aliases.
 → Smoothing filter bcoz ~~reduces~~ ^{tends to} smooth the corners on the step generated by ZOH.

Properties
 1. Difference
 due to

the

(n) is

z.

be of

a

Zero order Hold is represented as a sum of Rectangular pulses shifted by integer multiples of sampling interval

The o/p of ZOH ckt is

$$X_o(t) = \sum_{n=-\infty}^{\infty} x_p(n) \cdot h(t - nT_s)$$

$$\text{Let } h(t) = u(t) - u(t - T_s)$$

$$\begin{aligned} h(t - nT_s) &= u(t - nT_s) - u(t - nT_s - T_s) \\ &= u(t - nT_s) - u(t - (n+1)T_s) \end{aligned}$$

$$\therefore X_o(t) = \sum_{n=-\infty}^{\infty} x_p(n) \cdot [u(t - nT_s) - u(t - (n+1)T_s)]$$

Take LT on both sides.

$$\begin{aligned} X_o(s) &= \sum_{n=-\infty}^{\infty} x_p(n) \cdot \left[\frac{e^{-nT_s}}{s} - \frac{e^{-(n+1)T_s}}{s} \right] \\ &= \sum_{n=-\infty}^{\infty} x_p(n) \cdot \left[\frac{e^{-nT_s}}{s} (1 - e^{-T_s}) \right] \end{aligned}$$

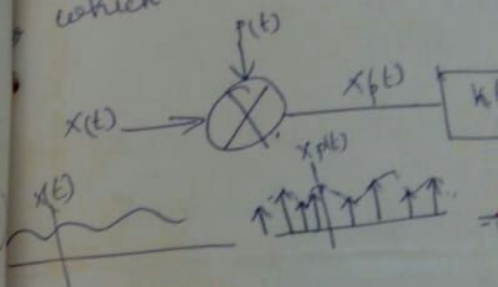
$$X_o(s) = \frac{1 - e^{-T_s}}{s} \sum_{n=-\infty}^{\infty} x_p(n) \cdot e^{-nT_s}$$

$$T.F = \frac{X_o(s)}{X_p(s)} = \frac{1 - e^{-T_s}}{s}$$

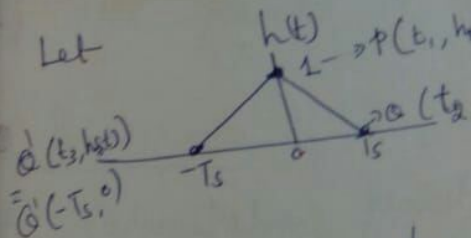
Zero order Hold is a discontinuous function

Linear Interpolation

In linear interpolation, samples or sample points are connected by straight lines. It produces a continuous function.



Let



$$h(t) = \begin{cases} 1 - \frac{t}{T_s} & 0 \leq t < T_s \\ 0 & \text{elsewhere} \end{cases}$$

$$\begin{aligned} H(\omega) &= \int_{-\infty}^{\infty} h(t) e^{j\omega t} dt \\ &= \int_0^{T_s} \left(1 - \frac{t}{T_s}\right) e^{j\omega t} dt \end{aligned}$$

huffed by
interval

Zero order Hold produces an output signal which is discontinuous.

Linear Interpolation (First order Hold)

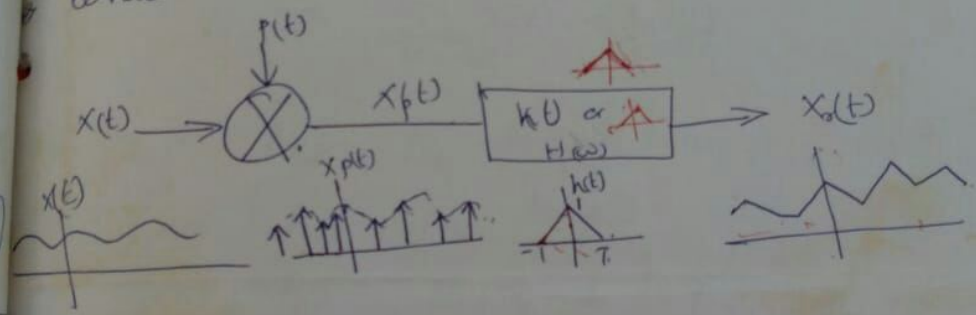
In linear Interpolation process, adjacent samples or sample points are connected by straight lines. It produces a reconstruction which is continuous.

nT_s)

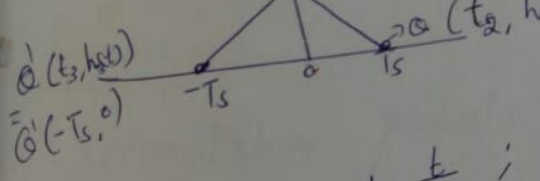
$(t - nT_s - T_s)$

$(t - (n+1)T_s)$

$u(t - (n+1)T_s)$



Let
 $h(t) \rightarrow p(t_1, h(t)) = p[0, 1]$
 $q(t_2, h(t)) = q(T_s, 0)$



$$h(t) = \begin{cases} 1 - \frac{t}{T_s} & ; 0 \leq t \leq T_s \\ 1 + \frac{t}{T_s} & ; -T_s \leq t \leq 0 \end{cases}$$

$$H(\omega) = \int_{-\infty}^{\infty} h(t) \cdot e^{-j\omega t} dt$$

$$\begin{aligned} H(\omega) &= \int_{-\infty}^{\infty} h(t) \cdot e^{-j\omega t} dt \\ &= \int_{-\infty}^0 \left(1 + \frac{t}{T_s}\right) \cdot e^{-j\omega t} dt + \int_0^{\infty} \left(1 - \frac{t}{T_s}\right) \cdot e^{-j\omega t} dt \end{aligned}$$

$$\frac{e^{-j\omega(n+1)T_s}}{s}$$

$$1 - e^{-j\omega T_s}$$

$$e^{-j\omega nT_s}$$

$$P(s)$$

$$= \frac{1}{T_s} \left[\frac{\sin(\omega T_s/2)}{(\omega/2)} \right]^2$$

Linear
The
input ;

⇒ x_1
h
y

⇒ Non

Prop
1. Co

Linear Convolution.

The response $y(n)$ of LTI s/n to arbitrary input;

$$y(n) = \sum_{k=-\infty}^{\infty} h(k) \cdot x(n-k).$$

$$\text{or} \\ = \sum_{k=-\infty}^{\infty} x(k) \cdot h(n-k).$$

→ $x_1(n)$ has N_1 samples

$h(n)$ has N_2 samples

$y(n)$ has $N_1 + N_2 - 1$ samples

→ Non-periodic or aperiodic convolution

Properties.

1. Commutative property:

$$y(n) = x(n) * h(n)$$

Proof: $y(n) = \underline{x(n) * h(n)} = \sum_{k=-\infty}^{\infty} x(k) \cdot h(n-k)$

Let $n-k = m$
 $k = n-m$

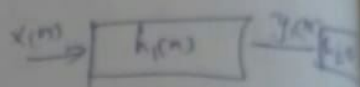
$$y(n) = \sum_{k=-\infty}^{\infty} x(n-m) \cdot h(m) = \underline{h(n) * x(n)}$$

2. Associative Property:

$$[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$$

Proof.

L.H.S $y_1(n) = x(n) * h_1(n)$



$$y(n) = y_1(n) * h_2(n)$$

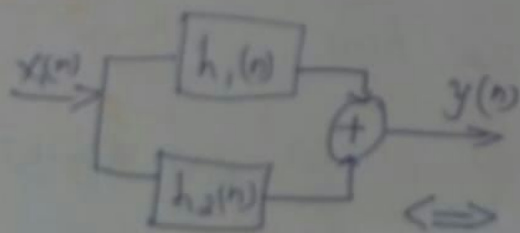
$$= [x(n) * h_1(n)] * h_2(n)$$

$$h(n) = h_1(n) * h_2(n)$$

$$y(n) = x(n) * h(n) = x(n) * [h_1(n) * h_2(n)]$$

3. Distributive Property:

$$x(n) * [h_1(n) + h_2(n)] = [x(n) * h_1(n)] + [x(n) * h_2(n)]$$

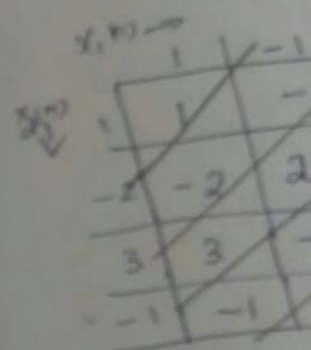


$$\Leftrightarrow x(n) \rightarrow [h_1(n) + h_2(n)] \rightarrow y(n)$$

Find the

$$x_1(n) = \{ \dots \}$$

$$x_2(n) = \{ \dots \}$$



$$y(n) = \{ \dots \}$$

$$= \{ \dots \}$$

If x

Th

$$Q. \quad x_1(n) = \dots$$

$$\downarrow 2$$

Find the convolution of sequences

$$x_1(n) = \{1, -1, 2, 3\}$$

$$N_1 = 4$$

$$x_2(n) = \{1, -2, 3, -1\}$$

$$N_2 = 4$$

$x_1(n) \rightarrow$

	1	-1	2	3
$x_2(n) \downarrow$	1	-1	2	3
-2	-2	2	-4	-6
3	3	-3	6	9
-1	-1	1	-2	-3

$$y(n) = N_1 + N_2 - 1 = 7$$

$$y(n) = \{1, (-2-1), (3+2+2), (-1-3-4+3), (1+6-6), (-2+9), -3\}$$

$$= \{1, -3, 7, -5, 1, 7, -3\}$$

If $x_1(n)$ starts at $n_1 = -1$, $x_2(n)$ starts at $n_2 = 0$

Then o/p $y(n)$ starts at $n_1 + n_2 = -1$

Q. $x_1(n) = \{1, -2, 3, 1\}$

$$x_2(n) = \{2, -3, -2\}$$

$x_1(n) \rightarrow$

	1	-2	3	1
$x_2(n) \downarrow$	2	-4	6	2
-3	-3	6	-9	-3
-2	-2	4	-6	-2

$$y(n) = \{2, -7, 10, -3, -9, -2\}$$

$$n_1 = 0$$

$$n_2 = 0$$

$$n = n_1 + n_2 = 0$$

Module 5.

Z-Transform - Region of convergence - Properties
of Z-Transform - Inverse ZT Methods
Z-Transform Function - Analysis of difference
equation of LTI systems - Basic idea on
stability and causality conditions.

- Z-Transform is an important tool in the
study of discrete time systems.

- Z-Transform of an arbitrary signal $x(n]$ is

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

Inverse Z-Transform of $X(z)$ is

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) \cdot z^{n-1} dz.$$

$\oint_C \Rightarrow$ Integration around the circle of
radius $|z|=r$ in a
counter clockwise direction.

Z-Transform $X(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_m z^{-m}}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-N}}$

Roots of Numerator polynomial are those values of z for which $X(z) = 0$; Zeros of $X(z)$

Roots of Denominator polynomial are poles for which $X(z)$ is infinity.

Zero location denoted by 'o';
pole location denoted by 'x'

Region of convergence:

- The values of z for which the sum in eq: $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$ converges.

Qn.

Find the Z-Transform & ROC for the signal $x(n) = a^n u(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n u(n) \cdot z^{-n}$$

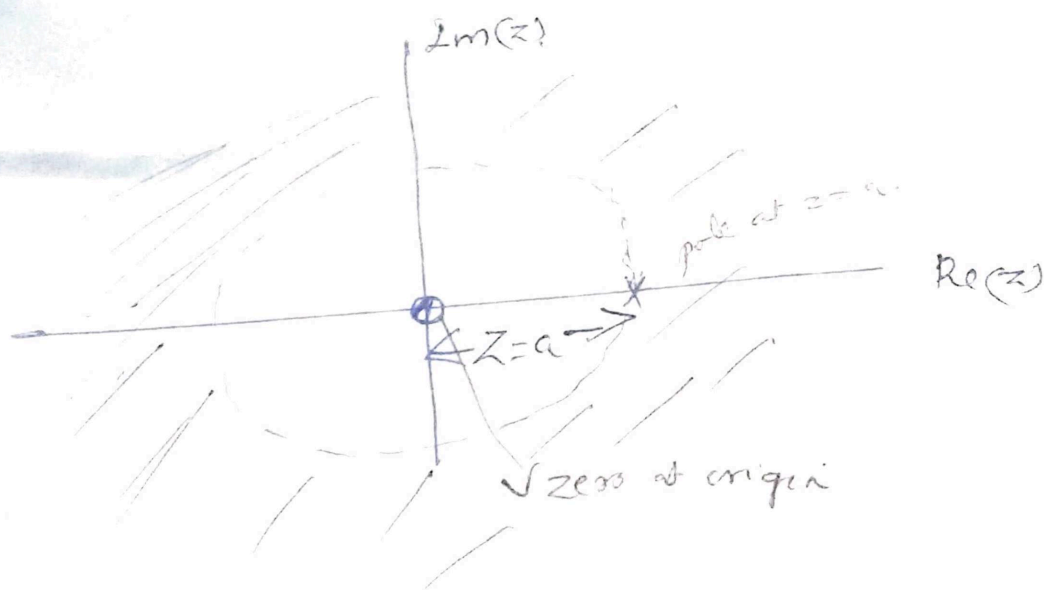
$$\left| \begin{array}{l} u(n) = 0; n < 0 \\ = 1; n \geq 0 \end{array} \right.$$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (a z^{-1})^n$$

