MODULE 3

ORDINARY DIFFERENTIAL EQUATIONS

Homogeneous linear ODES of second order

A second order ODE is called linear if it can be written as $y'' + p(x)y' + q(x)y = r(x)$. and non linear if it cannot be written in this form.

If $r(x) = 0$ the equation reduces to $y'' + p(x)y' + q(x) = 0$ and is called homogeneous. If $r(x) \neq 0$, the equation is called non homogeneous.

Fundamental theorem for the homogeneous linear ODE (Super position principle)

If y_1 and y_2 are solutions of the equation $y'' + p(x)y' + q(x) = 0$ then the linear combination $y = c_1y_1 + c_2y_2$ is again a solution. This solution is also called the general solution. y_1, y_2 are called a basis of solution.

Linear independence of solutions

Two solutions y_1 and y_2 of the equation $y'' + p(x)y' + q(x)y = 0$ are linearly dependent if $\frac{y_1}{y_2}$ = a constant. and is linearly independent if $\frac{y_1}{y_2} \neq$ a constant.

Wronskian

Wronskian of y_1, y_2 is obtained by

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

If $W = 0$ then y_1, y_2 are linearly dependent.

Scanned with CamScanner

If $w \neq 0$ then y_1, y_2 are linearly independent.

Problems

1. Find the wronskian of $e^{-.5x}$ and $e^{-2.5x}$. Show that they are linearly independent by using wronskian and quotients.

Ans: $y_1 = e^{-.5x}$ $\quad y_2 = e^{-2.5x}$

$y_1' = -.5e^{-.5x}$ $\quad y_2' = -2.5e^{-2.5x}$

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-.5x} & e^{-2.5x} \\ -.5e^{-.5x} & -2.5e^{-2.5x} \end{vmatrix}$$

$$= -2.5e^{-3x} + .5e^{-3x}$$

$$= -2e^{-3x}$$

$w \neq 0 \therefore y_1, y_2$ are linearly independent.

$$\frac{y_1}{y_2} = \frac{e^{-.5x}}{e^{-2.5x}} = e^{2x} \neq \text{constant}$$

$\therefore y_1, y_2$ are linearly independent.

2. Find the wronskian of x^3 and x^2. Show that they are linearly independent by using wronskian and quotients.

Ans: $y_1 = x^3$ $\quad y_2 = x^2$

$y_1' = 3x^2$ $\quad y_2' = 2x$

$$w = \begin{vmatrix} x^3 & x^2 \\ 3x^2 & 2x \end{vmatrix} = 2x^4 - 3x^4 = -x^4 \neq 0$$

$\therefore y_1, y_2$ are linearly independent.

$$\frac{y_1}{y_2} = \frac{x^3}{x^2} = x \neq \text{constant}$$ So they are linearly independent

Scanned with CamScanner

3. Find the wronskian of $\cosh(\frac{a}{2}x)$ and $\sinh(\frac{a}{2}x)$. Show that they are linearly independent

Ans $y_1 = \cosh(\frac{a}{2}x)$ $\qquad$ $y_2 = \sinh(\frac{a}{2}x)$

$y_1' = \frac{a}{2}\sinh(\frac{a}{2}x)$ $\qquad$ $y_2' = \frac{a}{2}\cosh(\frac{a}{2}x)$

$$W = \begin{vmatrix} \cosh(\frac{a}{2}x) & \sinh(\frac{a}{2}x) \\ \frac{a}{2}\sinh(\frac{a}{2}x) & \frac{a}{2}\cosh(\frac{a}{2}x) \end{vmatrix}$$

$$= \frac{a}{2}\cosh^2(\frac{a}{2}x) - \frac{a}{2}\sinh^2(\frac{a}{2}x)$$

$= \frac{a}{2} \neq 0$ so y_1, y_2 are linearly independent.

4. Find the wronskian of $e^{-x}\cos 5x$, $e^{-x}\sin 5x$

Ans: $y_1 = e^{-x}\cos 5x$ $\qquad$ $y_2 = e^{-x}\sin 5x$

$y_1' = e^{-x}\sin 5x(-5) + \cos 5x\, e^{-x}(-1)$ $\qquad$ $y_2' = 5e^{-x}\cos 5x - e^{-x}\sin 5x$

$$W = \begin{vmatrix} e^{-x}\cos 5x & e^{-x}\sin 5x \\ -5e^{-x}\sin 5x - e^{-x}\cos 5x & 5e^{-x}\cos 5x - e^{-x}\sin 5x \end{vmatrix}$$

$$= 5e^{-2x}\cos^2 5x - e^{-2x}\cos 5x\sin 5x + 5e^{-2x}\sin^2 5x + e^{-2x}\cos 5x\sin 5x$$

$$= \underline{\underline{5e^{-2x}}}$$

5) Show that e^{2x}, e^{3x} are linearly independent solutions of the differential equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$ in $-\infty < x < \infty$. What is its general solution.

Ans:

$y_1 = e^{2x}$ $\qquad$ $y_2 = e^{3x}$

$$\frac{y_1}{y_2} = \frac{e^{2x}}{e^{3x}} = e^{-x} \neq \text{constant}$$

$\therefore y_1, y_2$ are linearly independent.

Scanned with CamScanner

$y_1 = e^{2x}$ $\quad y_1' = 2e^{2x}$ $\quad y_1'' = 4e^{2x}$

$$\frac{d^2y_1}{dx^2} - 5\frac{dy_1}{dx} + 6y_1 = 4e^{2x} - 10e^{2x} + 6e^{2x} = 0$$

$\therefore y_1 = e^{2x}$ is a solution

$y_2 = e^{3x}$ $\quad y_2' = 3e^{3x}$ $\quad y_3'' = 9e^{3x}$

$$\frac{d^2y_2}{dx^2} - 5\frac{dy_2}{dx} + 6y_2 = 9e^{3x} - 15e^{3x} + 6e^{3x} = 0$$

$\therefore y_2 = e^{3x}$ is a solution.

