MODULE 2

VECTOR INTEGRAL THEOREMS

Greens Theorem (for simply connected domains, without proof) and applications to evaluating line integrals and finding areas. surface integrals over surfaces of the form $z = g(x,y)$, $y = g(x,z)$ or $x = g(y,z)$. Flux integrals over surfaces of the form $z = g(x,y)$ $y = g(x,z)$ or $x = g(y,z)$. Divergence theorem (without proof) and its applications to finding flux integrals, stokes theorem (without proof) and its applications to finding line integrals of vector fields and work done.

(Text 1 : Relevant topics from sections 15.4, 15.5, 15.6, 15.7, 15.8)

Greens Theorem

If $M(x,y)$ and $N(x,y)$ be continuous functions of x and y having continuous partial derivatives $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ in a region R of the xy plane bounded by a closed curve C, then

$$\oint_C (M\,dx + N\,dy) = \int_C (M\hat{i} + N\hat{j}) \cdot (dx\,\hat{i} + dy\,\hat{j})$$

$$= \int_C \vec{F} \cdot d\vec{r}$$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$$

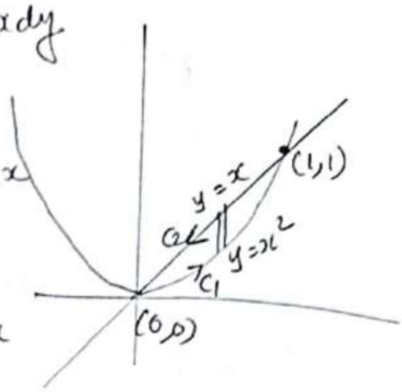
where C is taken in the anticlockwise direction.

Q1) Using Greens theorem, evaluate the line integral $\int_C (xy + y^2)\,dx + x^2\,dy$ where C is bounded by $y = x$ and $y = x^2$. Also check the answer by evaluating it directly.

Ans: Comparing with $\int_C M\,dx + N\,dy$ $\quad M = xy + y^2 \quad N = x^2$

$$\frac{\partial N}{\partial x} = 2x \qquad \frac{\partial M}{\partial y} = x+2y$$

$$\int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$



$$= \int_0^1 \int_{x^2}^{x} (x-2y)\, dy\, dx$$

$$= \int_0^1 \left(xy - y^2\right)_{x^2}^{x} dx$$

$$= \int_0^1 \left[(x^2 - x^2) - (x^3 - x^4)\right] dx$$

$$= \int_0^1 (x^4 - x^3)\, dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{5} - \frac{1}{4} = \frac{-1}{20}$$

$$\oint_C M dx + N dy = \int_{C_1} M dx + N dy + \int_{C_2} M dx + N dy \quad \text{— (1)}$$

Along C_1

$y = x^2 \qquad dy = 2x\,dx$

$$\int_{C_1} M dx + N dy = \int_{C_1} (xy + y^2) dx + x^2 dy$$

$$= \int_0^1 (x^3 + x^4)\, dx + x^2 \times 2x\, dx$$

$$= \int_0^1 (3x^3 + x^4)\, dx$$

$$= \left[\frac{3x^4}{4} + \frac{x^5}{5}\right]_0^1$$

$$= \frac{3}{4} + \frac{1}{5}$$

$$= \frac{19}{20}$$

Along C_2

$y = x \qquad dx = dy$

$$\int_{C_2} Mdx + Ndy = \int_{C_2} (xy + y^2)\,dx + x^2\,dy$$

$$= \int_1^0 (x^2 + x^2 + x^2)\,dx$$

$$= [x^3]_1^0 = -1$$

$$① \Rightarrow \int_C (xy + y^2)\,dx + x^2\,dy = \frac{19}{20} - 1 = \frac{-1}{20}$$

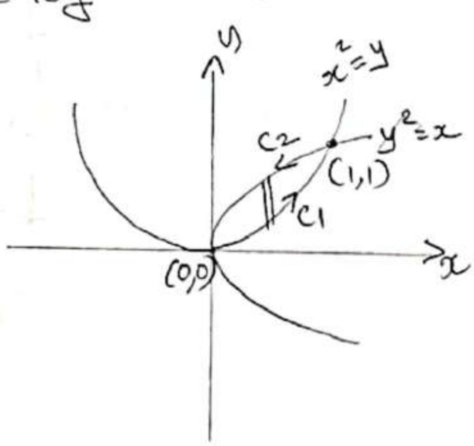
Hence Greens theorem is verified.