The summation in eqn: $\sum_{n=0}^{\infty} (a z^{-1})^n$ converges

$$|a z^{-1}| < 1 \quad \text{or} \quad |a| < |z| \quad \text{or} \quad |z| > |a|$$

$$X(z) = \frac{1}{1 - a z^{-1}} \quad \text{ROC: } \underline{|z| > |a|}$$



ROC is the exterior to the circle of radius 'a'

$$(2) x(n) = -b^n u(-n-1).$$

Ans:

$$u(-n-1) = 1, \quad -n-1 \geq 0$$

$$-n \geq +1$$

$$n \leq -1$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} -b^n u(-n-1) z^{-n}$$

$$= - \sum_{n=-\infty}^{-1} b^n z^{-n} = - \sum_{n=0}^{\infty} b^{-n} z^n$$

$$= - \sum_{n=0}^{\infty} b^{-n} z^n$$

$$= - \sum_{n=0}^{\infty} (b^{-1} z)^n$$

$$= - \left[\frac{1}{1 - b^{-1} z} - 1 \right] \quad (1)$$

$$= - \frac{b^{-1} z}{1 - b^{-1} z}$$

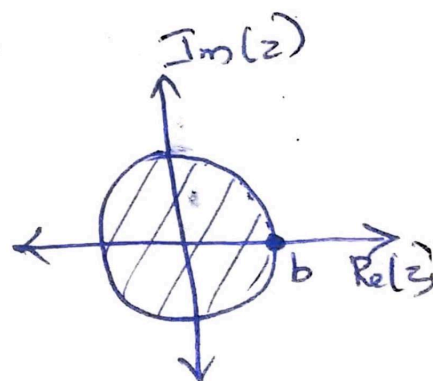
To find ROC:

From eq (1)

$$b^{-1} z < 1$$

$$\frac{z}{b} < 1$$

$$z < b$$



ROC is the interior to circle of radius b

$$(3) \quad x(n) = a^n u(n) - b^n u(-n-1) \\ = a^n u(n) + -b^n u(-n-1).$$

See the previous problems

$$x(z) = \frac{1}{1-az^{-1}} + \underbrace{\left[\frac{1}{1-b^{-1}z} - 1 \right]}_{\textcircled{1}}.$$

$$= \frac{z}{z-a} + \frac{z}{z-b}$$

To find ROC:

From eq. (1)

$$a z^{-1} < 1$$

$$\frac{a}{z} < 1$$

$$a < z$$

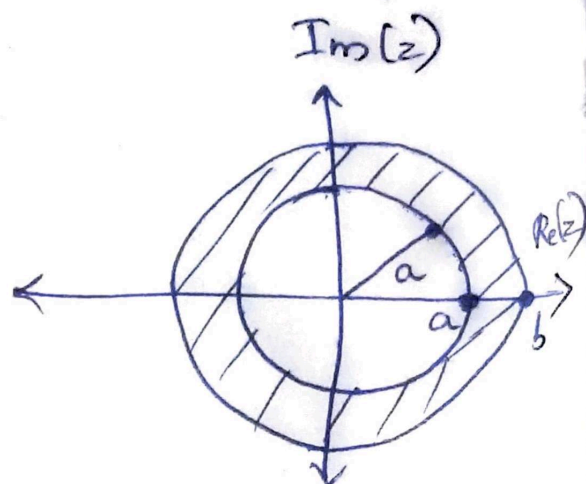
$$\Rightarrow z > a$$

$$b^{-1}z < 1$$

$$\frac{z}{b} < 1$$

$$\Rightarrow z < b$$

$$\Rightarrow a < z < b$$



Z-Transform and ROC of finite duration sequence

Right hand sequence:

- A right hand sequence is one which $x(n) = 0$ for all $n < n_0$, where n_0 is positive or negative but finite.
- For $n_0 \geq 0$: Resulting sequence is causal or a positive time sequence.
- ROC is entire z -plane except $z=0$

Qn. Find Z-Transform & ROC of Causal Sequence

$$x(n) = \{2, -1, 3, 2, 0, 1\}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

$$= x(-1)z + x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + x(4)z^{-4} + x(5)z^{-5}$$

$$= 2 + -z^{-1} + 3z^{-2} + 2z^{-3} + z^{-5}$$

$X(z)$ converges for all values of z except at $z=0$

Left Hand Sequence:

A left hand sequence is one for which $x(n) = 0$ for all $n \geq n_0$, where n_0 is positive or negative but finite. If $n_0 \leq 0$; Resulting sequence is Anticausal sequence.

- ROC is the entire z -plane except at $z = \infty$.

Q. Find the z -Transform & ROC of the anticausal sequence $x(n) = \{3, 2, -1, -4, 1\}$

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x(n) z^{-n} = x(-4) z^4 + x(-3) z^3 + x(-2) z^2 + x(-1) z + x(0) \\ &= 3z^4 + 2z^3 - z^2 - 4z + 1 \end{aligned}$$

$X(z)$ converges for all values of z except at $z = \infty$.

Two sided Sequence:

A signal that has finite duration on both the left and right sides.

- ROC is the entire z -plane except $z=0$ & $z=\infty$.

Q. Find z -Transform of

$$x(n) = \{1, 2, 0, -4, 3, 2, 1, 6, 5\}$$

$$\begin{aligned} X(z) &= x(4) z^4 + x(3) z^3 + x(-2) z^2 + x(-1) z + x(0) + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3} + x(4) z^{-4} \\ &= 1z^4 + 2z^3 - 4z + 3 + 2z^{-1} + z^{-2} + 6z^{-3} + 5z^{-4} \end{aligned}$$

ROC is entire z -plane except $z=0$ & $z=\infty$

Qa. Find the z-Transform

a) $x(n) = u(n) - u(n-3)$

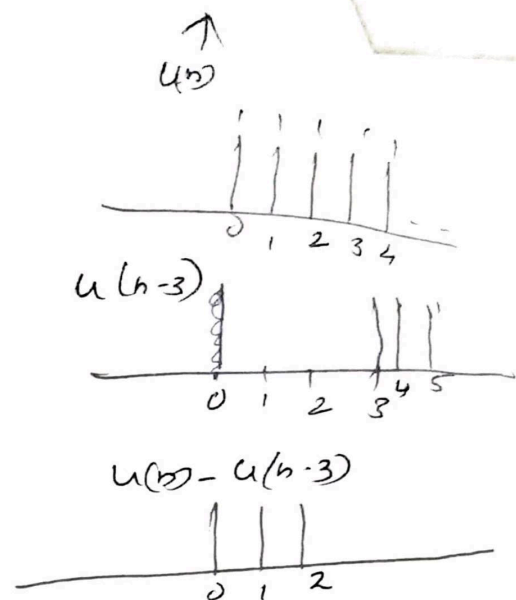
$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$= x(0) + x(1)z^{-1} + x(2)z^{-2}$$

$$= \underline{\underline{1 + z^{-1} + z^{-2}}}$$

ROC:

Entire z-plane except $z=0$



b) $x(n) = \{1, 2, 3, 2\}$

$$X(z) = 1 + 2z^{-1} + 3z^{-2} + 2z^{-3}$$

ROC: Entire z-plane except $z=0$

c) $x(n) = \{1, 2, \underset{\uparrow}{-1}, 2, 3\}$

$$X(z) = z^2 + 2z - 1 + 2z^{-1} + 3z^{-2}$$

ROC: Entire z-plane except $z=0$ & $z=\infty$

d) $x(n) = (e)^n u(n)$

$$X(z) = \frac{1}{1 - ez^{-1}}$$

ROC: $|z| > e$

e) $x(n) = (-1)^n u(n)$

$$X(z) = \frac{1}{1 + z^{-1}}$$

ROC: $|z| > 1$

f) $x(n] = (0.5)^n \text{ for } -2 < n < 0$
 $(0.2)^n \text{ for } n \geq 0$

$$\frac{1}{1 - 0.2z^{-1}} - \frac{1}{1 - 0.5z^{-1}}$$

ROC: $0.2 < |z| < 0.5$

Properties of ROC:

The ROC is a ring or disk in the z -plane centered at the origin

2. ROC cannot contain any poles
3. $x(n)$ is finite duration, Causal Sequence then ROC is the entire z -plane except $z=0$
4. If $x(n)$ is finite duration, anti-causal sequence, then ROC is the entire z -plane except at $z=\infty$
5. If $x(n)$ is finite duration, two sided sequence the ROC is the entire z -plane except at $z=0$ & $z=\infty$
6. If $x(n)$ is an infinite duration, two sided sequence the ROC will consist of a ring in the z -plane bounded on the interior & exterior by a pole, not containing any poles.
7. ROC of an LTI stable s/n contains Unit circle.
8. ROC must be a connected region

Qn Find z-Transform of

$x(n) = \sin \omega_0 n \cdot u(n)$ & find ROC

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

$$\left| \begin{array}{l} \sum_{n=0}^{\infty} 1 + r + r^2 + \dots \\ = \frac{1}{1-r} \end{array} \right| \quad |r| < 1$$

$$= \sum_{n=-\infty}^{\infty} \sin \omega_0 n \cdot u(n) \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} \sin \omega_0 n \cdot z^{-n} = \sum_{n=0}^{\infty} \left(\frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{2j} \right) \cdot z^{-n}$$

$$= \frac{1}{2j} \left[\sum_{n=0}^{\infty} e^{j\omega_0 n} z^{-n} - \sum_{n=0}^{\infty} e^{-j\omega_0 n} z^{-n} \right]$$

$$= \frac{1}{2j} \left[\sum_{n=0}^{\infty} \left(e^{j\omega_0} z^{-1} \right)^n - \sum_{n=0}^{\infty} \left(e^{-j\omega_0} z^{-1} \right)^n \right]$$

$$= \frac{1}{2j} \left[\frac{1}{1 - e^{j\omega_0} z^{-1}} - \frac{1}{1 - e^{-j\omega_0} z^{-1}} \right]$$

$$= \frac{1}{2j} \left[\frac{e^{j\omega_0} z^{-1} - e^{-j\omega_0} z^{-1}}{1 - e^{j\omega_0} z^{-1} - e^{-j\omega_0} z^{-1} + z^{-2}} \right]$$

$$= \frac{z^{-1} \sin \omega_0}{1 - 2 \cos \omega_0 z^{-1} + z^{-2}}$$

$$\text{ROC: } |z| > 1$$

$$\text{ROC: } |e^{j\omega_0} z^{-1}| < 1 \quad \& \quad |e^{-j\omega_0} z^{-1}| < 1$$

$$\text{i.e. } |z| > 1$$

$$\frac{1}{|z|} < 1$$

Properties of z-Transform:

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1. Linearity:

$$\text{If } \mathcal{Z}[x_1(n)] = X_1(z)$$

$$\mathcal{Z}[x_2(n)] = X_2(z)$$

$$\text{Then } \mathcal{Z}\{a x_1(n) + b x_2(n)\} = a X_1(z) + b X_2(z).$$

$$\begin{aligned} \text{Proof: } \mathcal{Z}\{a x_1(n) + b x_2(n)\} &= \sum_{n=-\infty}^{\infty} a x_1(n) \cdot \bar{z}^n + \sum_{n=-\infty}^{\infty} b x_2(n) \cdot \bar{z}^n \\ &= a X_1(z) + b X_2(z) \end{aligned}$$

2. Time Shift:

$$\text{If } \mathcal{Z}[x(n)] = X(z)$$

$$\text{Then } \mathcal{Z}[x(n-m)] = \bar{z}^{-m} X(z).$$

$$\begin{aligned} \text{Proof: } \mathcal{Z}\{x(n-m)\} &= \sum_{n=-\infty}^{\infty} x(n-m) \cdot \bar{z}^n \\ &= \sum_{n=-\infty}^{\infty} x(p) \cdot \bar{z}^{(m+p)} \quad \left| \begin{array}{l} n-m=p \\ n=m+p \end{array} \right. \\ &= \bar{z}^{-m} X(z) \end{aligned}$$

Note: If $x(n)$ is causal:

$$X_+(z) = \mathcal{Z}\{x(n)\}.$$

$$\text{Then } \mathcal{Z}\{x(n+m)\} = \bar{z}^m \left[X_+(z) - \sum_{k=0}^{m-1} x(k) \bar{z}^k \right]$$

$$(2) \quad \mathcal{Z}\{x(n-m)\} = z^{-m} \left[X_+(z) + \sum_{k=1}^m x(k) \cdot z^{-k} \right]$$

III Multiplication by an Exponential Sequence:

If $\mathcal{Z}\{x(n)\} = X(z)$

Then $\mathcal{Z}\{a^n x(n)\} = X[\bar{a}^{-1}z]$

Proof:

$$\begin{aligned} \mathcal{Z}\{a^n x(n)\} &= \sum_{n=-\infty}^{\infty} a^n x(n) \cdot z^{-n} \\ &= \sum_{n=-\infty}^{\infty} x(n) \cdot (\bar{a}^{-1}z)^{-n} \\ &= \underline{\underline{X(\bar{a}^{-1}z)}} \end{aligned}$$