general solution $\quad y = c_1y_1 + c_2y_2$

$$= c_1e^{2x} + c_2e^{3x}$$

6 Find the wronskian of $e^x\cos 2x$ and $e^x\sin 2x$

Ans: $y_1 = e^x\cos 2x$ $\qquad y_2 = e^x\sin 2x$

$y_1' = e^x\sin 2x(-2) + e^x\cos 2x$ $\qquad y_2' = 2e^x\cos 2x + e^x\sin 2x.$

$$W = \begin{vmatrix} e^x\cos 2x & e^x\sin 2x \\ -2e^x\sin 2x + e^x\cos 2x & 2e^x\cos 2x + e^x\sin 2x \end{vmatrix}$$

$$= 2e^{2x}\cos^2 2x + e^{2x}\cos 2x\sin 2x + 2e^{2x}\sin^2 2x - e^{2x}\cos 2x\sin 2x$$

$$= 2e^{2x}$$

METHOD OF REDUCTION OF ORDER

This method is used to find a basis if one solution is known. The general form of second order ODE is

$y'' + P(x)y' + Q(x)y = 0$ — ① Let the 1st solution can be obtained by inspection and then take $y_2 = uy_1$ where u is a function of x and $u = \int \frac{1}{(y_1)^2} e^{-\int P dx} dx$ where P is the coefficient of y' in ①

Scanned with CamScanner

7) Solve the differential equation $xy'' = 2y'$ by reducing it to first order

Ans: $xy'' = 2y'$

$$y'' - \frac{2}{x}y' = 0 \quad \text{———} \quad (1)$$

$$P = \frac{-2}{x}$$

Let $y_1 = 1$ be a solution

$$y_1' = 0 \quad y_1'' = 0$$

(1) $y'' - \frac{2}{x}y' = 0 - 0 = 0$ is true.

$$\int P\,dx = -2\int \frac{1}{x}dx = -2\log x \quad \cancel{= \log x^{-2}}$$

$$e^{-\int P dx} = e^{+2\log x} = x^2$$

$$u = \int \frac{1}{(y_1)^2} e^{-\int P dx} dx = \int \frac{1}{1} \times x^2 dx$$

$$= \frac{x^3}{3}$$

$$y_2 = u y_1 = \frac{x^3}{3}$$

general solution $y = c_1 + c_2 \frac{x^3}{3}$ //

8) Find a basis of solutions of the ODE $(x^2 - x)y'' - xy' + y = 0$

Given $y_1 = x$ is a solution

Ans $$y'' - \frac{x}{(x^2-x)}y' + \frac{y}{(x^2-x)} = 0$$

$$P = \frac{-x}{x^2-x} = \frac{-1}{x-1}$$

$$\int P\,dx = -\int \frac{1}{x-1}dx = -\log(x-1)$$

$$e^{-\int P dx} = e^{+\log(x-1)} = x-1$$

Scanned with CamScanner

$$u = \int \frac{1}{y_1^2} e^{-\int P dx} dx$$

$$= \int \frac{1}{x^2} (-x-1) dx$$

$$= \int \left(\frac{1}{x} - x^{-2}\right) dx$$

$$= \log x - \frac{x^{-1}}{-1}$$

$$= \log x + \frac{1}{x}$$

$$y_2 = u y_1 = x \log x + 1$$

$$y = c_1 x + c_2 [x \log x + 1]$$

9) Reduce to first order and solve $x^2 y'' + x y' - 4y = 0$ Given $y_1 = x^2$ is a solution

Ans:

$$y'' + \frac{y'}{x} - \frac{4y}{x^2} = 0$$

$$P = \frac{1}{x}$$

$$\int P dx = \log x$$

$$e^{-\int P dx} = e^{-\log x} = x^{-1}$$

$$u = \int \frac{1}{(x^4)} \times \frac{1}{x} dx$$

$$= \int x^{-5} dx$$

$$= -\frac{1}{4x^4}$$

$$y_2 = u y_1 = \left(\frac{-1}{4x^4}\right)(x^2) = \frac{-1}{4x^2}$$

$$y = c_1 x^2 - c_2 \frac{1}{4x^2}$$

Scanned with CamScanner

Homogeneous linear ODE's with constant coefficients

$y'' + ay' + by = 0$ — ① be a second order homogeneous linear ODE whose coefficients a and b are constants.

characteristic equation or auxiliary equation is

$$m^2 + am + b = 0$$

Let its roots be m_1, m_2

Case I

Two distinct real roots m_1, m_2

The general solution is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$ are the basis

Case II : Real double roots $m_1 = m_2 = m$

The general solution is $y = (c_1 + c_2 x) e^{mx}$

$y_1 = e^{mx}$ and $y_2 = x e^{mx}$ are the basis

Case III Complex roots $(\alpha \pm i\beta)$

The general solution is $y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$

$y = e^{\alpha x}\cos\beta x$, $y_2 = e^{\alpha x}\sin\beta x$ are the basis.

(6) Find the general solution of the following ODE's.

$y'' - y = 0$

$\Rightarrow$ $D^2 y - y = 0$

$(D^2 - 1) y = 0$

AE is $m^2 - 1 = 0$

$m = \pm 1$ $\qquad y = c_1 e^{x} + c_2 e^{-x}$

Scanned with CamScanner

17) $y'' + 6y' + 9y = 0$

$D^2y + 6Dy + 9y = 0$

$(D^2 + 6D + 9)y = 0$

$m^2 + 6m + 9 = 0$

$m = -3, -3$

$y = (c_1 + c_2 x)e^{-3x}$

18) $y'' + y = 0$

$D^2y + y = 0$

$(D^2 + 1) = 0$

$m^2 + 1 = 0$

$m^2 = -1$

$m = \pm i$

$y = c_1 \cos x + c_2 \sin x$

19) $3y'' + y' - y = 0$

$(3D^2 + D - 1)y = 0$

$3m^2 + m - 1 = 0$

$m = \dfrac{-1 \pm \sqrt{1 + 12}}{6}$

$= \dfrac{-1 \pm \sqrt{13}}{6}$

$= \dfrac{-1 + \sqrt{13}}{6}, \dfrac{-1 - \sqrt{13}}{6}$

g.s $y = c_1 e^{\left(\frac{-1+\sqrt{13}}{6}\right)x} + c_2 e^{\left(\frac{-1-\sqrt{13}}{6}\right)x}$