Q2) Using Greens theorem evaluate the line integral $\int_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$ where C is the boundary of the region defined by (a) $y = \sqrt{x}$, $y = x^2$ and oriented counterclockwise. (b) $x = 0$, $y = 0$, $x + y = 1$ and positively oriented. Also check the answer by evaluating it directly.

(a) ~~$y = \sqrt{x}$~~ $M = 3x^2 - 8y^2 \qquad N = 4y - 6xy$

$$\frac{\partial M}{\partial y} = -16y \qquad \frac{\partial N}{\partial x} = -6y$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -6y + 16y = 10y$$



$$\int_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\,dx$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} 10y \; dy\,dx$$

$$= 5\int_0^1 [y^2]_{x^2}^{\sqrt{x}} \, dx$$

$$= 5\int_0^1 (x - x^4)\,dx$$

$$= 5\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = 5\left[\frac{1}{2} - \frac{1}{5}\right] = \frac{3}{2}$$

LHS

$$\int_C Mdx + Ndy = \int_{C_1} Mdx + Ndy + \int_{C_2} Mdx + Ndy$$

Along $x^2 = y$

$2x\,dx = dy$ x varies from 0 to 1

$$\int_{C_1} Mdx + Ndy = \int_{C_1} (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$

$$= \int_0^1 (3x^2 - 8x^4)\,dx + (4x^2 - 6x^3)\,2x\,dx$$

$$= \int_0^1 \left[3x^2 + 8x^3 - 20x^4\right] dx$$

$$= \left[x^3 + 2x^4 - 4x^5\right]_0^1$$

$$= 1 + 2 - 4$$

$$= -1$$

Along $y^2 = x$ $2y\,dy = dx$

$$\int_{C_2} Mdx + Ndy = \int_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$

$$= \int_1^0 (3y^4 - 8y^2)\,2y\,dy + (4y - 6y^3)\,dy$$

$$= \int_1^0 \left[6y^5 - 22y^3 + 4y\right] dy$$

$$= \left[\frac{\cancel{8}y^6}{\cancel{5}} - \frac{22y^4}{4} + 2y^2\right]_1^0$$

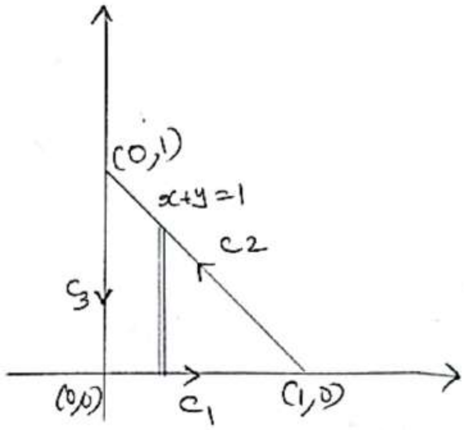
$$= 0 - \left[1\cancel{\tfrac{8}{5}} - \frac{22}{4} + 2\right]$$

$$= \cancel{\frac{11}{2} - \frac{3}{5} - 2} = \frac{11}{2} - 3 = \frac{5}{2}$$

$$\int_C M\,dx + N\,dy = -1 + \frac{5}{2} = \underline{\underline{\frac{3}{2}}}$$

Hence Greens theorem is verified.

(b) RHS



$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\,dx$$

$$= \int_0^1 \int_0^{1-x} 10y \, dy\,dx$$

$$= \int_0^1 5\left[y^2\right]_0^{1-x} dx$$

$$= 5\int (1-x)^2 \, dx$$

$$= 5\left[\frac{(1-x)^3}{-3}\right]_0^1$$

$$= -\frac{5}{3}\left[0-1\right] = \underline{\underline{\frac{5}{3}}}$$

LHS

$$\int_C (M\,dx + N\,dy) = \int_{C_1} (M\,dx + N\,dy) + \int_{C_2} (M\,dx + N\,dy) + \int_{C_3} (M\,dx + N\,dy)$$

Along C_1 $\quad y = 0 \quad dy = 0 \quad x$ varies from 0 to 1

$$\int_{C_1} (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy = \int_0^1 3x^2\,dx = \left[x^3\right]_0^1 = \underline{1}$$

Along C_2 $\quad y = 1-x \quad dy = -dx \quad x$ varies from 1 to 0

$$\int_{C_2} (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy = \int_1^0 3x^2 - 8(1-x)^2 dx + \left[4(1-x) - 6x(1-x)\right]dx$$

$$= \int_1^0 \left[3x^2 - 8(1-2x+x^2) - 4(1-x) + 6x(1-x)\right] dx$$

$$= \int_1^0 (3x^2 - 8 + 16x - 8x^2 - 4 + 4x + 6x - 6x^2)\,dx$$

$$= \int_1^0 (-11x^2 + 26x - 12)\,dx$$

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$$= \left[-\frac{11x^3}{3} + 13x^2 - 12x\right]_1^0$$

$$= -\left[-\frac{11}{3} + 13 - 12\right] = -\left[-\frac{11}{3} + 1\right] = \frac{8}{3}$$