ROC: $|a|/R_1 < |z| < |a|/R_2$

4. Time Reversal:

If $\mathcal{Z}\{x(n)\} = X(z)$

$\mathcal{Z}\{x(-n)\} = X(\bar{z}^{-1})$

Proof:

$$\begin{aligned} \mathcal{Z}\{x(-n)\} &= \sum_{n=-\infty}^{\infty} x(-n) \cdot \bar{z}^n \\ &= \sum_{l=-\infty}^{\infty} x(l) \cdot \bar{z}^{-(-l)} \\ &= \underline{\underline{X(\bar{z}^{-1})}} \end{aligned}$$

put
 $l = -n$

5. Differentiation of $X(z)$

If $\mathcal{Z}\{x(n)\} = X(z)$ Then

$$\mathcal{Z}\{n x(n)\} = -z \frac{d}{dz} X(z)$$

6. Convolution Theorem:

$\mathcal{Z}\{x(n)\} = X(z)$ & $\mathcal{Z}\{h(n)\} = H(z)$

$$\mathcal{Z}\{x(n) * h(n)\} = X(z) \cdot H(z)$$

7. Initial value Theorem

$$x(0) = \lim_{z \rightarrow \infty} X_+(z)$$

Proof:

$$X_+(z) = \sum_{n=0}^{\infty} x(n) \cdot z^{-n}$$

$$\lim_{z \rightarrow \infty} X_+(z) = \lim_{z \rightarrow \infty} \sum_{n=0}^{\infty} x(n) \cdot z^{-n}$$

$$\lim_{z \rightarrow \infty} X_+(z) = x(0)$$

8. Final value Theorem

$$x(\infty) = \lim_{z \rightarrow 1} \left(\frac{1-z^{-1}}{z-1} \right) X_+(z) \quad \text{or} \quad \lim_{z \rightarrow 1} (z-1) X(z)$$

Proof: $y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k) \cdot h(n-k)$

$\mathcal{Z}$ Transform on both sides

$$Y(z) = \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x(k) \cdot h(n-k) \right] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x(k) \cdot h(n-k) \right] \cdot z^{-n} \cdot z^k \cdot z^{-k}$$

$$= \sum_{k=-\infty}^{\infty} x(k) \cdot z^{-k} \cdot \sum_{l=-\infty}^{\infty} h(l) \cdot z^{-l} = X(z) \cdot H(z)$$

5. Multiplication by 'n' (Differentiation property)
 [Differentiation property]
 If $\mathcal{Z}\{x(n)\} = X(z)$, then

$$\mathcal{Z}\{n x(n)\} = -z \frac{d}{dz} X(z)$$

Proof:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

$$\mathcal{Z}\{n x(n)\} = \sum_{n=-\infty}^{\infty} n x(n) \cdot z^{-n}$$

$$= z \sum_{n=-\infty}^{\infty} n x(n) \cdot z^{-n-1}$$

$$= z \sum_{n=-\infty}^{\infty} x(n) \cdot \left[n z^{-[n+1]} \right]$$

$$= z \sum_{n=-\infty}^{\infty} x(n) \left\{ -\frac{d}{dz} (z^{-n}) \right\}$$

$$= -z \sum_{n=-\infty}^{\infty} x(n) \cdot \left\{ \frac{d}{dz} (z^{-n}) \right\}$$

$$= -z \frac{d}{dz} \left\{ \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n} \right\}$$

$$= -z \frac{d}{dz} X(z)$$

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$$\mathcal{Z}\{n^k x(n)\} = \left(-z \frac{d}{dz} \right)^k X(z)$$

$$\left[\frac{d}{dx} (x^n) = n x^{n-1} \right]$$

Conclusion:

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Most important property.

$$\mathcal{Z}\{x(n) * h(n)\} = X(z) H(z)$$

Let Proof:

$$\begin{aligned} y(n) &= x(n) * h(n) \\ &= \sum_{k=-\infty}^{\infty} x(k) \cdot h(n-k) \end{aligned}$$

Take 'z' Transform on both sides;

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x(k) h(n-k) \right] z^{-n} \\ &= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x(k) \cdot h(n-k) \cdot z^{-n} \\ &= \sum_{k=-\infty}^{\infty} x(k) \cdot z^{-k} \sum_{n=-\infty}^{\infty} h(n-k) \cdot z^{-(n-k)} \\ &= \sum_{k=-\infty}^{\infty} x(k) \cdot z^{-k} \sum_{l=-\infty}^{\infty} h(l) \cdot z^{-l} \end{aligned}$$

$$\underline{\underline{Y(z) = X(z) \cdot H(z)}}$$

7. Initial value Theorem

$$x(0) = \lim_{z \rightarrow \infty} X_+(z)$$

Proof:

$$X_+(z) = \sum_{n=0}^{\infty} x(n) \cdot z^{-n}$$

$$\lim_{z \rightarrow \infty} X_+(z) = \lim_{z \rightarrow \infty} \sum_{n=0}^{\infty} x(n) \cdot z^{-n}$$

$$\lim_{z \rightarrow \infty} X_+(z) = x(0)$$

8. Final value Theorem

$$x(\infty) = \lim_{z \rightarrow 1} \left(\frac{1-z^{-1}}{z-1} \right) X_+(z) \quad \text{as} \quad \lim_{z \rightarrow 1} (z-1) X(z)$$

Find z-Transform of the sequence

$$x(n) = a^{n-1} u(n-1) \text{ using property.}$$

Let

$$x(n) = a^n u(n)$$

$$X(z) = \frac{1}{1 - az^{-1}} \quad \text{ROC: } |z| > |a|.$$

Using Time Shifting property;

$$Z\{x(n-k)\} = z^{-k} X(z)$$

$$Z\{a^{n-1} u(n-1)\} = \frac{z^{-1}}{1 - az^{-1}} = \frac{1/z}{1 - a/z} = \underline{\underline{\frac{1}{z-a}}}$$

Qn. Find the z-Transform of

$$x(n) = a^n \cos(n\pi/2)$$

$$\boxed{\begin{aligned} Z[\cos \omega_0 n \cdot u(n)] &= \frac{1 - \cos \omega_0 z^{-1}}{1 - (2 \cos \omega_0) z^{-1} + z^{-2}} \\ \omega_0 &= \pi/2 \\ Z[\cos n\pi/2 \cdot u(n)] &= \underline{\underline{\frac{1}{1 + z^{-2}}}} \end{aligned}}$$

Property $Z[a^n x(n)] = X[a^{-1} z]$

$$Z[a^n \cos \frac{n\pi}{2}] = \underline{\underline{\frac{1}{1 + (a^{-1} z)^{-2}}}} \quad \text{*}$$

$$= \frac{1}{1 + \frac{a^2}{z^2}} = \frac{z^2}{\underline{\underline{z^2 + a^2}}}$$

Qn. Determine Z Transform of

$$X(n) = n U(n)$$

$$Z[U(n)] = \frac{z}{z-1}$$

$$\begin{aligned} Z[n U(n)] &= -z \frac{d}{dz} \left(\frac{z}{z-1} \right) \\ &= -z \left[\frac{(z-1) \cdot 1 - z}{(z-1)^2} \right] \\ &= \underline{\underline{\frac{z}{(z-1)^2}}} \end{aligned}$$

Inverse Z-Transform

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- The process of finding $x(n)$ from its Z-Transform $X(z)$ is called Inverse Z-Transform

$$x(n) = z^{-1} [X(z)]$$

4 Methods:

1. Partial Fraction Method
2. Long division or power series Method
3. Residue Method.
4. Convolution Method.

1. Partial Fraction Method:

$$X(z) = \frac{N(z)}{D(z)}$$

• Distinct poles

$$\frac{X(z)}{z} = \frac{C_1}{z-p_1} + \frac{C_2}{z-p_2} + \dots + \frac{C_N}{z-p_N}$$

$$C_k = (z-p_k) \cdot \frac{X(z)}{z} \bigg|_{z=p_k}$$

where $k=1, 2, 3, \dots, N$.

Multiple poles:

$$\frac{X(z)}{z} = \frac{C_1}{z-p_1} + \frac{C_2}{z-p_2} + \frac{C_3}{(z-p_2)^2}$$

$$C_1 = (z-p_1) \cdot \frac{X(z)}{z} \bigg|_{z=p_1}$$

$$C_2 = \frac{1}{1!} \frac{d}{dz} (z-p_2)^2 \frac{X(z)}{z} \bigg|_{z=p_2}$$

$$C_3 = (z-p_2)^2 \frac{X(z)}{z} \bigg|_{z=p_2}$$

Qn. Find Inverse Z-Transform of

$$X(z) = \frac{1 + 3z^{-1}}{1 + 3z^{-1} + 2z^{-2}} ; |z| > 2$$

$$= \frac{1 + 3/z}{1 + 3/z + 2/z^2} = \left(\frac{z+3}{z^2 + 3z + 2} \right) \cdot z$$

$$\frac{X(z)}{z} = \frac{z+3}{(z+2)(z+1)} = \frac{A}{z+2} + \frac{B}{z+1}$$

$$A = -1$$

$$B = 2$$

$$\frac{X(z)}{z} = \frac{2}{z+1} - \frac{1}{z+2}$$

$$x[a^n u(n)] = \frac{z}{z-a}$$

$$X(z) = 2 \frac{z}{z+1} - \frac{z}{z+2}$$

$$x(n) = 2(-1)^n u(n) - (-2)^n u(n)$$

$$\text{ROC} \quad |z| > 2$$

$$Q_n: X(z) = \frac{z(z^2 - 4z + 5)}{(z-3)(z-1)(z-2)}$$

$$\text{for } ① 2 < |z| < 3$$

$$② |z| > 3$$

$$③ |z| < 1$$

$$\frac{X(z)}{z} = \frac{z^2 - 4z + 5}{(z-3)(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{z-3}$$

$$A=1 \quad B=-1 \quad C=1$$

$$\frac{X(z)}{z} = \frac{1}{z-1} - \frac{1}{z-2} + \frac{1}{z-3}$$

$$X'(z) = \frac{z}{z-1} - \frac{z}{z-2} + \frac{z}{z-3}$$

(i) ROC $2 < |z| < 3$.

$$X(n) = u(n) - 2^n u(n) - (3)^n u(-n-1)$$

(ii) ROC $|z| > 3$

$$X(n) = u(n) - 2^n u(n) + 3^n u(n)$$

(iii) $|z| < 1$

$$X(n) = -u(-n-1) + 2^n u(-n-1) - 3^n u(-n-1)$$

Note:

$$z[a^n u(n)] = \frac{z}{z-a}$$

$$|z| > a$$

$$z[-b^n u(n-1)] = \frac{z}{z-b}$$

$$|z| < b$$

$$X(z) = \frac{z^2}{(1-az)(z-a)}$$

$$ROC: a < |z| < \frac{1}{a} \quad 16$$

$$X(z) = \frac{z^2}{-a(z-\frac{1}{a})(z-a)}$$

$$\frac{X(z)}{z} = \frac{(-\frac{z}{a})}{(z-\frac{1}{a})(z-a)} = \frac{A}{(z-\frac{1}{a})} + \frac{B}{(z-a)}$$

$$A = \frac{\frac{-1/a}{a}}{(1/a-a)} = \frac{-1/a^2}{(1-a^2)/a} = \underline{\underline{\frac{-1}{a(1-a^2)}}}$$

$$B = \frac{-1}{(a^2-1)/a} = \frac{-a}{a^2-1} = \underline{\underline{\frac{a}{1-a^2}}}$$

$$\frac{X(z)}{z} = \frac{-1/a(1-a^2)}{(z-\frac{1}{a})} + \frac{(a/1-a^2)}{(z-a)}$$

$$X(z) = \frac{-1}{a(1-a^2)} \cdot \frac{z}{(z-\frac{1}{a})} + \left(\frac{a}{1-a^2}\right) \cdot \frac{z}{z-a}$$

$$X(n) = \underline{\underline{\frac{-1}{a(1-a^2)} \cdot \left(\frac{1}{a}\right)^n u(-n-1) + \frac{a}{1-a^2} \cdot a^n u(n)}}$$

$$X(z) = \frac{1}{1+z^{-1}+z^{-2}} = \frac{z^2}{z^2+z+1}$$

$$\frac{X(z)}{z} = \frac{z}{z^2+z+1} = \frac{z}{(z+0.5+j0.866)(z+0.5-j0.866)}$$

$$= \frac{A}{[z - (-0.5+j0.866)]} + \frac{B}{z - [-0.5-j0.866]}$$

$$A = 0.5 + j0.288$$

$$B = 0.5 - j0.288$$

$$\frac{X(z)}{z} = \frac{0.5 + j0.288 \cdot z}{z - (-0.5 + j0.866)} + \frac{0.5 - j0.288 \cdot z}{z - [-0.5 - j0.866]}$$

$$= (0.5 + j0.288) \cdot (-0.5 + j0.866)^n u(n) +$$

$$(0.5 - j0.288) \cdot (-0.5 - j0.866)^n u(n)$$

$$= (0.5 + j0.288) \left[\cos \frac{2n\pi}{3} + j \sin \frac{2n\pi}{3} \right]^n u(n) +$$