Scanned with CamScanner

20) $4y'' - 25y = 0$

$(4D^2 - 25)y = 0$

$4m^2 - 25 = 0$

$4m^2 = 25$

$m^2 = \frac{25}{4}$ $\quad m = \frac{5}{2}, -\frac{5}{2}$

$y = c_1 e^{\frac{5}{2}x} + c_2 e^{-\frac{5}{2}x}$

21) $y'' + 4.5y' = 0$

$(D^2 + 4.5D)y = 0$

$m^2 + 4.5m = 0$

$m(m + 4.5) = 0$

$m = 0, -4.5$

$y = c_1 e^{0x} + c_2 e^{-4.5x}$

$= c_1 + c_2 e^{-4.5x}$

22) Solve the initial value problem $y'' + 4y' + 5y = 0$ $\quad y(0) = 2, y'(0) = -5$

Ans: $y'' + 4y' + 5y = 0$

$m^2 + 4m + 5 = 0$

~~m+5~~ $\quad m = -2 \pm i$

$y = e^{-2x}[c_1 \cos x + c_2 \sin x]$

$y'(x) = e^{-2x}[-c_1 \sin x + c_2 \cos x] + [c_1 \cos x + c_2 \sin x]e^{-2x}(-2)$

$= e^{-2x}[-c_1 \sin x + c_2 \cos x] - 2e^{-2x}[c_1 \cos x + c_2 \sin x]$

Scanned with CamScanner

given $y(0) = 2 \Rightarrow 2 = c_1$

$y'(0) = -5 \Rightarrow -5 = c_2 - 2c_1$

$-5 = c_2 - 4$

$\therefore c_2 = -1$

$\therefore y(x) = e^{-2x}[2\cos x - \sin x]$

23) Solve the initial value problem $9y'' - 30y' + 25y = 0$ $y(0) = 3$ $y'(0) = 10$

Ans:

$(9D^2 - 30D + 25)y = 0$

$9m^2 - 30m + 25 = 0$

$$m = \frac{30 \pm \sqrt{900 - 900}}{18}$$

$m = \frac{5}{3}, \frac{5}{3}$

$$g.s = y(x) = (c_1 + c_2 x)e^{5/3 x}$$

$$= c_1 e^{5/3 x} + c_2 x e^{5/3 x}$$

given $y(0) = 3 \Rightarrow 3 = c_1$

$$y'(x) = \frac{5}{3} c_1 e^{\frac{5}{3}x} + c_2 \left[\frac{5}{3} x e^{5/3 x} + e^{5/3 x}\right]$$

given $y'(0) = 10 \Rightarrow 10 = 5 + c_2$

$c_2 = 5$

$$\therefore y(x) = (3 + 5x)e^{5/3 x}$$

24) Solve the initial value problem $y'' - 4y' + 13y = 0$ with $y(0) = -1$ and $y'(0) = 2$.

Ans: $(D^2 - 4D + 13)y = 0$

Scanned with CamScanner

$m^2 - 4m + 13 = 0$

BE $\Rightarrow$ $m = \dfrac{4 \pm \sqrt{16-52}}{2} = \dfrac{4 \pm 6i}{2} = 2 \pm 3i$

$y(x) = e^{2x}[c_1 \cos 3x + c_2 \sin 3x]$

$y(0) = -1 \Rightarrow \quad -1 = c_1$

$y'(x) = e^{2x}[3c_1 \sin 3x + 3c_2 \cos 3x] + [c_1 \cos 3x + c_2 \sin 3x]e^{2x}(2$

$y'(0) = 2 \Rightarrow \quad 2 = 3c_2 + 2c_1$

$2 = 3c_2 - 2$

$c_2 = 4/3$

$\therefore y(x) = e^{2x}[-\cos 3x + \frac{4}{3}\sin 3x]$

25) Solve the boundary value problem $y'' - 10y' + 25y = 0$

$y(0) = 1, \quad y(1) = 0.$

Ans: $m^2 - 10m + 25 = 0$

$(m-5)^2 = 0$

$m = 5, 5$

$y(x) = (c_1 + c_2 x)e^{5x}$

$y(0) = 1 \Rightarrow \quad 1 = c_1$

$y(1) = 0 \Rightarrow \quad 0 = (1 + c_2)e^5$

$c_2 = -1$

$\therefore y(x) = (1 - x)e^{5x}$

26) Solve the initial value problem $4y'' - 25y = 0 \quad y(0) = 0, y'(0) = -5$

Ans: $4m^2 - 25 = 0$

$m = \pm 5/2$

Scanned with CamScanner

$$y = c_1 e^{5/2 x} + c_2 e^{-5/2 x}$$

$y(0)=0 \Rightarrow 0 = c_1 + c_2$ —— ①

$y'(x) = \frac{5}{2} c_1 e^{5/2 x} - \frac{5}{2} c_2 e^{-5/2 x}$

$y'(0) = -5 \Rightarrow \frac{5}{2} c_1 - \frac{5}{2} c_2 = -5$ —— ②

① $\times \frac{5}{2} \Rightarrow \frac{5}{2} c_1 + \frac{5}{2} c_2 = 0$

$5c_1 = -5$

$c_1 = \underline{-1}$

$\therefore c_2 = 1$

$\therefore y(x) = -e^{5/2 x} + e^{-5/2 x}$

Q7 Find a second order homogeneous linear ODE for which e^{-x} and e^{-2x} are the solutions.

Ans: general solution $y = c_1 e^{-x} + c_2 e^{-2x}$

$m_1 = -1$ & $m_2 = -2$

AE is $(m+1)(m+2) = 0$

$m^2 + 3m + 2 = 0$

$(D^2 + 3D + 2)y = 0$

$D^2 y + 3Dy + 2y = 0$

$y'' + 3y' + 2y = 0$

28) Show that the functions $e^{-x}\cos x$ and $e^{-x}\sin x$ are linearly independent. Form a second order linear ODE having these functions are solutions.

Ans: $y_1 = e^{-x}\cos x$ $\quad y_2 = e^{-x}\sin x$.

$\frac{y_1}{y_2} = \frac{e^{-x}\cos x}{e^{-x}\sin x} = \cot x \neq$ a constant

Scanned with CamScanner

$\therefore$ y_1, y_2 are linearly independent.

g.s $\quad y = e^{-x}[c_1 \cos x + c_2 \sin x]$

$m_1 = -1+i \quad m_2 \; -1-i$

AE is $\quad [m-(-1+i)][m-(-1-i)] = 0$

$[(m+1)-i][(m+1)+i] = 0$

$(m+1)^2 - (i)^2 = 0$

$m^2 + 2m + 1 + 1 = 0$

$m^2 + 2m + 2 = 0$

$(D^2 + 2D + 2)y = 0$

$D^2y + 2Dy + 2y = 0$

$y'' + 2y' + 2y = 0$

29) Show that $y_1(x) = e^x$ and $y_2(x) = xe^x$ form a fundamental system (basis) for the differential equation $y'' - 2y' + y = 0$

Ans:

$y'' - 2y' + y = 0$

$(D^2 - 2D + 1)y = 0$

AE $\Rightarrow$ $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

$m = 1, 1$

$\therefore$ soln $y(x) = (c_1 + c_2 x)e^x$

$\therefore$ e^x and xe^x are the basis.