Along C_3: $x = 0$, $dx = 0$ y varies from 1 to 0

$$\int_{C_3} (3x^2 - 8y^2)dx + (4y - 6xy)\,dy = \int_1^0 4y\,dy$$

$$= \left[2y^2\right]_1^0$$

$$= 0 - 2 = -2$$

LHS

$$\int_C M\,dx + N\,dy = 1 + \frac{8}{3} - 2 = \frac{5}{3}$$

Hence Greens Theorem is verified.

University Questions

Q3) Using Greens theorem evaluate $\int_C x\,dy - y\,dx$ where C is the circle $x^2 + y^2 = 4$ (May 2019)

Q4) Using Greens theorem evaluate $\int_C (xy + y^2)\,dx + x^2 dy$ where C is the boundary of the region bounded by $y = x^2$ and $x = y^2$ (May 2019)

Q5) Evaluate $\int_C (x^2 - 3y)\,dx + 3x\,dy$ where C is the circle $x^2 + y^2 = 4$ (DEC 201)

Q6) Use Greens theorem to evaluate $\int_C \log(1+y)\,dx - \frac{xy}{(1+y)}\,dy$ where C is the triangle with vertices (0,0) (2,0) and (0,4)

Q7) Verify Greens theorem for $\int_C (xy + y^2)\,dx + x^2 dy$ where C is bounded by $y = x$ and $y = x^2$ (July 2017)

Q8) Apply Greens theorem to evaluate $\int_C (2x^2 - y^2)\,dx + (x^2 + y^2)\,dy$ where C is the boundary of the area enclosed by the x axis and the upper half of the circle $x^2+y^2=a^2$ (July 2017)

Q9) using Greens theorem evaluate $\int y\,dx + x\,dy$ where C is the unit circle oriented counter clockwise (Dec 2016)

Q10) using Greens theorem evaluate $\oint_C (e^x+y^2)\,dx + (e^y+x^2)\,dy$ where C is the boundary of the region between $y=x^2$ and $y=2x$ (Dec 2016)

Answers

3) By greens theorem $\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$M = -y \qquad N = x$

$\frac{\partial M}{\partial y} = -1 \qquad \frac{\partial N}{\partial x} = 1$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2$

$$\int_C x\,dy - y\,dx = \iint 2\,dx\,dy$$
$$= 2\iint dx\,dy$$
$$= 2 \times \text{Area of the circle}$$
$$= 2 \times \pi \times 4$$
$$= \underline{\underline{8\pi}}$$

Q5) $\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

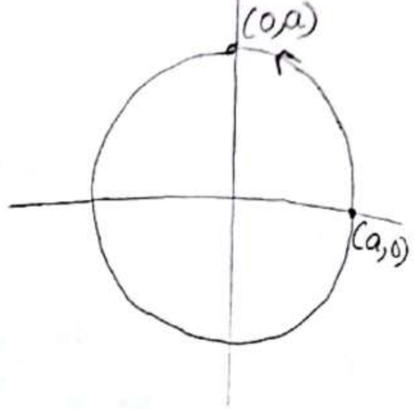
$M = x^2 - 3y \qquad N = 3x$

$\frac{\partial M}{\partial y} = -3 \qquad \frac{\partial N}{\partial x} = 3$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 6$

$$\therefore \int_C (x^2-3y)\,dx + 3x\,dy = 6\iint dx\,dy = 6 \times 4\pi = \underline{\underline{24\pi}}$$

Q8) $\int_C (2x^2-y^2)\,dx + (x^2+y^2)\,dy$



$M = 2x^2-y^2 \qquad N = x^2+y^2$

$\frac{\partial M}{\partial y} = -2y \qquad \frac{\partial N}{\partial x} = 2x$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2(x+y)$

By Greens theorem $\int_C (2x^2-y^2)\,dx + (x^2+y^2)\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx$

$$= \int \int 2(x+y)\,dx\,dy$$

$$= 2\int_0^{\pi}\int_0^{a} (\cos\theta + \sin\theta)\, r^2\,dr\,d\theta$$

$$= 2\int_0^{\pi} [\cos\theta + \sin\theta]\, \frac{a^3}{3}\,d\theta$$

$$= \frac{2a^3}{3}\left[\sin\theta - \cos\theta\right]_0^{\pi}$$

$$= \frac{2a^3}{3}\left[(0+1)-(0-1)\right]$$

$$= \underline{\underline{\frac{4a^3}{3}}}$$

Q9) $\int_C y\,dx + x\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$M = y \qquad N = x$