$$(0.5 - j0.288) \left[\cos \frac{2n\pi}{3} - j \sin \frac{2n\pi}{3} \right]^n u(n)$$

$$= (0.5 + j0.288) \left[\cos \frac{2n\pi}{3} + j \sin \frac{2n\pi}{3} \right] u(n) +$$

$$(0.5 - j0.288) \left[\cos \frac{2n\pi}{3} - j \sin \frac{2n\pi}{3} \right] u(n)$$

$$\begin{aligned}
 &= \left[\cos \frac{2n\pi}{3} - 0.576 \sin \frac{2n\pi}{3} \right] \text{ ucm} \\
 &= 1.15 \left[0.866 \cos \frac{2n\pi}{3} - 0.5 \sin \frac{2n\pi}{3} \right] \\
 &= 1.15 \left[\cos \left(\frac{2n\pi}{3} + \frac{\pi}{6} \right) \right] \text{ ucm}
 \end{aligned}$$

$$3. \quad X(z) = \frac{z(z-1)}{(z+2)^3(z+1)}$$

$$\frac{X(z)}{z} = \frac{z-1}{(z+2)^3(z+1)}$$

$$= \frac{A}{z+2} + \frac{B}{(z+2)^2} + \frac{C}{(z+2)^3} + \frac{D}{z+1}$$

$$A = \underline{\underline{2}}$$

$$B = \underline{\underline{2}}$$

$$D = \underline{\underline{-2}}$$

$$C = \frac{-2-1}{-1} = \underline{\underline{3}}$$

$$z-1 = A(z+2)^2(z+1) + B(z+2)(z+1) + C(z+1) + D(z+2)^3$$

$$z-1 = A(z+2)^2(z+1) + B(z+2)(z+1) + (3z+3) - 2(z+2)^3$$

$$\underline{\underline{z=0}} \quad -1 = 4A + 2B + 3 - 16$$

$$4A + 2B - 13 = -1$$

$$4A + 2B = 12$$

$$6A + 2B = 16$$

$$\underline{\underline{-2A = -4}}$$

$$A = \underline{\underline{2}}$$

$$6 + B = 8 \quad B = \underline{\underline{2}}$$

$$\underline{\underline{z=1}} \quad 0 = A(18) + B(6) + 6 - 54$$

$$48 = 18A + 6B$$

$$8 = 3A + B$$

$$A = \frac{1}{2!} \frac{d^2}{dz^2} \left[(z+2)^3 \cdot \frac{X(z)}{z} \right] \bigg|_{z=-2}$$

$$= \underline{\underline{2}}$$

$$B = \frac{1}{1!} \frac{d}{dz} \left[(z+2)^2 \cdot \frac{X(z)}{z} \right] \bigg|_{z=-2}$$

$$= \underline{\underline{2}}$$

Note:

$$\frac{z}{z-a}$$

$$a^n u(n)$$

$$\frac{z}{(z-a)^2}$$

$$n a^{n-1} u(n)$$

$$\frac{z}{(z-a)^3}$$

$$\frac{n(n-1)}{2!} a^{n-2} u(n)$$

$$\frac{z}{(z-a)^4}$$

$$\frac{n(n-1)(n-2)}{3!} a^{n-3} u(n)$$

⋮

$$\frac{z}{(z-a)^k}$$

$$\frac{n(n-1) \dots [n-(k-2)]}{(k-1)!} a^{n-k+1} u(n)$$

$$f(z) = \frac{2}{z+2} + 2 \frac{z}{(z+2)^2} + 3 \frac{z}{(z+2)^3} - 2 \frac{z}{z+1}$$

$$f(n) = 2(-2)^n u(n) + 2 \cdot n (-2)^{n-1} u(n) + 3 \frac{n(n-1)}{2!} (-2)^{n-2} u(n) - 2 (-1)^n u(n)$$

$$4. \quad X(z) = \frac{8z-19}{z^2-5z+6}$$

$$\frac{X(z)}{z} = \frac{8z-19}{z(z^2-5z+6)} = \frac{A}{z} + \frac{B}{(z-3)} + \frac{C}{(z-2)}$$

$$A = \underline{\underline{-19/6}} \quad B = \frac{24-19}{3} = \underline{\underline{5/3}}$$

$$C = \frac{16-19}{2 \times (-1)} = \frac{-3}{-2} = \underline{\underline{3/2}}$$

$$\frac{X(z)}{z} = -19/6 \cdot \frac{1}{z} + 5/3 \cdot \frac{1}{z-3} + 3/2 \cdot \frac{1}{z-2}$$

$$X(z) = -19/6 \cdot \frac{z}{z} + 5/3 \cdot \frac{z}{z-3} + 3/2 \cdot \frac{z}{z-2}$$

$$= -19/6 + 5/3 \cdot \frac{z}{z-3} + 3/2 \cdot \frac{z}{z-2}$$

$$X(n) = -19/6 \delta(n) + 5/3 (3)^n u(n) + \underline{\underline{3/2 (2)^n u(n)}}$$

Using long division, Find Inverse z -Transform

$$X(z) = \frac{1 + 2z^{-1}}{1 - 2z^{-1} + z^{-2}}$$

If a) $x(n)$ is causal b) $x(n)$ is anticausal.

a) Given $X(z) = \frac{1 + 2z^{-1}}{1 - 2z^{-1} + z^{-2}}$

The sequence $x(n]$ is causal & $X(z)$ is expressed in negative powers of z . $\therefore$ We begin the division with a constant

$$\begin{array}{r}
 1 + 4z^{-1} + 7z^{-2} + 10z^{-3} \\
 \hline
 1 - 2z^{-1} + z^{-2} \overline{) 1 + 2z^{-1}} \\
 \underline{1 - 2z^{-1} + z^{-2}} \phantom{+ 10z^{-3}} \\
 4z^{-1} - z^{-2} \\
 \underline{4z^{-1} - 8z^{-2} + 4z^{-3}} \\
 7z^{-2} - 4z^{-3} \\
 \underline{7z^{-2} - 14z^{-3} + 7z^{-4}} \\
 10z^{-3} - 7z^{-4} \\
 \underline{10z^{-3} - 20z^{-4} + 10z^{-5}} \\
 \phantom{10z^{-3} - 20z^{-4} + 10z^{-5}}
 \end{array}$$

$$X(z) = 1 + 4z^{-1} + 7z^{-2} + 10z^{-3} + 13z^{-4} + \dots$$

$$x(n) = \{ 1, 4, 7, 10, 13, 16, \dots \}$$

5) $X(z)$ is Non-causal. we begin the division with highest negative of

$$\begin{array}{r}
 2z + 5z^2 + 8z^3 + 11z^4 \\
 \hline
 z^{-2} - 2z^{-1} + 1 \quad \left) \begin{array}{r} 2z^{-1} + 1 \\ 2z^{-1} - 4z^0 \\ \hline 5 - 2z \\ 5 - 10z + 5z^2 \\ \hline 8z - 5z^2 \\ 8z - 16z^2 + 8z^3 \\ \hline 11z^2 - 8z^3 \\ 11z^2 - 22z^3 + 11z^4 \\ \hline 14z^3 - 11z^4
 \end{array}
 \end{array}$$

$$X(z) = 2z + 5z^2 + 8z^3 + 11z^4 + 14z^5 + \dots$$

$$X(n) = \{ \dots, 14, 11, 8, 5, 2, 0 \}$$

↑

Qn. Using long division;

$$X(z) = \frac{z+1}{z^2-3z+2}$$

1) If $X(z)$ is causal

2) $X(z)$ is anticausal

Note: * If $x(n)$ is causal and ~~we~~ positive powers of z we begin with highest powers of z .
 * If $x(n)$ is anticausal and +ve powers of z we begin with lowest powers of z .

$$\begin{array}{r}
 2 - 3z + z^2 \overline{) 1 + z} \\
 \underline{z^{-1} + 4z^{-2} + 10z^{-3} + 22z^{-4}} \\
 z + 1 \\
 \underline{z - 3 + 2z^{-1}} \\
 4 - 2z^{-1} \\
 \underline{4 - 12z^{-1} + 8z^{-2}} \\
 10z^{-1} - 8z^{-2} \\
 \underline{10z^{-4} - 30z^{-2} + 20z^{-3}} \\
 22z^{-2} - 20z^{-3}
 \end{array}$$

$$X(z) = \{ 0, 1, 4, 10, 22, \dots \}$$

↑

b) anticausal.

$$\begin{array}{r}
 2 - 3z + z^2 \overline{) 0.5 + 1.25z} \\
 \underline{1 + z} \\
 1 - 1.5z + 0.5z^2 \\
 \underline{2.5z - 0.5z^2} \\
 2.5z - 3.75z^2 + 1.25z^3
 \end{array}$$

$$X(z) = 0.5 + 1.25z + 1.625z^2 + 1.4825z^3 + \dots$$

$$X(n) = \{ \dots, 1.8125, 1.625, 1.25, 0.5 \}$$

↑

1.1.4
13. Find the Inverse z -Transform of

$$X(z) = \log \left(\frac{1}{1 - a\bar{z}^{-1}} \right) \quad \text{ROC: } |z| > |a|$$

$$= -\log(1 - a\bar{z}^{-1})$$

$$= - \left[-a\bar{z}^{-1} - \frac{(a\bar{z}^{-1})^2}{2} - \frac{(a\bar{z}^{-1})^3}{3} - \frac{(a\bar{z}^{-1})^4}{4} - \dots \right]$$

$$= a\bar{z}^{-1} + \frac{a^2 \bar{z}^{-2}}{2} + \frac{a^3 \bar{z}^{-3}}{3} + \frac{a^4 \bar{z}^{-4}}{4} + \dots$$

$$= \sum_{n=1}^{\infty} \frac{a^n \bar{z}^{-n}}{n}$$

$$X(n) = \sum_{n=1}^{\infty} \frac{a^n}{n} \cdot \delta[n]$$

Long division Method (power Series Method)

$$1. \quad \frac{1}{1 - (a\bar{z})} = \sum_{n=0}^{\infty} (a\bar{z})^n = 1 + a\bar{z} + a^2 \bar{z}^2 + a^3 \bar{z}^3 + \dots$$

$$2. \quad \cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots$$

$$3. \quad \sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots$$

$$4. \quad e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

$$5. \quad \log(1 + a\bar{z}) = a\bar{z} - \frac{(a\bar{z})^2}{2} + \frac{(a\bar{z})^3}{3} - \frac{(a\bar{z})^4}{4} + \dots$$

$$\ln(1 + a) = a - \frac{a^2}{2} + \frac{a^3}{3} - \frac{a^4}{4} + \dots$$

$$6. \quad \log(1 - a\bar{z}) = -(a\bar{z}) - \frac{(a\bar{z})^2}{2} - \frac{(a\bar{z})^3}{3} - \frac{(a\bar{z})^4}{4} - \dots$$

$$\ln(1 - a) = -a - \frac{a^2}{2} - \frac{a^3}{3} - \frac{a^4}{4} - \dots$$

→ Transform

Q. Find the Inverse z -Transform of
 ROC: $|z| > |a|$

$$\begin{aligned}
 X(z) &= \log \left(\frac{1}{1 - a\bar{z}^{-1}} \right) \\
 &= -\log (1 - a\bar{z}^{-1}) \\
 &= - \left[-a\bar{z}^{-1} - \frac{(a\bar{z}^{-1})^2}{2} - \frac{(a\bar{z}^{-1})^3}{3} - \frac{(a\bar{z}^{-1})^4}{4} - \dots \right] \\
 &= a\bar{z}^{-1} + \frac{a^2 \bar{z}^{-2}}{2} + \frac{a^3 \bar{z}^{-3}}{3} + \frac{a^4 \bar{z}^{-4}}{4} + \dots \\
 &= \sum_{n=1}^{\infty} \frac{a^n \bar{z}^{-n}}{n}
 \end{aligned}$$

$$\underline{\underline{X(n) = \sum_{n=1}^{\infty} \frac{a^n}{n}}} \quad \text{for } |z| > |a|$$

Solution of Difference Equation using Z-Transform

1) Find the Impulse Response & Step response for the given S/ms.

$$y(n) - \frac{3}{4} y(n-1] + \frac{1}{8} y(n-2) = x(n)$$

$$Y(z) - \frac{3}{4} [\bar{z}^{-1} Y(z) + y(-1)] + \frac{1}{8} [\bar{z}^{-2} Y(z) + \bar{z}^{-1} Y(z) + y(-2)] = X(z)$$
$$= X(z)$$

$$Y(z) - \frac{3}{4} \bar{z}^{-1} Y(z) + \frac{1}{8} \bar{z}^{-2} Y(z) = X(z)$$

Impulse response: $x(n] = \delta(n]$ $\therefore X(z) = 1$

$$\frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{3}{4} \bar{z}^{-1} + \frac{1}{8} \bar{z}^{-2}} = \frac{z^2}{z^2 - \frac{3}{4} z + \frac{1}{8}}$$

$$\frac{Y(z)}{z} = \frac{z}{(z - \frac{1}{2})(z - \frac{1}{4})} = \frac{A}{z - \frac{1}{2}} + \frac{B}{z - \frac{1}{4}}$$

$$A = 2 \quad B = -1$$

$$\frac{Y(z)}{z} = 2 \frac{z}{z - \frac{1}{2}} + -1 \frac{z}{z - \frac{1}{4}}$$

$$y(n) = \underline{\underline{2 \left(\frac{1}{2}\right)^n u(n) - \left(\frac{1}{4}\right)^n u(n)}}$$