30) Show that $y_1(x) = e^{-4x}$ and $y_2(x) = xe^{-4x}$ are solutions of the differential equation $\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 16y = 0$. Are they linearly

Scanned with CamScanner

Ans: $\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 16y = 0$

$m^2 + 8m + 16 = 0$

$m = -4, -4$

$g.s = (c_1 + c_2 x)e^{-4x}$

$\therefore$ basis e^{-4x} and xe^{-4x}

$\frac{y_1}{y_2} = \frac{e^{-4x}}{xe^{-4x}} = \frac{1}{x} \neq$ a constant $\therefore y_1, y_2$ are linearly independent

31) Verify that the given functions $x^{3/2}, x^{-1/2}$ are linearly independent and form a basis of solution space of given ODE $4x^2y'' - 3y = 0$

Ans: $y_1 = x^{3/2}$ $\quad y_1' = \frac{3}{2}x^{1/2}$ $\quad y_1'' = \frac{3}{4}x^{-1/2}$

$4x^2y_1'' - 3y_1 = 4x^2 \times \frac{3}{4}x^{-1/2} - 3x^{3/2}$

$= 3x^{3/2} - 3x^{3/2}$

$= \underline{\underline{0}}$

$\therefore y_1 = x^{3/2}$ is a solution.

$y_2 = x^{-1/2}$ $\quad y_2' = -\frac{1}{2}x^{-3/2}$ $\quad y_2'' = \frac{3}{4}x^{-5/2}$

$4x^2y_2'' - 3y_2 = 4x^2 \times \frac{3}{4}x^{-5/2} - 3x^{-1/2}$

$= 3x^{-1/2} - 3x^{-1/2}$

$= \underline{\underline{0}}$

$\therefore y_2 = x^{-1/2}$ is a solution

$\therefore y_1 = x^{3/2}$ and $y_2 = x^{-1/2}$ form a basis

$\frac{y_1}{y_2} = \frac{x^{3/2}}{x^{-1/2}} = x^2 \neq$ a constant.

$\therefore y_1, y_2$ are linearly independent.

Scanned with CamScanner

32) Prove that $y_1(x) = e^x$ and $y_2(x) = e^{4x}$ form a basis for the differential equation $y'' - 5y' + 4y = 0$ Can $5e^x - 2e^{4x}$ be a solution of the differential equation Explain?

Ans: $y'' - 5y' + 4y = 0$

$m^2 - 5m + 4 = 0$

$m = 1, 4$

$y(x) = c_1 e^{4x} + c_2 e^x$

$\therefore$ e^{4x}, e^x form a basis.

$y(x) = 5e^x - 2e^{4x}$ is a solution since it is a linear combination of e^x and e^{4x}.

HOMOGENEOUS LINEAR ODES WITH CONSTANT COEFFICIENTS OF HIGHER ORDER

An nth order homogeneous linear ODE with constant coefficient is $y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$

The characteristic equation is

$$m^{(n)} + a_{n-1} m^{(n-1)} + \cdots + a_1 m + a_0 = 0$$

I Distinct Real roots:

If all the nth roots $m_1, m_2 \ldots m_n$ are real and different, the general solution is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$

II Multiple real roots:

If m is a real root of order m, the general soln is

$$y = (c_1 + c_2 x + c_3 x^2 + \cdots + c_n x^{n-1}) e^{mx}$$

III Multiple Complex roots

If $m = \alpha \pm i\beta$ is a complex double root, the general soln is

Scanned with CamScanner

$$y = e^{\alpha x}\left[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x\right]$$

If $m = \alpha \pm i\beta$ is a complex triple root. The general soln is

$$y = e^{\alpha x}\left[(c_1 + c_2 x + c_3 x^2)\cos\beta x + (c_4 + c_5 x + c_6 x^2)\sin\beta x\right]$$

Q) Solve the following ODE's

33) $y''' - y' = 0$

$m^3 - m = 0$

$m(m^2 - 1) = 0$

$m = 0, +1, -1$

$y(x) = c_1 + c_2 e^{x} + c_3 e^{-x}$

34) $y''' + 6y'' + 12y' + 8y = 0$

$(m^3 + 6m^2 + 12m + 8)y = 0$

$m = -2, -2, -2$

$y(x) = (c_1 + c_2 x + c_3 x^2)e^{-2x}$

35) $\frac{d^3y}{dx^3} - 8y = 0$

$m^3 - 8 = 0$

$m = 2, -1 \pm \sqrt{3}i$

$y(x) = c_1 e^{2x} + e^{-x}\left[c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x\right]$

36) $\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 2y = 0$

$(m^3 - 4m^2 + 5m - 2) = 0$

$m = 1, 1, 2$

$y(x) = (c_1 + c_2 x)e^{x} + c_3 e^{2x}$

Scanned with CamScanner

37) $y''' - y'' + 4y' = 0$

$m^3 - m^2 + 4m = 0$

$m = 0, \ \frac{1 \pm i\sqrt{15}}{2}$

$y(x) = c_1 + e^{\frac{1}{2}x}\left[c_2 \cos \frac{\sqrt{15}}{2}x + c_3 \sin \frac{\sqrt{15}}{2}x\right]$

38) $y''' - 3y'' - 4y' + 6y = 0$

$m^3 - 3m^2 - 4m + 6 = 0$

$m = 1, \ 1+\sqrt{14}, \ 1-\sqrt{14}$

$y(x) = c_1 e^{x} + c_2 e^{(1+\sqrt{14})x} + c_3 e^{(1-\sqrt{14})x}$

39) $y^{IV} - 5y'' + 4y = 0$

$m^4 - 5m^2 + 4 = 0$

$(m^2 - 1)(m^2 - 4) = 0$

$m^2 = 1, 4$

$m = \pm 1, \pm 2$

$y(x) = c_1 e^{x} + c_2 e^{-x} + c_3 e^{2x} + c_4 e^{-2x}$

40) $y^{V} - 3y^{IV} + 3y''' - y'' = 0$

$m^5 - 3m^4 + 3m^3 - m^2 = 0$

$m^2(m^3 - 3m^2 + 3m - 1) = 0$

$m = 0, 0 \qquad m^3 - 3m^2 + 3m - 1 = 0$

$m = 0, 0, 1, 1, 1$

$y(x) = (c_1 + c_2 x) + (c_3 + c_4 x + c_5 x^2)e^{x}$

41) $(D^4 + 3D^2 - 4)y = 0$

$m^4 + 3m^2 - 4 = 0 \qquad (m^2+4)(m^2-1) = 0$

[redacted]