$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = 1$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$

$\therefore \int_C y\,dx + x\,dy = \iint_R 0\,dx\,dy$

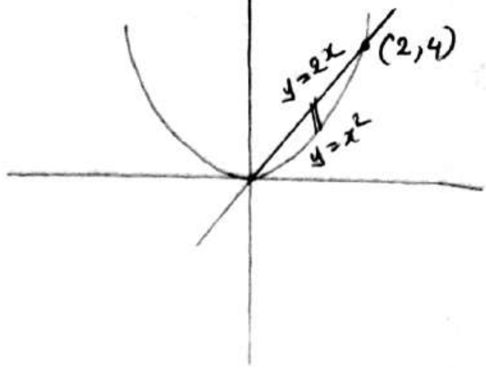
$$= \underline{\underline{0}}$$

Q10) $\int_C (e^x + y^2)\,dx + (e^y + x^2)\,dy$

$M = e^x + y^2 \qquad N = e^y + x^2$

$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = 2x$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2(x-y)$



$$\int_C (e^x + y^2)\,dx + (e^y + x^2)\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\,dx$$

$$= 2\int_0^2 \int_{x^2}^{2x} (x-y)\,dy\,dx$$

$$= 2\int_0^2 \left[xy - \frac{y^2}{2}\right]_{x^2}^{2x} dx$$

$$= 2\int_0^2 (2x^2 - 2x^2) - \left(x^3 - \frac{x^4}{2}\right) dx$$

$$= -2\int_0^2 \left(x^3 - \frac{x^4}{2}\right) dx$$

$$= -2\left[\frac{x^4}{4} - \frac{x^5}{10}\right]_0^2$$

$$= -2\left[\left(4 - \frac{16}{5}\right) - 0\right]$$

$$= \frac{-8}{5}$$

Application of Greens theorem to compute area

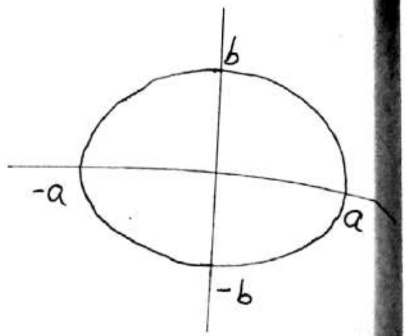
Let $x = x(t)$ and $y = y(t)$ be the parametric representation of a simple closed curve C, in the XY plane enclosing a simply connected region R having area A. Then

$$\text{Area} \quad A = \iint_R dx\,dy$$

$$= \int_C \left(\frac{1}{2}x\,dy - \frac{1}{2}y\,dx\right) = \frac{1}{2}\int_C (x\,dy - y\,dx)$$

use line integral to evaluate the area.

Q Find the area bounded by the ellipse with axes $2a$ and



Ans: equation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

parametric equation $x = a\cos\theta \quad y = b\sin\theta$

$dx = -a\sin\theta\, d\theta \quad dy = b\cos\theta\, d\theta$

$$\text{Area} = \frac{1}{2}\int_C (x\,dy - y\,dx)$$

$$= \frac{1}{2}\int_0^{2\pi}(a\cos\theta\, b\cos\theta + b\sin\theta\, a\sin\theta)\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} ab\, d\theta$$

$$= \frac{ab}{2}\left[\theta\right]_0^{2\pi}$$

$$= \frac{ab}{2}\left[2\pi - 0\right] = \underline{\pi ab}.$$

Q If R is a region bounded by a simple closed curve then using Greens theorem S.T the area of R is given by $\frac{1}{2}\oint_C -y\,dx + x\,dy$.

Ans: Comparing with $\int_C M\,dx + N\,dy$

$M = -y \qquad N = x$

$\frac{\partial M}{\partial y} = -1 \qquad \frac{\partial N}{\partial x} = 1$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1 - -1 = 2$

By Greens theorem $\frac{1}{2}\oint_C -y\,dx + x\,dy = \frac{1}{2}\iint_R 2\, dx\,dy$

$= \iint_R dx\,dy$

$=$ Area of the reg

Surface Integral

Let S be a differentiable parametric surface with vector equation $\vec{r} = x(u,v)i + y(u,v)j + z(u,v)k$ where (u,v) varies over a region R. Let $f(x,y,z)$ be a continuous function defined at all points in S. Then the surface integral is computed by

$$\iint_S f(x,y,z)\,ds = \iint_R f[x(u,v), y(u,v), z(u,v)] \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| dA$$