Step response:

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$$X(n) = u(n)$$

$$X(z) = \frac{z}{z-1}$$

$$\frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$$

$$\frac{Y(z)}{z} = \left(\frac{z}{z-1} \right) \cdot \left(\frac{z^2}{z^2 - \frac{3}{4}z + \frac{1}{8}} \right)$$

$$\frac{Y(z)}{z} = \frac{z^2}{(z-1)(z-\frac{1}{4})(z-\frac{1}{2})}$$

$$= \frac{A}{z-1} + \frac{B}{z-\frac{1}{2}} + \frac{C}{z-\frac{1}{4}}$$

$$A = \frac{8}{3} \quad B = -2 \quad C = \frac{1}{3}$$

$$\frac{Y(z)}{z} = \frac{\frac{8}{3}}{z-1} - \frac{2}{z-\frac{1}{2}} + \frac{\frac{1}{3}}{z-\frac{1}{4}}$$

$$Y(z) = \frac{8}{3} \cdot \frac{z}{z-1} - 2 \cdot \frac{z}{z-\frac{1}{2}} + \frac{1}{3} \cdot \frac{z}{z-\frac{1}{4}}$$

$$y(n) = \frac{8}{3} u(n) - 2 \cdot \left(\frac{1}{2}\right)^n u(n) + \frac{1}{3} \left(\frac{1}{4}\right)^n u(n)$$

Qn. Find the output $y(n)$ of a linear time invariant discrete time system specified by the eqn:

$$y(n] - \frac{3}{2}y(n-1) + \frac{1}{2}y(n-2) = 2x(n) + \frac{3}{2}x(n-1)$$

when initial conditions are $y(-1) = 0$

$$y(-2) = 1 \quad x(-1) = 0$$

and the input $x(n] = \left(\frac{1}{4}\right)^n u(n]$ $X(z) = \frac{1}{1 - \frac{1}{4}z^{-1}}$

Ans:
$$Y(z) - \frac{3}{2}[z^{-1}Y(z) + y(-1)] + \frac{1}{2}[z^{-2}Y(z) + z^{-1}y(-1) + y(-2)] = 2X(z) + \frac{3}{2}[z^{-1}X(z) + x(-1)]$$

$$Y(z) - \frac{3}{2}[z^{-1}Y(z)] + \frac{1}{2}[z^{-2}Y(z) + 1] =$$

$$2X(z) + \frac{3}{2}z^{-1}X(z)$$

$$Y(z) \left[1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2} \right] = \frac{-1}{2} + \left(2 + \frac{3}{2}z^{-1} \right) X(z)$$

$$Y(z) = \frac{-\frac{1}{2} + \left(\frac{1}{1 - \frac{1}{4}z^{-1}} \right) \left(2 + \frac{3}{2}z^{-1} \right)}{\left(1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2} \right)}$$

$$= \frac{-\frac{1}{2} \left(1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2} \right) + \frac{\left(2 + \frac{3}{2}z^{-1} \right)}{\left(1 - \frac{1}{4}z^{-1} \right) \left(1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2} \right)}}{\left(1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2} \right)}$$

$$= \frac{-z^2}{2(z^2 - \frac{3}{2}z + \frac{1}{2})} + \frac{z^2(2z + \frac{3}{2})}{\left(z - \frac{1}{4} \right) \left(z^2 - \frac{3}{2}z + \frac{1}{2} \right)}$$

$$= Y_1(z) + Y_2(z)$$

$$Y_1(z) = \frac{-z^2}{2(z-1)(z-\frac{1}{2})}$$

$$\frac{Y_1(z)}{z} = \frac{-z}{2(z-1)(z-\frac{1}{2})}$$

$$= \frac{A}{z-1} + \frac{B}{z-\frac{1}{2}}$$

$$A = \underline{\underline{-1}}$$

$$B = \underline{\underline{\frac{1}{2}}}$$

$$Y_1(z) = \frac{-z}{z-1} + \frac{\frac{1}{2}z}{z-\frac{1}{2}} = -u(n) + 0.5\left(\frac{1}{2}\right)^n u(n)$$

$$Y_2(z) = \frac{z^2(2z+\frac{3}{2})}{(z-\frac{1}{4})(z^2-\frac{3}{2}z+\frac{1}{2})}$$

$$\frac{Y_2(z)}{z} = \frac{z(2z+\frac{3}{2})}{(z-\frac{1}{4})(z-1)(z-\frac{1}{2})}$$

$$= \frac{A_1}{z-\frac{1}{4}} + \frac{B_1}{z-1} + \frac{C_1}{z-\frac{1}{2}}$$

$$A_1 = \underline{\underline{8/3}} \quad B_1 = \underline{\underline{28/3}} \quad C_1 = \underline{\underline{-10}}$$

$$Y_2(z) = 8/3 \cdot \frac{z}{z-\frac{1}{4}} + 28/3 \cdot \frac{z}{z-1} + -10 \cdot \frac{z}{z-\frac{1}{2}}$$

$$Y_2(z) = 8/3 \left(\frac{1}{4}\right)^n u(n) + \frac{28}{3} \cdot \underline{\underline{u(n)}} - 10 \left(\frac{1}{2}\right)^n u(n)$$

$$y(n) = \frac{28}{3} u(n) - \frac{10}{2} \left(\frac{1}{2}\right)^n u(n) + \frac{8}{3} \left(\frac{1}{4}\right)^n u(n)$$

Fourier Representation of Discrete Time Signals

Discrete Time Fourier Series - Properties

Frequency Response of Simple DT Systems

Basics of Non-Linear Systems - Types & Properties

Introduction to Random Signals and Processes

Frequency Domain Representation of Discrete time signals

- Discrete time signals are analysed with the help of Fourier Transform and Fourier series, then they are known as DTFT & DTFS

Discrete time Fourier series (DTFS)

$$X(n) = \sum_{k=0}^{N-1} C_k \cdot e^{j 2\pi k n / N} \quad k=0, 1, \dots, N-1$$

Fundamental period $N = \frac{2\pi}{\omega_0}$ $\left\{ \omega = \frac{2\pi}{T} \right.$

$$C_k = \frac{1}{N} \sum_{n=0}^{N-1} X(n) \cdot e^{-j 2\pi k n / N} \quad ; k=0, 1, \dots, N-1$$

C_k = DTFS coefficients.

$$\rightarrow \text{or } X(n) = \sum_{k=-(N/2-1)}^{N/2} C_k \cdot e^{j 2\pi k n / N} = \sum_{k=-(N/2-1)}^{N/2} C_k \cdot e^{j \omega_0 k n}$$

Qn. Find the Spectral Coefficient for

$$x(n) = \cos(\pi/4 n)$$

$$N = \frac{2\pi}{\omega_0} = \frac{2\pi}{(\frac{\pi}{4})} = 8$$

$$\omega_0 = \frac{2\pi}{N}$$

$$\cos(\pi/4 n) = \frac{e^{j(\pi/4)n} + e^{-j(\pi/4)n}}{2}$$

— (1)

DFTS

$$x(n) = \sum_{k=-3}^4 C_k e^{j\omega_0 kn}$$

$$x(n) = \sum_{k=-3}^4 C_k e^{j \frac{2\pi}{N} kn}$$

$$= \sum_{k=-3}^4 C_k e^{j \frac{\pi}{4} kn}$$

$$= \frac{1}{2} [e^{j\pi n/4} + e^{-j\pi n/4}] = C_{-3} e^{-j\frac{3\pi n}{4}} + C_{-2} e^{-j\frac{2\pi n}{4}} + C_{-1} e^{-j\frac{\pi n}{4}} + C_0 + C_1 e^{j\frac{\pi n}{4}} + C_2 e^{j\frac{2\pi n}{4}} + C_3 e^{j\frac{3\pi n}{4}} + C_4 e^{j\frac{4\pi n}{4}}$$

$$C_1 = C_{-1} = \underline{\underline{1/2}}$$

— (2)

$$C_{-3} = C_{-2} = C_0 = C_2 = C_3 = C_4 = \underline{\underline{0}}$$

Find the DTFS representation for
 $x(n) = 5 + \sin(n\pi/2) + \cos(n\pi/4)$ and
 sketch amplitude & phase spectra.

$$N_1 = \frac{2\pi}{\pi/2} = 4$$

$$N = \text{LCM}(4, 8) = \underline{8}$$

$$N_2 = \frac{2\pi}{\pi/4} = \underline{8}$$

$$x(n) = 5 + \left[\frac{e^{jn\pi/2} - e^{-jn\pi/2}}{2j} \right] + \left(\frac{e^{jn\pi/4} + e^{-jn\pi/4}}{2} \right)$$

$$= 5 + \frac{e^{jn\pi/2}}{2j} - \frac{e^{-jn\pi/2}}{2j} + \frac{e^{jn\pi/4}}{2} + \frac{e^{-jn\pi/4}}{2}$$

$$x(n) = \sum_{K=-3}^4 C_K \cdot e^{j\frac{2\pi}{N}Kn} = \sum_{K=-3}^4 C_K \cdot e^{j\frac{\pi}{4}Kn}$$

$$5 + \frac{e^{jn\pi/2}}{2j} - \frac{e^{-jn\pi/2}}{2j} + \frac{e^{jn\pi/4}}{2} + \frac{e^{-jn\pi/4}}{2} = C_3 e^{j\frac{3\pi}{4}n} + C_{-2} e^{-j\frac{2\pi}{4}n} + C_{-1} e^{-j\frac{\pi}{4}n} \\ + C_0 + C_1 e^{j\frac{\pi}{4}n} + C_2 e^{j\frac{2\pi}{4}n} + C_3 e^{j\frac{3\pi}{4}n} + C_4 e^{j\frac{4\pi}{4}n}$$

$$C_0 = 5$$

$$C_{-3} = C_3 = 0$$

$$C_{-2} = -\frac{1}{2}j$$

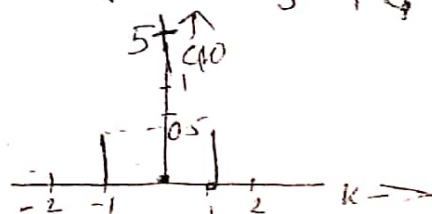
$$C_4 = 0$$

$$C_2 = \frac{1}{2}j$$

$$C_{-1} = \frac{1}{2}$$

$$C_1 = \frac{1}{2}$$

$$|C_K| = \begin{cases} \frac{1}{2} & K=2 \text{ \& } -2 \\ \frac{1}{2} & K=1 \text{ \& } -1 \\ 5 & K=0 \end{cases}$$



Properties of DFS (Discrete time Fourier series)

1. Linearity

$$x_1(n) \xleftrightarrow{\text{DFS}} C_k$$

$$x_2(n) \xleftrightarrow{\text{DFS}} D_k$$

$$\text{DFS}[a_1 x_1(n) + b_1 x_2(n)] \longleftrightarrow a_1 C_k + b_1 D_k$$

2. Time shifting

$$x(n) \xleftrightarrow{\text{DFS}} C_k$$

$$x(n - n_0) \xleftrightarrow{\text{DFS}} C_k e^{-jk \left(\frac{2\pi}{N}\right) n_0}$$

3. Frequency shifting

$$x(n) \xleftrightarrow{\text{DFS}} C_k$$

$$e^{j\omega_0 n} x(n) \xleftrightarrow{\text{DFS}} C_{k-M}$$

4. Time Reversal

$$x(-n) \xleftrightarrow{\text{DFS}} C_{-k}$$

5. Periodic Convolution

$$\sum_{r=-\infty}^{\infty} x(r) y(n-r) = N C_k D_k$$

6. Parseval's Theorem

$$\frac{1}{N} \sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{k=-\infty}^{\infty} |C_k|^2$$

Q. Sketch the sequence, $x(n) = \sum_{k=-\infty}^{\infty} \delta(n-3k)$ & find its discrete Fourier series.

$$x(n) = \sum_{k=-\infty}^{\infty} \delta(n-3k)$$

$$N = 6$$

$$= \dots + \delta(n+6) + \delta(n+3) + \delta(n) + \delta(n-3) + \delta(n-6) + \dots$$

$$C_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) \cdot e^{-j\omega_k n}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} x(n) \cdot e^{-j\frac{2\pi}{N}kn}$$

$$= \frac{1}{3} \sum_{n=0}^2 x(n) \cdot e^{-j\frac{2\pi}{3}kn}$$

$$C_k = \frac{1}{3} \left[x(0) + x(1) \cdot e^{-j\frac{2\pi k}{3}} + x(2) \cdot e^{-j\frac{4\pi k}{3}} \right]$$

$$C_k = \frac{1}{3} x(0) = \frac{1}{3} \text{ for all 'k'}$$

==

Discrete - Time Fourier Transform (DTFT)

$$\rightarrow X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n}$$

$$\text{DTFT of } x(n) = X(e^{j\omega}) \Rightarrow x(n) \xleftrightarrow{\text{DTFT}} X(e^{j\omega})$$