Scanned with CamScanner

$m^2 = -4, 1$

$m = \pm 2i \quad \pm 1$

$y(x) = c_1 e^{x} + c_2 e^{-x} + c_3 \cos 2x + c_4 \sin 2x$

42) $y'''' + 2y'' + y = 0$

$(m^4 + 2m^2 + 1) = 0$

$(m^2+1)^2 = 0$

$m^2 = -1, -1$

$m = \pm i, \pm i$

$y = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x$

43) $y''' + y = 0$

$m^3 + 1 = 0$

$m = -1, \frac{1 \pm i\sqrt{3}}{2}$

$y(x) = c_1 e^{-x} + e^{\frac{x}{2}}\left[c_2 \cos \frac{\sqrt{3}}{2}x + c_3 \sin \frac{\sqrt{3}}{2}x\right]$

44) $y^{(4)} + 6y^{(3)} + 9y^{(2)} = 0$

$m^4 + 6m^3 + 9m^2 = 0$

$m^2(m^2 + 6m + 9) = 0$

$m = 0, 0 -3, -3$

$y = (c_1 + c_2 x) + (c_3 + c_4 x)e^{-3x}$

45) $y''' - 8y'' + 37y' - 50y = 0$

$m^3 - 8m^2 + 37m - 50 = 0$

$m = 2, 3 \pm 4i$

$y = c_1 e^{2x} + e^{3x}(c_2 \cos 4x + c_3 \sin 4x)$

Scanned with CamScanner

NONHOMOGENEOUS ODES

The general form of nth order non homogeneous linear ODE with constant coefficient is

$$y^{(n)} + a_{n-1} y^{(n-1)} + a_{n-2} y^{(n-2)} + \cdots + a_1 y' + a_0 y = r(x)$$

The solution of this equation contains two parts called complimentary function and particular integral. To find the complimentary function we have to find the general solution of the homogeneous equation

$$y^{(n)} + a_{n-1} y^{(n-1)} + a_{n-2} y^{(n-2)} + \cdots + a_1 y' + a_0 y = 0$$

The PI is a particular solution of the given equation

$\therefore$ The general solution $y = CF + PI$

(e) $y = y_h + y_p$.

METHOD OF UNDETERMINED COEFFICIENTS

The method of finding the solution y_p (particular integral) is called method of undetermined coefficients. Choose a form of y_p similar to $r(x)$, but with unknown coefficients to be determined by substituting that y_p and its derivatives in to the ODE. The choice of y_p for important forms of $r(x)$ are

1. If $r(x) = ke^{ax}$ then $y_p = Ae^{ax}$
2. If $r(x) = kx^m$ then $y_p = A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m$
3. If $r(x) = k \sin \omega x$ then $y_p = A \cos \omega x + B \sin \omega x$
4. If $r(x) = k \cos \omega x$ then $y_p = A \cos \omega x + B \sin \omega x$
5. If $r(x) = ke^{ax} \sin \omega x$ then $y_p = e^{ax}[A \cos \omega x + B \sin \omega x]$

Scanned with CamScanner

6. If $r(x) = ke^{ax}\cos\omega x$ then $y_p = e^{ax}[A\cos\omega x + B\sin\omega x]$

If the choice for y_p is a solution of the homogeneous ODE, multiply this term by x. If the choice for y_p is a double solution of the homogeneous ODE, multiply this term by x^2. If $r(x)$ is a sum of functions, choose for y_p the sum of the functions in the corresponding form.

<u>Problems</u>

1. Solve $y'' + 5y' + 6y = 2e^{-x}$. —①

$y'' + 5y' + 6y = 0$

$m^2 + 5m + 6 = 0$

$m = -2, -3$

$y_h = c_1e^{-2x} + c_2e^{-3x}$

Let $y_p = Ae^{-x}$

$y_p' = -Ae^{-x}$

$y_p'' = +Ae^{-x}$

Sub in ① $y_p'' + 5y_p' + 6y_p = 2e^{-x}$

$Ae^{-x} - 5Ae^{-x} + 6Ae^{-x} = 2e^{-x}$

$A - 5A + 6A = 2$

$2A = 2 \quad \therefore A = 1$

$\therefore y_p = e^{-x}$

Soln $y = y_h + y_p$

$= c_1e^{-2x} + c_2e^{-3x} + e^{-x}$

Scanned with CamScanner

2. Solve $(D^2+3D+2)y = 5$ —— ①

$m^2+3m+2 = 0$

$m = -2, -1$

$y_h = c_1e^{-2x} + c_2e^{-x}$

Let $y_p = Ae^{0x} = A$

$y_p' = 0 \qquad y_p'' = 0$

Sub in ① $\quad y_p'' + 3y_p' + 2y_p = 5$

$0 + 0 + 2A = 5$

$A = 5/2$

$\therefore y_p = 5/2$

Soln $\quad y = c_1e^{-2x} + c_2e^{-x} + \frac{5}{2}$

3. Solve $(D^2+3D+2)y = 3$ —— ①

$y_h = c_1e^{-2x} + c_2e^{-x}$

Let $y_p = Ae^{0x} = A \qquad y_p' = 0 \qquad y_p'' = 0$

Sub in ① $\quad y_p'' + 3y_p' + 2y_p = 3$

$0 + 0 + 2A = 3$

$A = 3/2$

$\therefore y_p = 3/2$

Soln $\quad y = c_1e^{-2x} + c_2e^{-x} + 3/2$

4. Solve $y'' + 6y' + 9y = e^{3x}$

$y_h = (c_1 + c_2x)e^{3x}$

Scanned with CamScanner

Let $y_p = Ax^2e^{3x}$ [$\because e^{3x}$ is a double root of y_h]