→ Inverse DTFT

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) \cdot e^{j\omega n} \cdot d\omega$$

→ Major difference between CTFT & DTFT

CTFT

1. Frequency spectrum is not periodic

2. Spectrum in the range of $-\infty$ to $+\infty$.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t} \cdot d\omega$$

DTFT

1. It is periodic with period 2π

2. Synthesis Equation (IDFT) involves integration over a frequency interval produces distinct complex exponentials.

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) \cdot e^{j\omega n} \cdot d\omega$$

Properties of Discrete time Fourier Transform

1. Periodicity

It is periodic with period 2π

$$\text{Let } X(\omega) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n}$$

$$X(\omega + 2\pi k) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j(\omega + 2\pi k)n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n} \cdot \underbrace{e^{-j2\pi kn}}_1$$

$$X(e^{j(\omega + 2\pi k)}) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n} = X(\omega)$$

2. Linearity

$$x_1(n) \xleftrightarrow{\text{DTFT}} X_1(e^{j\omega})$$

$$x_2(n) \xleftrightarrow{\text{DTFT}} X_2(e^{j\omega})$$

$$a x_1(n) + b x_2(n) \xleftrightarrow{\text{DTFT}} a X_1(e^{j\omega}) + b X_2(e^{j\omega})$$

3. Time Shifting property

$$x(n) \xleftrightarrow{\text{DTFT}} X(e^{j\omega})$$

$$x(n - n_0) \xleftrightarrow{\text{DTFT}} e^{-j\omega n_0} X(e^{j\omega})$$

~~Modulation property~~ Frequency shifting property

$$X(n) \xleftrightarrow{\text{DTFT}} X(e^{j\omega})$$

$$e^{j\omega_0 n} X(n) \xleftrightarrow{\text{DTFT}} X[e^{j(\omega - \omega_0)}]$$

5. Frequency Differentiation

$$\text{If } X(n) \xleftrightarrow{\text{DTFT}} X(e^{j\omega})$$

$$n X(n) \xleftrightarrow{\text{DTFT}} j \frac{d}{d\omega} X(e^{j\omega})$$

6. Time Reversal

$$X(n) \xleftrightarrow{\text{DTFT}} X(e^{j\omega})$$

$$X(-n) \xleftrightarrow{\text{DTFT}} X(e^{-j\omega})$$

7. Convolution property

$$\text{If } X(n) \xleftrightarrow{\text{DTFT}} X(e^{j\omega})$$

$$Y(n) \xleftrightarrow{\text{DTFT}} Y(e^{j\omega})$$

$$X(n) * Y(n) \xleftrightarrow{\text{DTFT}} X(e^{j\omega}) \cdot Y(e^{j\omega})$$

8. Parseval's Theorem

$$X(n) \xleftrightarrow{\text{DTFT}} X(e^{j\omega})$$

$$E = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

9 Modulation

$$x(n) \cos \omega_0 n \xleftrightarrow{\text{DTFT}} \frac{1}{2} X(e^{j\omega+\omega_0}) + \frac{1}{2} X(e^{j\omega-\omega_0})$$

Scaling

$$X(pn) \xleftrightarrow{\text{DTFT}} \frac{1}{p} X\left(\frac{\omega}{p}\right)$$

Parseval's Theorem or Rayleigh's Theorem

→ Total Energy of a discrete-time signal $x(n)$ may be determined by the knowledge of its DTFT $X(e^{j\omega})$.

$$\text{If } x(n) \xleftrightarrow{\text{DTFT}} X(e^{j\omega})$$

$$E = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

Proof: $E = \sum_{n=-\infty}^{\infty} |x(n)|^2$

$$E = \sum_{n=-\infty}^{\infty} x(n) x^*(n)$$

$$E = \sum_{n=-\infty}^{\infty} x(n) \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(e^{j\omega}) \cdot e^{j\omega n} d\omega \right]$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(e^{j\omega}) \cdot \left[\sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n} \right] d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(e^{j\omega}) X(e^{j\omega}) d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$\text{Since } |x(n)|^2 = x(n) x^*(n)$$

Inverse DTFT
 $x^*(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(e^{j\omega}) e^{j\omega n} d\omega$

Find Fourier Transform of following

$x(n) = \delta(n)$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \delta(n) \cdot e^{-j\omega n}$$

b) $x(n) = u(n)$

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} u(n) \cdot e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} 1 \cdot e^{-j\omega n} = 1 + e^{-j\omega} + e^{-2j\omega} + \dots + \infty$$

$$= \frac{1}{1 - e^{-j\omega}}$$

$| 1 + e^{-j\omega} + e^{-2j\omega} + \dots = \frac{1}{1 - e^{-j\omega}} |$

c) $x(n) = \delta(n-k)$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta(n-k) \cdot e^{-j\omega n}$$

$$\left| \begin{array}{l} \delta(n-k) = 1 \text{ for } n=k \\ \quad \quad \quad = 0 \text{ for } n \neq k \end{array} \right.$$

$$= e^{-j\omega k}$$

d) $x(n) = u(n-k)$

$$= \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n} = \sum_{n=k}^{\infty} 1 \cdot e^{-j\omega n}$$

$$= e^{-j\omega k} + e^{-j\omega(k+1)} + \dots$$

$$= e^{-j\omega k} [1 + e^{-j\omega} + e^{-2j\omega} + \dots]$$

$$= e^{-j\omega k} \cdot \frac{1}{1 - e^{-j\omega}}$$

$$x(n) = x^n u(n)$$

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=0}^{\infty} a^n \cdot e^{-j\omega n} = 1 + a e^{-j\omega} + a^2 e^{-2j\omega} + \dots \\ &= \frac{1}{1 - a e^{-j\omega}} \quad |a| < 1 \end{aligned}$$

$$\begin{aligned} x(n) &= \delta(n+2) - \delta(n-2) \\ &= \sum_{n=-\infty}^{\infty} \delta(n+2) \cdot e^{j\omega n} - \sum_{n=-\infty}^{\infty} \delta(n-2) \cdot e^{j\omega n} \quad \left| \begin{array}{l} \delta(n+2)=1, n=-2 \\ \delta(n-2)=1, n=2 \end{array} \right. \\ &= e^{2j\omega} - e^{-2j\omega} = \underline{\underline{2j \sin 2\omega}} \end{aligned}$$

$$x(n) = \{1, -1, 2, 2\}$$

$$x(0) = 1 \quad x(1) = -1 \quad x(2) = 2 \quad x(3) = 2$$

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n} = \sum_{n=0}^3 x(n) e^{-j\omega n} \\ &= x(0) + x(1) \cdot e^{-j\omega} + x(2) \cdot e^{-2j\omega} + x(3) \cdot e^{-3j\omega} \\ &= \underline{\underline{1 - e^{-j\omega} + 2e^{-2j\omega} + 2e^{-3j\omega}}} \end{aligned}$$

$$X(n) = 2^n u(n).$$

Not absolutely summable
FT does not exist

$$X(n) = (0.5)^n u(n) + 2^n u(-n-1)$$

$$= \sum_{n=-\infty}^{\infty} (0.5)^n u(n) e^{-j\omega n} + \sum_{n=-\infty}^{\infty} 2^n u(-n-1) e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (0.5)^n e^{-j\omega n} + \sum_{n=-\infty}^{-1} 2^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (0.5 e^{-j\omega})^n + \sum_{n=1}^{\infty} 2^{-n} e^{j\omega n}$$

$$= \sum_{n=0}^{\infty} (0.5 e^{-j\omega})^n + \sum_{n=1}^{\infty} (2^{-1} e^{j\omega})^n$$

$$= \frac{1}{1 - 0.5 e^{-j\omega}} + \left(\frac{1}{1 - 0.5 e^{j\omega}} - 1 \right) \Big|_Z$$

$$= \frac{1 - 0.5 e^{j\omega} + 1 - 0.5 e^{-j\omega}}{1 - 0.5 e^{j\omega} - 0.5 e^{-j\omega} + 0.25}$$

$$= \frac{1}{1 - 0.5 e^{-j\omega}} + \frac{0.5 e^{j\omega}}{1 - 0.5 e^{j\omega}}$$

$$= \frac{1 - 0.15 e^{j\omega} + 0.5 e^{j\omega} - 0.25}{1 - 0.5 [e^{j\omega} + e^{-j\omega}] + 0.25}$$

$$0.75$$

$$1.25 - \cos \omega$$

3. Find the Inverse Fourier Transform of the following

$$X(e^{j\omega}) = \begin{cases} 1 & \pi/4 \leq \omega \leq 3\pi/4 \\ 0 & 3\pi/4 \leq \omega \leq \pi, \quad 0 \leq \omega \leq \pi/4 \end{cases}$$

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) \cdot e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \left[\int_{-\pi/4}^{-\pi/4} e^{j\omega n} d\omega + \int_{\pi/4}^{3\pi/4} e^{j\omega n} d\omega \right]$$

$$= \frac{1}{2\pi} \left[\left[\frac{e^{j\omega n}}{jn} \right]_{-\pi/4}^{-\pi/4} + \left[\frac{e^{j\omega n}}{jn} \right]_{\pi/4}^{3\pi/4} \right]$$

$$= \frac{1}{2\pi j n} \left[e^{-j\pi/4 n} - e^{-3j\pi/4 n} + e^{3j\pi/4 n} - e^{j\pi/4 n} \right]$$

$$= \frac{1}{2\pi j n} \left[2j \left[\sin \frac{3\pi}{4} n \right] - 2j \left[\sin \frac{\pi}{4} n \right] \right]$$

$$= \frac{1}{\pi n} \left[\sin \frac{3\pi}{4} n - \sin \frac{\pi}{4} n \right]$$

Find the Inverse Fourier Transform of

$$X(e^{j\omega}) = 1 + 2e^{-j\omega} + 2e^{-2j\omega} + 3e^{-3j\omega}$$

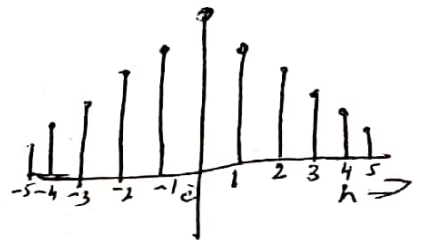
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=0}^3 x(n) \cdot e^{-j\omega n} = x(0) + x(1)e^{-j\omega} + x(2)e^{-2j\omega} + x(3)e^{-3j\omega}$$

$$x(0) = 1 \quad x(1) = 2 \quad x(2) = 2 \quad x(3) = 3$$

$$x(n) = \{1 \quad 2 \quad 2 \quad 3\}$$

Q2. Find FT of $x(n) = a^{|n|}$



$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^{|n|} \cdot e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{-1} a^{|n|} e^{-j\omega n} + \sum_{n=0}^{\infty} a^{|n|} e^{-j\omega n}$$

$$= \sum_{n=-1}^{-\infty} a^{-n} e^{-j\omega n} + \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$= \sum_{n=-1}^{-\infty} (a e^{j\omega})^{-n} + \sum_{n=0}^{\infty} (a e^{-j\omega})^n$$

$$= \sum_{n=1}^{\infty} (a e^{j\omega})^n + \sum_{n=0}^{\infty} (a e^{-j\omega})^n$$

$$= \frac{a e^{j\omega}}{1 - a e^{j\omega}} + \frac{1}{1 - a e^{-j\omega}} = \frac{1 - a^2}{1 - 2a \cos \omega + a^2}$$

Q. Find FT of $x(n) = \sin\left(\frac{\pi n}{2}\right) u(n)$.

$$\begin{aligned}
 X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n} \\
 &= \sum_{n=0}^{\infty} \left[\frac{e^{j\pi/2} - e^{-j\pi/2}}{2j} \right] \cdot e^{-j\omega n} \\
 &= \frac{1}{2j} \sum_{n=0}^{\infty} \left[e^{j(\pi/2 - \omega)n} - e^{-j(\pi/2 + \omega)n} \right] \\
 &= \frac{1}{2j} \left[\frac{1}{1 - e^{j\pi/2} \cdot e^{-j\omega}} - \frac{1}{1 - e^{-j\pi/2} \cdot e^{-j\omega}} \right] \\
 &= \frac{1}{2j} \left[\frac{e^{-j\omega} [e^{j\pi/2} - e^{-j\pi/2}]}{1 - 2\cos(\pi/2) \cdot e^{-j\omega} + e^{-2j\omega}} \right] \\
 &= \frac{1}{2j} \left[\frac{e^{-j\omega} \cdot \sin(\pi/2) \cdot 2j}{1 + e^{-2j\omega}} \right] \\
 &= \underline{\underline{\frac{e^{-j\omega}}{1 + e^{-2j\omega}}}}
 \end{aligned}$$

Q. $X(e^{j\omega}) = e^{-j\omega} \quad -\pi \leq \omega \leq \pi$

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) \cdot e^{j\omega n} d\omega$$

$$\begin{aligned}
 x(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-j\omega} \cdot e^{j\omega n} \cdot d\omega \\
 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(n-1)\omega} \cdot d\omega \\
 &= \left[\frac{e^{j(n-1)\omega}}{2\pi j(n-1)} \right]_{-\pi}^{\pi} = \frac{e^{j(n-1)\pi} - e^{-j(n-1)\pi}}{2\pi j(n-1)} \\
 &= \frac{\sin \pi(n-1) \times 2j}{2j \pi(n-1)} = \underline{\underline{\frac{\sin \pi(n-1)}{\pi(n-1)}}}
 \end{aligned}$$