$$y_p' = A[3x^2e^{3x} + 2xe^{3x}]$$

$$= e^{3x}[3Ax^2 + 2Ax]$$

$$y_p'' = e^{3x}[6Ax + 2A] + 3e^{3x}[3Ax^2 + 2Ax]$$

Sub in ① $y_p'' - 6y_p' + 9y_p = e^{3x}$

$$e^{3x}[6Ax + 2A] + 3e^{3x}[3Ax^2 + 2Ax] - 6e^{3x}[3Ax^2 + 2Ax] + 9Ax^2e^{3x} = e^{3x}$$

$$6Ax + 2A + 9Ax^2 + 6Ax - 18Ax^2 - 12Ax + 9Ax^2 = 1$$

$$2A = 1 \quad \therefore A = 1/2$$

$$\therefore y_p = \frac{x^2e^{3x}}{2}$$

Soln $y = (c_1 + c_2x)e^{3x} + \frac{x^2e^{3x}}{2}$

5. Solve $y'' + y = \cosh 5x$

$$y'' + y = \frac{e^{5x}}{2} + \frac{e^{-5x}}{2} \quad \text{—— ①}$$

$$y'' + y = 0$$

$$m^2 + 1 = 0$$

$$m = \pm i$$

$$y_h = c_1\cos x + c_2\sin x.$$

Let $y_{p_1} = Ae^{5x}$

$$y_{p_1}' = 5Ae^{5x}$$

$$y_{p_1}'' = 25Ae^{5x}$$

Scanned with CamScanner

Sub in ① $y_{p_1}'' + y_{p_1} = \frac{e^{5x}}{2}$

$$25Ae^{5x} + Ae^{5x} = \frac{e^{5x}}{2}$$

$$26A = \frac{1}{2} \quad \therefore A = \frac{1}{52}$$

$$y_{p_1} = \frac{e^{5x}}{52}$$

Let $y_{p_2} = Ae^{-5x}$

$y_{p_2}' = -5Ae^{-5x}$ $\quad y_{p_2}'' = 25Ae^{-5x}$

sub in ① $\quad y_{p_2}'' + y_{p_2} = \frac{e^{-5x}}{2}$

$$25Ae^{-5x} + Ae^{-5x} = \frac{e^{-5x}}{2}$$

$$26A = \frac{1}{2}$$

$$A = \frac{1}{52}$$

$$y_{p_2} = \frac{e^{-5x}}{52}$$

general soln $\quad y = y_h + y_{p_1} + y_{p_2}$

$$y = c_1\cos x + c_2\sin x + \frac{e^{5x}}{52} + \frac{e^{-5x}}{52}$$

6 Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = \sinh 2x$

$$y'' + 4y' + 4y = \frac{e^{2x}}{2} - \frac{e^{-2x}}{2} \quad \text{——— ①}$$

$$y'' + 4y' + 4y = 0$$

$$m^2 + 4m + 4 = 0$$

$$m = -2, -2$$

$$y_h = (c_1 + c_2x)e^{-2x}$$

Let $y_{p_1} = Ae^{2x}$ $\quad y_{p_1}' = 2Ae^{2x}$ $\quad y_{p_1}'' = 4Ae^{2x}$.

Scanned with CamScanner

Sub in ① $y_{P_1}'' + 4y_{P_1}' + 4y_{P_1} = \frac{e^{2x}}{2}$

$$4Ae^{2x} + 8Ae^{2x} + 4Ae^{2x} = \frac{e^{2x}}{2}$$

$$16A = \frac{1}{2}$$

$$A = \frac{1}{32}$$

$$\therefore y_{P_1} = \frac{e^{2x}}{32}$$

Let $y_{P_2} = Ax^2e^{-2x}$

$$y_{P_2}' = A\left[2x^2e^{-2x} + 2xe^{-2x}\right]$$

$$= e^{-2x}\left[-2Ax^2 + 2Ax\right]$$

$$y_{P_2}'' = e^{-2x}\left[-4Ax + 2A\right] + \left[-2Ax^2 + 2xA\right]e^{-2x}(-2)$$

$$= e^{-2x}\left[-4Ax + 2A + 4Ax^2 - 4xA\right]$$

Sub in ① $y_{P_2}'' + 4y_{P_2}' + 4y_{P_1} = \frac{e^{-2x}}{2}$

$$e^{-2x}\left[-4Ax + 2A + 4Ax^2 - 4Ax\right] + 4e^{-2x}\left[-2Ax^2 + 2Ax\right] + 4Ax^2e^{-2x} = \frac{e^{-2x}}{2}$$

$$4Ax^2 - 8Ax + 2A - 8Ax^2 + 8Ax + 4Ax^2 = \frac{1}{2}$$

$$2A = \frac{1}{2}$$

$$A = 1/4$$

$$\therefore y_{P_2} = \frac{x^2e^{-2x}}{4}$$

Soln, $y = y_h + y_{P_1} - y_{P_2}$

$$= (c_1 + c_2x)e^{-2x} + \frac{e^{2x}}{32} - \frac{x^2e^{-2x}}{4}$$

Scanned with CamScanner

7) Solve $y'' - 4y' - 5y = 4\cos 2x$ — ①

$m^2 - 4m - 5 = 0$

$m = 5, -1$

$y_h = c_1 e^{5x} + c_2 e^{-x}$

Let $y_p = A\cos 2x + B\sin 2x$.

$y_p' = -2A\sin 2x + 2B\cos 2x$

$y_p'' = -4A\cos 2x - 4B\sin 2x$.