Qn. Obtain Impulse response of S/M;

$$\begin{aligned}
 H(e^{j\omega}) &= 1 \quad \text{for } |\omega| \leq \omega_c \\
 &= 0 \quad \omega_c \leq |\omega| \leq \pi
 \end{aligned}$$

$$\begin{aligned}
 h(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) \cdot e^{j\omega n} \cdot d\omega \\
 &= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} 1 \cdot e^{j\omega n} \cdot d\omega = \frac{1}{2\pi} \left[\frac{e^{j\omega n}}{jn} \right]_{-\omega_c}^{\omega_c} \\
 &= \frac{1}{2\pi} \left[\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{jn} \right] \\
 &= \frac{1}{\pi n} \left[\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j} \right] = \underline{\underline{\frac{\sin \omega_c n}{\pi n}}}
 \end{aligned}$$

Qn. Find FT of $x(n) = \left(\frac{1}{2}\right)^{n-1} u(n-1)$

$$\begin{aligned}
 X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n} \\
 &= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} e^{-j\omega n} \\
 &= e^{-j\omega} + \frac{1}{2} \cdot e^{-2j\omega} + \left(\frac{1}{2}\right)^2 \cdot e^{-3j\omega} + \dots \\
 &= e^{-j\omega} \left[1 + \frac{1}{2} \cdot e^{-j\omega} + \left(\frac{1}{2}\right)^2 e^{-2j\omega} + \dots \right] \\
 &= e^{-j\omega} \frac{1}{\left[1 - \frac{1}{2} e^{-j\omega}\right]} = \frac{e^{-j\omega}}{1 - \frac{1}{2} e^{-j\omega}}
 \end{aligned}$$

Qn. Consider $x(n) = A$ for $|n| \leq N$
 $= 0$ for $|n| > N$

Find $X(e^{j\omega})$

$$\begin{aligned}
 X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n} = \sum_{n=-N}^{-1} A \cdot e^{-j\omega n} + \sum_{n=0}^N A e^{-j\omega n} \\
 &= \sum_{n=1}^N A \cdot e^{j\omega n} + \sum_{n=0}^N A e^{-j\omega n} \\
 &= A \left[e^{j\omega} + e^{2j\omega} + e^{3j\omega} + \dots + e^{j\omega N} \right] + A \left[1 + e^{-j\omega} + e^{-2j\omega} + e^{-3j\omega} + \dots + e^{-j\omega N} \right]
 \end{aligned}$$

$$= A e^{j\omega} \left[1 + e^{j\omega} + e^{2j\omega} + \dots + e^{j\omega(N-1)} \right] \\ + A \left[1 + e^{-j\omega} + e^{-2j\omega} + e^{-3j\omega} + \dots + e^{-j\omega N} \right]$$

$$= \sum_{n=1}^N A \cdot e^{j\omega n} + \sum_{n=0}^N A \cdot e^{-j\omega n} \\ = A \left[\left(\frac{1 - e^{+j\omega(N+1)}}{1 - e^{j\omega}} - 1 \right) + \frac{1 - e^{-j\omega(N+1)}}{1 - e^{-j\omega}} \right]$$

$$= A \left[\frac{1 - e^{j\omega N} \cdot e^{j\omega} - 1 + e^{j\omega}}{1 - e^{j\omega}} + \frac{1 - e^{-j\omega(N+1)}}{1 - e^{-j\omega}} \right] \\ = A e^{j\omega} \left[\frac{1 - e^{j\omega N}}{1 - e^{j\omega}} \right] + A \left[\frac{1 - e^{-j\omega(N+1)}}{1 - e^{-j\omega}} \right]$$

Compute DTFT of a sequence

$y(n) = (n-1)^2 x(n)$. Use DTFT properties

$$F[n x(n)] = j \frac{d}{d\omega} X(e^{j\omega})$$

$$F[n^2 x(n)] = \left(j \frac{d}{d\omega}\right)^2 X(e^{j\omega}) = -\frac{d^2}{d\omega^2} X(e^{j\omega})$$

$$\begin{aligned} F[(n-1)^2 x(n)] &= F[n^2 x(n)] - 2F[n x(n)] + F[x(n)] \\ &= -\frac{d^2}{d\omega^2} X(e^{j\omega}) - 2j \frac{d}{d\omega} X(e^{j\omega}) + X(e^{j\omega}) \end{aligned}$$

2n) Find the convolution of signals using FT.

$$x_1(n) = \left(\frac{1}{2}\right)^n u(n) \quad x_2(n) = \left(\frac{1}{3}\right)^n u(n)$$

$$x_1(n) * x_2(n) \xleftrightarrow{\text{DFT}} X_1(e^{j\omega}) \cdot X_2(e^{j\omega})$$

$$X_1(e^{j\omega}) = \frac{1}{1 - \frac{1}{2} e^{-j\omega}} =$$

$$X_2(e^{j\omega}) = \frac{1}{1 - \frac{1}{3} e^{-j\omega}}$$

$$\begin{aligned} Y(e^{j\omega}) &= X_1(e^{j\omega}) \cdot X_2(e^{j\omega}) = \frac{1}{1 - \frac{1}{2} e^{-j\omega}} \cdot \frac{1}{1 - \frac{1}{3} e^{-j\omega}} \\ &= \frac{e^{j\omega}}{(e^{j\omega} - \frac{1}{2})(e^{j\omega} - \frac{1}{3})} = \frac{A}{e^{j\omega} - \frac{1}{2}} + \frac{B}{e^{j\omega} - \frac{1}{3}} \end{aligned}$$

$$\frac{Y(e^{j\omega})}{e^{j\omega}} = \frac{A}{e^{j\omega} - \frac{1}{2}} + \frac{B}{e^{j\omega} - \frac{1}{3}} = \frac{e^{j\omega}}{(e^{j\omega} - \frac{1}{2})(e^{j\omega} - \frac{1}{3})}$$

$$A[e^{j\omega} - \frac{1}{3}] + B[e^{j\omega} - \frac{1}{2}] = e^{j\omega}$$

$$\begin{aligned} A + B &= 1 \\ -\frac{1}{3}A - \frac{1}{2}B &= 0 \end{aligned}$$

$$\begin{cases} A = 3 \\ B = -2 \end{cases}$$

$$\frac{Y(e^{j\omega})}{e^{j\omega}} = \frac{3}{(e^{j\omega} - \frac{1}{2})} - \frac{2}{e^{j\omega} - \frac{1}{3}}$$

$$Y(e^{j\omega}) = \frac{3e^{j\omega}}{(e^{j\omega} - \frac{1}{2})} - 2 \frac{e^{j\omega}}{(e^{j\omega} - \frac{1}{3})}$$

$$= 3 \frac{1}{1 - \frac{1}{2}e^{-j\omega}} - 2 \frac{1}{1 - \frac{1}{3}e^{-j\omega}}$$

$$\underline{\underline{y(n) = 3 \left(\frac{1}{2}\right)^n u(n) - 2 \left(\frac{1}{3}\right)^n u(n)}}$$

Find the Frequency Response of 1st order system described by difference eqn:

$$y(n) = a y(n-1) + x(n] \text{ or plot Mag. \& phase response } h(n) = a^n u[n]$$

FT on both sides

$$Y e^{j\omega} = a e^{-j\omega} Y e^{j\omega} + X e^{j\omega}$$

$$Y [e^{j\omega}] [1 - a e^{-j\omega}] = X (e^{j\omega})$$

$$\frac{Y(e^{j\omega})}{X(e^{j\omega})} = \overset{\text{Frequency Response}}{H(e^{j\omega})} = \frac{1}{1 - a e^{-j\omega}}$$

$$= \frac{1}{1 - a [\cos \omega + j \sin \omega]}$$

$$= \frac{1}{1 - a \cos \omega + j a \sin \omega}$$

$$H(e^{j\omega}) = \frac{1}{(1 - a \cos \omega) + j (a \sin \omega)}$$

$$|H(e^{j\omega})| = \frac{1}{[(1 - a \cos \omega)^2 + a^2 \sin^2 \omega]^{1/2}}$$

$$= \frac{1}{(1 + a^2 - 2a \cos \omega)^{1/2}}$$

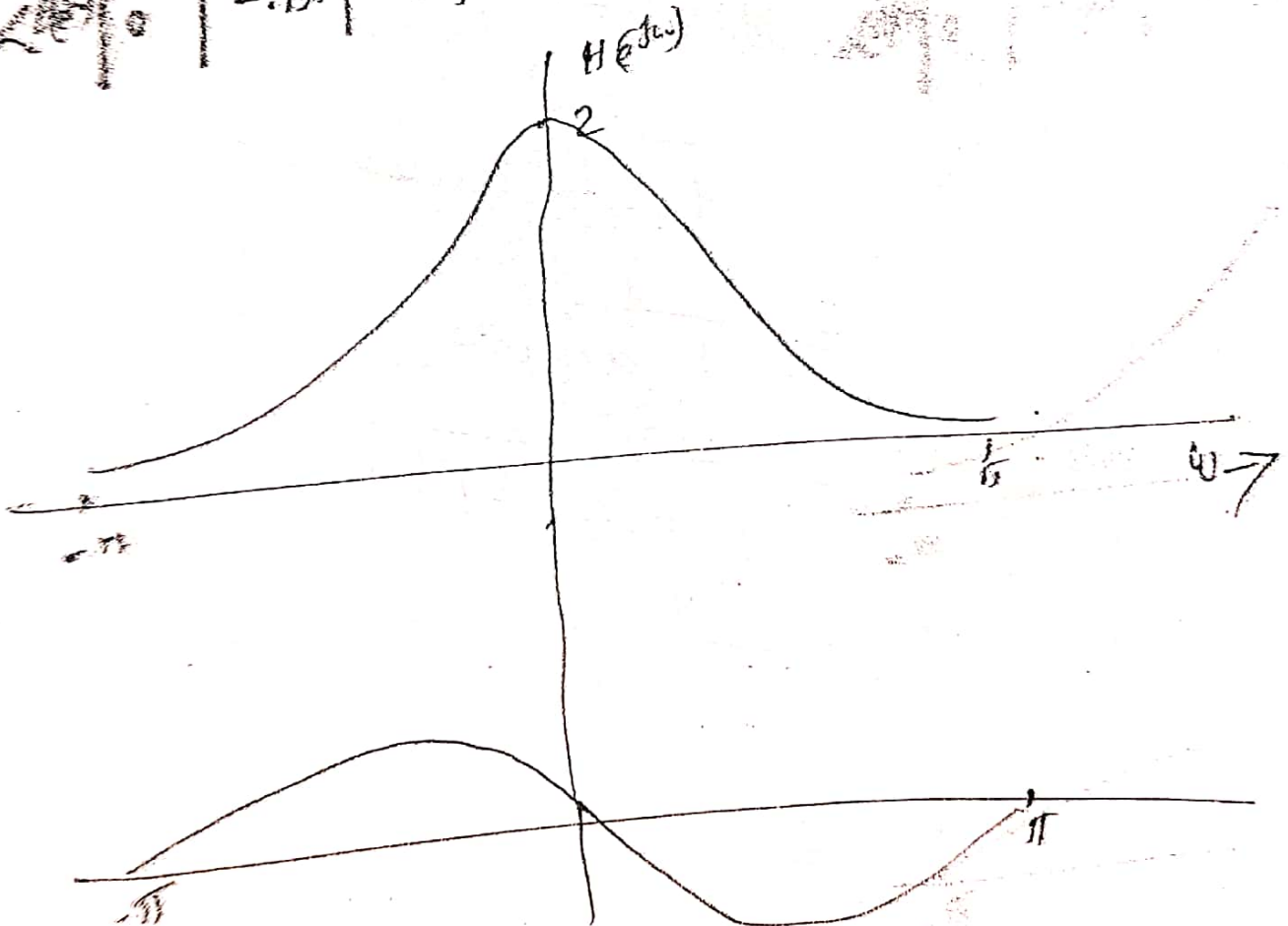
$$\angle H(e^{j\omega}) = -\tan^{-1} \left(\frac{a \sin \omega}{1 - a \cos \omega} \right)$$

Mag & phase Response $q = 0.5$

$$|H(j\omega)| = \frac{1}{(1.25 - \cos\omega)^{1/2}}$$

$$\angle H(j\omega) = -\tan^{-1} \left(\frac{0.5 \sin\omega}{1 - 0.5 \cos\omega} \right)$$

ω	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$ H(j\omega) $	2	1.61	1.36	1.15	0.8	0.75	0.71	0.68	0.66
$\angle H(j\omega)$	0	-0.13π	-0.21π	-0.36π	-0.47π	-0.66π	-0.81π	-0.95π	$-\pi$



Qa: Consider a s/n whose i/p $x(n)$ & o/p $y(n)$ are given by:

$$y(n) = \frac{1}{6} y(n-1) - \frac{1}{6} y(n-2) + x(n)$$

- a) Determine Frequency Response $H(e^{j\omega})$ for the s/n
- b) Determine Impulse Response $h(n)$ for the s/n

$$y(n) = \frac{1}{6} y(n-1) - \frac{1}{6} y(n-2) + x(n)$$

$$Y(e^{j\omega}) - \frac{1}{6} e^{-j\omega} Y(e^{j\omega}) - \frac{1}{6} e^{-2j\omega} Y(e^{j\omega}) = X(e^{j\omega})$$

$$Y(e^{j\omega}) \left[1 - \frac{1}{6} e^{-j\omega} - \frac{1}{6} e^{-2j\omega} \right] = X(e^{j\omega})$$

$$\frac{Y(e^{j\omega})}{X(e^{j\omega})} = H(e^{j\omega}) = \frac{1}{1 - \frac{1}{6} e^{-j\omega} - \frac{1}{6} e^{-2j\omega}} = \text{Frequency Resp.}$$

$$= \frac{e^{2j\omega}}{e^{2j\omega} - \frac{1}{6} e^{j\omega} - \frac{1}{6}}$$

$$\frac{H(e^{j\omega})}{e^{j\omega}} = \frac{e^{j\omega}}{\left(e^{j\omega} - \frac{1}{2}\right)\left(e^{j\omega} + \frac{1}{3}\right)}$$

$$= \frac{A}{\left(e^{j\omega} - \frac{1}{2}\right)} + \frac{B}{\left(e^{j\omega} + \frac{1}{3}\right)} = \frac{3/5}{e^{j\omega} - \frac{1}{2}} + \frac{2/5}{\left(e^{j\omega} + \frac{1}{3}\right)}$$

$$e^{j\omega} = A \left(e^{j\omega} + \frac{1}{3}\right) + B \left(e^{j\omega} - \frac{1}{2}\right)$$

$$1 = A + B$$

$$0 = \frac{1}{3} A - \frac{1}{2} B$$

$$\left. \begin{array}{l} 1 = A + B \\ 0 = \frac{1}{3} A - \frac{1}{2} B \end{array} \right\} \begin{array}{l} A = \frac{3}{5} \\ B = \frac{2}{5} \end{array}$$

$$H(\omega) = \frac{3}{5} \cdot \left(\frac{e^{j\omega}}{e^{j\omega} - \frac{1}{2}} \right) + \frac{2}{5} \cdot \left(\frac{e^{j\omega}}{e^{j\omega} + \frac{1}{3}} \right)$$

$$H(\omega) = \frac{3}{5} \cdot \left[\frac{1}{1 - \frac{1}{2} e^{j\omega}} \right] + \frac{2}{5} \cdot \left[\frac{1}{1 + \frac{1}{3} e^{j\omega}} \right]$$

$$\underline{\underline{h(n) = \frac{3}{5} \left(\frac{1}{2} \right)^n u(n) + \frac{2}{5} \left(-\frac{1}{3} \right)^n u(n)}}$$

Ques: Find the Frequency Response

$$y(n) = \frac{1}{2} x(n) + x(n-1) + \frac{1}{2} x(n-2)$$

$$Y(e^{j\omega}) = \frac{1}{2} X(e^{j\omega}) + e^{-j\omega} X(e^{j\omega}) + \frac{1}{2} e^{-2j\omega} X(e^{j\omega})$$

$$\begin{aligned} \underline{\underline{H(\omega) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}}} &= \frac{\frac{1}{2} + e^{-j\omega} + \frac{1}{2} e^{-2j\omega}}{1} = 0.5 + e^{-j\omega} + 0.5 e^{-2j\omega} \\ &= e^{-j\omega} [0.5 e^{j\omega} + 1 + 0.5 e^{-j\omega}] \\ &= e^{-j\omega} [1 + 0.5 \times 2 \cos \omega] \\ &= \underline{\underline{e^{-j\omega} [1 + \cos \omega]}} \end{aligned}$$

$$2) \quad y(n) - \frac{1}{4} y(n-1) - \frac{3}{8} y(n-2) = x(n) + x(n-1)$$

$$Y(e^{j\omega}) - \frac{1}{4} e^{-j\omega} Y(e^{j\omega}) - \frac{3}{8} e^{-2j\omega} Y(e^{j\omega}) = X(e^{j\omega}) + e^{-j\omega} X(e^{j\omega})$$

$$Y(e^{j\omega}) \left[1 - \frac{1}{4} e^{-j\omega} - \frac{3}{8} e^{-2j\omega} \right] = X(e^{j\omega}) [1 + e^{-j\omega}]$$

$$\frac{Y(e^{j\omega})}{X(e^{j\omega})} = H(e^{j\omega}) = \frac{1 + e^{-j\omega}}{1 - \frac{1}{4}e^{-j\omega} - \frac{3}{8}e^{-2j\omega}}$$

Q2.) Consider DTLTI with impulse response $h(n) = (\frac{1}{2})^n u(n)$.
 Use FT to determine the response of the S/M to
 i/p.

$$\frac{Y(e^{j\omega})}{X(e^{j\omega})} = H(e^{j\omega})$$

a) $x(n) = (\frac{3}{4})^n u(n)$

$$Y(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{j\omega}) = \frac{1}{1 - \frac{3}{4}e^{-j\omega}} \cdot \frac{1}{1 - \frac{1}{2}e^{-j\omega}}$$

$$Y(e^{j\omega}) = \left(\frac{e^{j\omega}}{e^{j\omega} - \frac{3}{4}} \right) \left(\frac{e^{j\omega}}{e^{j\omega} - \frac{1}{2}} \right)$$

$$\frac{Y(e^{j\omega})}{(e^{j\omega})} = \frac{e^{j\omega}}{(e^{j\omega} - \frac{3}{4})(e^{j\omega} - \frac{1}{2})} = \frac{A}{e^{j\omega} - \frac{3}{4}} + \frac{B}{e^{j\omega} - \frac{1}{2}}$$

$$e^{j\omega} = A(e^{j\omega} - \frac{1}{2}) + B(e^{j\omega} - \frac{3}{4})$$

$$1 = A + B$$

$$0 = -\frac{1}{2}A - \frac{3}{4}B$$

$$\begin{cases} A = 3 \\ B = -2 \end{cases}$$

$$\frac{Y(e^{j\omega})}{(e^{j\omega})} = \frac{3}{1 - \frac{3}{4}e^{-j\omega}} - \frac{2}{1 - \frac{1}{2}e^{-j\omega}}$$

$$Y(e^{j\omega}) = 3 \cdot \frac{1}{1 - \frac{3}{4}e^{-j\omega}} - 2 \cdot \frac{1}{1 - \frac{1}{2}e^{-j\omega}}$$

$$Y(n) = 3 \left(\frac{3}{4} \right)^n u(n) - 2 \left(\frac{1}{2} \right)^n u(n)$$

Random signal:

- If the input signal involves certain amount of uncertainty and unpredictability
- Precise value of Random signal cannot be predicted ahead of time.

Eg: Speech and Music signals

output voltage of a solar detector.

→ Statistical measures are used to characterize a Random signal.

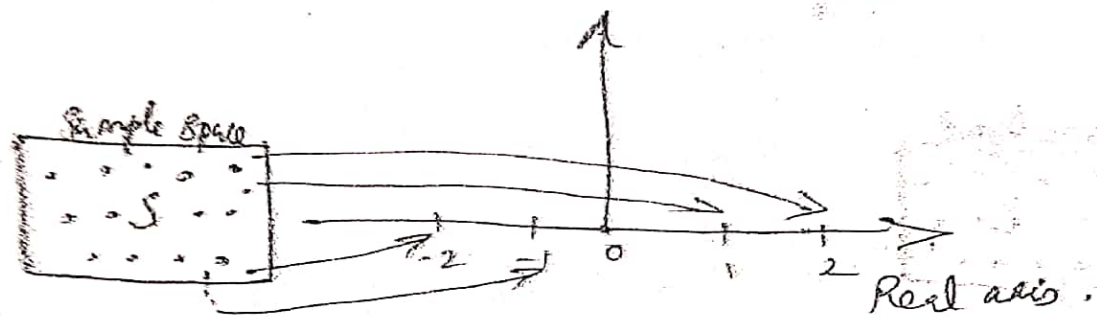
- Complete statistical characterization of Random signal is provided by Probability density & distribution function.

- Random signals have infinite Energy. Hence Transform & z -~~do~~ transform does not exist.

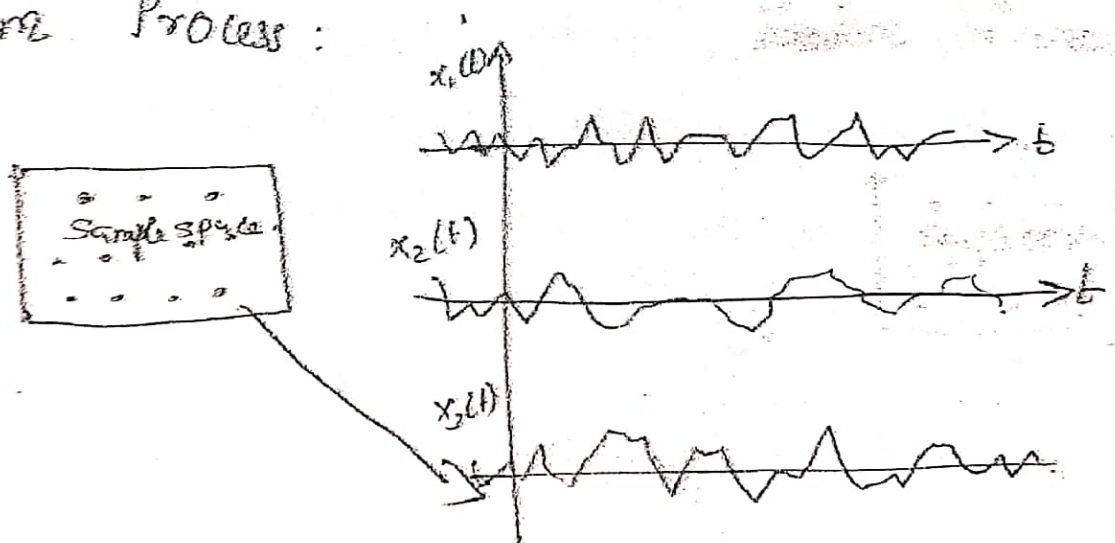
- Random signals in terms of finite Energy sequence called auto-correlation for which z -Transform & Fourier Transform exists.

→ Fourier transform of auto correlation sequence is Power density spectrum describes how average power is distributed across frequencies

- A Random Variable assigns a unique number to the outcome of a Random Experiment.
- Random Variable is a real function that maps the elements of sample space S into points of the Real axis.



Random Process :



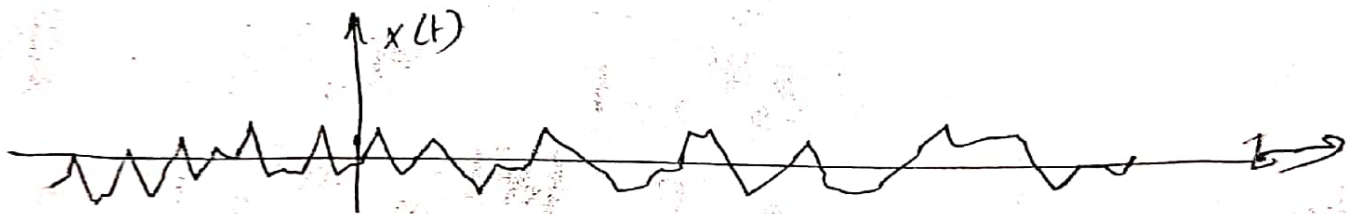
Random variable is a mapping of elements of Sample space S into points on Real axis.

- Random process is a collection of Random variables with 't' as a parameter.
- Random variable assigns a unique number to the outcome of a Random experiment.

Random processes are classified as :-

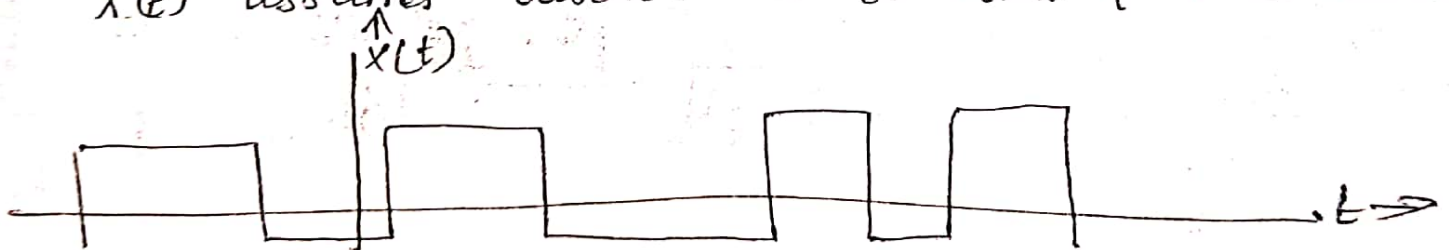
1. Continuous state & continuous time

- both $X(t)$ & t have a continuum values.



2. Discrete state & continuous time

- $X(t)$ assumes discrete set of values & 't' is continuous.



3. Continuous state & discrete time:

- $X(t)$ assumes continuum variable while 't' assume discrete set of values



Discrete state & time

→ ~~state~~ a time t is a discrete set of values