Sub in ① $y_p'' - 4y_p' - 5y_p = 4\cos 2x$

$[-4A\cos 2x - 4B\sin 2x] - 4[-2A\sin 2x + 2B\cos 2x] - 5[A\cos 2x + B\sin 2x]$
$= 4\cos 2x$

equating coefficients of $\cos 2x$

$-4A - 8B - 5A = 4$

$-9A - 8B = 4$ — ②

equating coefficients of $\sin 2x$

$-4B + 8A - 5B = 0$

$8A - 9B = 0$ — ③

Solving ② & ③ $B = \frac{-32}{145}$ & $A = \frac{-36}{145}$

$\therefore y_p = \frac{-36\cos 2x - 32\sin 2x}{145}$

Solution $y = c_1 e^{5x} + c_2 e^{-x} - \frac{(36\cos 2x + 32\sin 2x)}{145}$

8) Solve $y'' + y = \sin x$

AE $\Rightarrow m^2 + 1 = 0$

$m = \pm i$

$y_h = c_1\cos x + c_2\sin x$

Scanned with CamScanner

Let $y_p = x(A\cos x + B\sin x)$

$y_p' = x(-A\sin x + B\cos x) + (A\cos x + B\sin x)$

$= (-Ax + B)\sin x + (Bx + A)\cos x$

$y_p'' = (-Ax + B)\cos x + (-A)\sin x + -(Bx + A)\sin x + B\cos x$

Substituting in ①

$(-Ax + 2B)\cos x + (-Bx - 2A)\sin x + xA\cos x + xB\sin x = \sin x$

equating coefficients of $\cos x$

$-Ax + 2B + xA = 0$

$2B = 0 \quad \therefore B = 0$

equating coefficients of $\sin x$

$-Bx - 2A + xB = 1$

$A = -1/2$

$\therefore y_p = -\frac{x\cos x}{2}$

Soln $y = c_1\cos x + c_2\sin x - \frac{x\cos x}{2}$

9. Solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 4\cos^2 x$

$(D^2 + 3D + 2)y = \frac{4(1 + \cos 2x)}{2}$

$(D^2 + 3D + 2)y = 2 + 2\cos 2x$ —— ①

$m^2 + 3m + 2 = 0$

$m = -1, -2$

$y_h = c_1 e^{-x} + c_2 e^{-2x}$

Scanned with CamScanner

$y_{P_1} = Ae^{0x} = A$

$y_P' = 0, \; y_{P_1}'' = 0$

① $\Rightarrow$ $2A = 2$ $\therefore A = 1$ $\therefore y_{P_1} = 1$

Let $y_{P_2} = A\cos 2x + B\sin 2x$

$y_{P_2}' = -2A\sin 2x + 2B\cos 2x$

$y_{P_2}'' = -4A\cos 2x - 4B\sin 2x$

Sub in ① $\Rightarrow$ $-4A\cos 2x - 4B\sin 2x - 6A\sin 2x + 6B\cos 2x$

$+ 2A\cos 2x + 2B\sin 2x = 2\cos 2x$

equating coefficients of $\cos 2x \Rightarrow -4A + 6B + 2A = 2$

$-2A + 6B = 2$ —— ②

equating coefficients of $\sin 2x \Rightarrow -4B - 6A + 2B = 0$

$-6A - 2B = 0$ —— ③

Solving ② and ③ $B = \frac{3}{10}$ $\therefore A = -\frac{1}{10}$

$$y_{P_2} = \frac{-\cos 2x + 3\sin 2x}{10}$$

Sol $y = y_h + y_{P_1} + y_{P_2}$

$$= c_1e^{-x} + c_2e^{-2x} + 1 - \frac{\cos 2x + 3\sin 2x}{10}$$

10 Solve $(D^2 - 4D + 4)y = \sin^2 x$

$m^2 - 4m + 4 = 0 \Rightarrow (m-2)^2 = 0 \Rightarrow m = 2, 2$

$y_h(x) = (c_1 + c_2x)e^{2x}$

$y'' - 4y' + 4y = \frac{1}{2} - \frac{\cos 2x}{2}$ —— ①

$y_{P_1} = A$ $y_{P_1}' = 0$ $y_{P_1}'' = 0$

Scanned with CamScanner

① $\Rightarrow$ $4A = 1/2$ $\therefore A = 1/8$

Let $y_{P_2} = A\cos 2x + B\sin 2x$

$y'_{P_2} = -2A\sin 2x + 2B\cos 2x$

$y''_{P_2} = -4A\cos 2x + -4B\sin 2x$

Sub in ①

$$y''_{P_2} - 4y'_{P_2} + 4y_{P_2} = \frac{\cos 2x}{2}$$

$$-4A\cos 2x - 4B\sin 2x + 8A\sin 2x - 8B\cos 2x + 4A\cos 2x + 4B\sin 2x = \frac{\cos 2x}{2}$$

comparing coefficients of $\cos 2x \Rightarrow -4A - 8B + 4A = 1/2$

$B = -\frac{1}{16}$

equating coefficients of $\sin 2x \Rightarrow -4B + 8A + 4B = 0$ $\therefore A = 0$

$\therefore y_{P_2} = -\frac{\sin 2x}{16}$

Soln $\Rightarrow$ $y = (c_1 + c_2 x)e^{2x} + \frac{1}{8} + \frac{\sin 2x}{16}$

Scanned with CamScanner

EULER CAUCHY EQUATIONS

An equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = r(x)$$

is called cauchys homogeneous linear equations. This equation can be reduced to linear differential equations with constant coefficients by putting

$x = e^z$ or $z = \ln x$ then

$x\frac{dy}{dx} = D_1 y$ where $D_1 = \frac{d}{dz}$

$x^2 \frac{d^2 y}{dx^2} = D_1(D_1 - 1)y$

Here D_1 represents differentiation wrt z.

Problems

1) Find the general solution of $x^2 y'' + (1.5-1)y' - .5y = 0$ is a cauchys homogeneous equation

$\therefore$ put $x = e^z$ or $z = \ln x$

$x y' = D_1 y$

$x^2 y'' = D_1(D_1 - 1)y$

① $\Rightarrow$ $D_1(D_1-1)y + .5D_1 y - .5y = 0$

$(D_1^2 - D_1 + .5D_1 - .5)y = 0$

Scanned with CamScanner

$(D_1^2 - .5D_1 + .5)y = 0$

$m^2 - .5m + .5 = 0$

$m = .5, -1$

$y_h = c_1 e^{.5z} + c_2 e^{-z}$

$= c_1 x^{.5} + \frac{c_2}{x}$

2) Find the general solution of $x^2y'' - 5xy' + 9y = 0$ — ①

put $x = e^z$ or $z = \ln x$

$xy' = D_1 y \quad x^2y'' = D_1(D_1 - 1)y$

① $\Rightarrow$ $D_1(D_1 - 1)y - 5D_1 y + 9y = 0$

$(D_1^2 - 6D_1 + 9)y = 0$

AE $\Rightarrow$ $m^2 - 6m + 9 = 0$

$m = 3, 3$

$y_h = (c_1 + c_2 z)e^{3z}$

$= (c_1 + c_2 \ln x)x^3$

3) Find the general solution of $(x^2D^2 - 3xD + 10I)y = 0$

ie) $x^2y'' - 3xy' + 10y = 0$ — ①

$D_1(D_1 - 1)y - 3D_1 y + 10y = 0$

$(D_1^2 - 4D_1 + 10)y = 0$

$m^2 - 4m + 10 = 0$

$m = \frac{4 \pm \sqrt{-24}}{2} = \frac{4 \pm 2\sqrt{6}i}{2} = 2 \pm \sqrt{6}i$

$y_h = e^{2z}[c_1 \cos\sqrt{6}z + c_2 \sin\sqrt{6}z]$

Scanned with CamScanner

$$= x^2\left[c_1\cos(\sqrt{6}\ln x) + c_2\sin(\sqrt{6}\ln x)\right]$$

4) Solve $(x^2D^2 + xD - 9)y = \ln x$ —— ①

$$x^2y'' + xy' - 9y = \ln x$$

$$D_1(D_1 - 1)y + D_1y - 9y = z$$

$$(D_1^2 - 9)y = z \quad \text{—— ②}$$

AE $\Rightarrow$ $m^2 - 9 = 0$ $\quad m = \pm 3$

$$y_h = c_1e^{3z} + c_2e^{-3z}$$

$$= c_1x^3 + \frac{c_2}{x^3}$$

$$y_p = A_0 + A_1z$$

$$y_p' = A_1$$

$$y_p'' = 0$$

② $\Rightarrow$ $0 - 9(A_0 + A_1z) = z$

$$-9A_1 = 1 \quad A_1 = -\frac{1}{9} \quad \& \; A_0 = 0$$

$$\therefore y_p = 0 + -\frac{1}{9}z$$

$$y_p = \frac{-\ln x}{9}$$

Soln $y = c_1x^3 + \dfrac{c_2}{x^3} - \dfrac{\ln x}{9}$

Scanned with CamScanner

Solution by variation of parameters.

Consider the non homogeneous linear ODE

$$y'' + p(x)y' + q(x)y = r(x) \quad \text{—} \; (1)$$

The method of variation of parameter is for finding the particular solution of the ODE

working rule

(i) Find the homogeneous solution of (1) which is of the form $c_1 y_1 + c_2 y_2$

(ii) Find the wronskian $w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

(iii) PI $= y_p = -y_1 \int \frac{y_2 r(x)}{w} dx + y_2 \int \frac{y_1 r(x)}{w} dx$

Problems

(1) Solve $y'' + y = \sec x$ using method of variation of parameters

$m^2 + 1 = 0$

$m = \pm i$

$y_h = c_1 \cos x + c_2 \sin x$ $\qquad y_1 = \cos x, \quad y_2 = \sin x$

$$w = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$y_p = -y_1 \int \frac{y_2 r(x)}{w} dx + y_2 \int \frac{y_1 r(x)}{w} dx$$

$$= -\cos x \int \sin x \sec x \, dx + \sin x \int \cos x \sec x \, dx$$

Scanned with CamScanner

$$= -\cos x \int \tan x \, dx + \sin x \int dx$$

$$= -\cos x - \log|\cos x| + x \sin x$$

$$= \cos x \ln|\cos x| + x \sin x$$

$$y = c_1 \cos x + c_2 \sin x + \cos x \ln|\cos x| + \underline{x \sin x}$$

2. Solve $y'' + y = \tan x$ using method of variation of parameters.

$$y_h = c_1 \cos x + c_2 \sin x$$

$$w = 1$$

$$y_p = -\cos x \int \sin x \tan x \, dx + \sin x \int \cos x \tan x \, dx$$

$$= -\cos x \int \frac{\sin^2 x}{\cos x} dx + \sin x \int \sin x \, dx$$

$$= -\cos x \int \left(\frac{1 - \cos^2 x}{\cos x} \right) dx + -\sin x \cos x$$

$$= -\cos x \int (\sec x - \cos x) \, dx - \sin x \cos x$$

$$= -\cos x \left[\log|\sec x + \tan x| - \sin x \right] - \sin x \cos x$$

$$= -\cos x \ln|\sec x + \tan x|$$

$$y = c_1 \cos x + c_2 \sin x - \cos x \ln|\sec x + \underline{\tan x}|$$

3) Solve $y'' + y = \operatorname{cosec} x$ using variation of parameters.

$$y_h = c_1 \cos x + c_2 \sin x$$

$$w = 1$$

$$y_p = -\cos x \int \sin x \operatorname{cosec} x \, dx + \sin x \int \cos x \operatorname{cosec} x \, dx$$

Scanned with CamScanner

$$= -\cos x \int dx + \sin x \int \cot x \, dx$$

$$= -x\cos x + \sin x \ln|\sin x|$$

Soln $y = c_1\cos x + c_2 \sin x - x\cos x + \sin x \ln|\sin x|$

4) By method of variation of parameters solve $y'' + y = x\sin x$

$y_h = c_1\cos x + c_2 \sin x$

$w = 1$

$$y_p = -\cos x \int \sin x \, x \sin x \, dx + \sin x \int \cos x \, x \sin x \, dx$$

$$= -\cos x \int x \sin^2 x \, dx + \frac{\sin x}{2} \int x \sin 2x \, dx$$

$$= -\frac{\cos x}{2} \int x(1 - \cos 2x) \, dx + \frac{\sin x}{2} \int x \sin 2x \, dx$$

$$= -\frac{\cos x}{2} \int (x - x\cos 2x) \, dx + \frac{\sin x}{2} \int x \sin 2x \, dx$$

$$= -\frac{\cos x}{2} \left[\frac{x^2}{2} - \left(x \frac{\sin 2x}{2} + \frac{\cos 2x}{4} \right) \right] + \frac{\sin x}{2} \left[x\left(\frac{-\cos 2x}{2}\right) \right.$$

$$= -\frac{x^2 \cos x}{4} + \frac{x\cos x \sin 2x}{4} + \frac{\cos x \cos 2x}{8} - \frac{x \sin x \cos x}{4} + \frac{\sin x \sin 2x}{8}$$

Soln $y = y_h + y_p$

$$= c_1\cos x + c_2 \sin x - \frac{x^2 \cos x}{4} + \frac{x\cos x \sin 2x}{4} + \frac{\cos x \cos 2x}{8} - \frac{x \sin x \cos 2x}{4} + \frac{\sin x \sin 2x}{8}$$

Scanned with CamScanner