Q: Evaluate the surface integral $\iint_\sigma \frac{x^2+z^2}{y} ds$ over the surface σ represented by the vector valued function $r(u,v) = 2\cos v\, i + u j + 2\sin v\, k$

$1 \leq u \leq 3$, $0 \leq v \leq \pi$

Ans: $\vec{r}(u,v) = 2\cos v\, i + u j + 2\sin v\, k. \Rightarrow$ $x = 2\cos v$, $y = u$, $z = 2\sin v$

$$\frac{\partial \vec{r}}{\partial u} = 0i + j + 0k$$

$$\frac{\partial \vec{r}}{\partial v} = -2\sin v\, i + 0j + 2\cos v\, k$$

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ -2\sin v & 0 & 2\cos v \end{vmatrix} = i[2\cos v] - j[0] + k[2\sin v]$$

$$\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| = \sqrt{4\cos^2 v + 4\sin^2 v} = 2$$

$$\iint_S f(x,y,z)\,ds = \iint_R f[x(u,v), y(u,v), z(u,v)] \left| \frac{\partial \vec{r}}{\partial u} + \frac{\partial \vec{r}}{\partial v} \right| dA$$

$$\therefore \iint_\sigma \frac{x^2+z^2}{y} ds = \int_0^\pi \int_1^3 \frac{4\cos^2 v + 4\sin^2 v}{u} \times 2\,du\,dv$$

$$= \int_0^\pi \int_1^3 \frac{8}{u}\,du\,dv = \int_0^\pi 8[\log u]_1^3\,dv$$

$$= 8\log 3 \int_0^\pi dv = 8\pi\log 3$$

Surface integrals over $z = g(x,y)$, $y = g(x,z)$ and $x = g(y,z)$

(a) Let σ be a surface with equation $z = g(x,y)$ and let R be its projection on the xy plane. If g has continuous first partial derivatives on R and $f(x,y,z)$ is continuous on σ then

$$\iint_\sigma f(x,y,z)\,dS = \iint_R f(x,y,g(x,y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\; dA \quad (dxdy$$

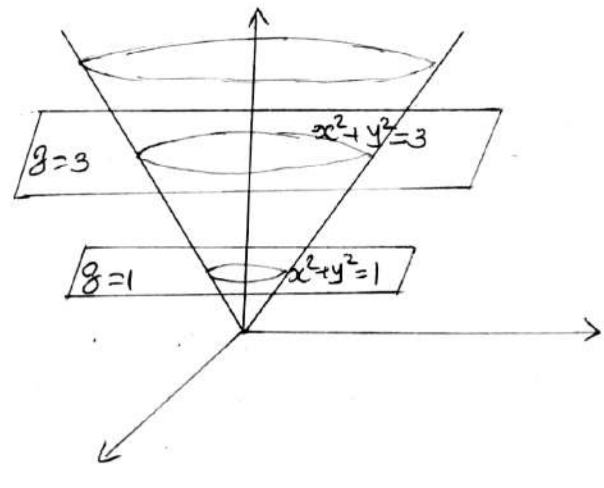
(b) Let σ be a surface with equation $y = g(x,z)$ and let R be its projection on xz plane. If g has continuous first partial derivatives on R and $f(x,y,z)$ is continuous on σ, then

$$\iint_\sigma f(x,y,z)\,dS = \iint_R f[x, g(x,z), z]\sqrt{\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + 1}\; dA. \quad (dxdz$$

(c) Let σ be a surface with equation $x = g(y,z)$ and let R be its projection on the yz-plane. If g has continuous first partial derivatives on R and $f(x,y,z)$ is continuous on σ, then

$$\iint_\sigma f(x,y,z)\,dS = \iint_R f[g(y,z), y, z]\sqrt{\left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 + 1}\; dA. \quad (dydz$$

Q Evaluate the surface integral $\iint_\sigma z^2\,dS$ where σ is the portion of the con $z = \sqrt{x^2+y^2}$ between the planes $z=1$ and $z=3$.



Surface integrals over $z = g(x,y)$, $y = g(x,z)$ and $x = g(y,z)$

(a) Let σ be a surface with equation $z = g(x,y)$ and let R be its projection on the xy plane. If g has continuous first partial derivatives on R and $f(x,y,z)$ is continuous on σ then

$$\iint_\sigma f(x,y,z)\,ds = \iint_R f(x,y,g(x,y)) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\; dA \quad (d$$

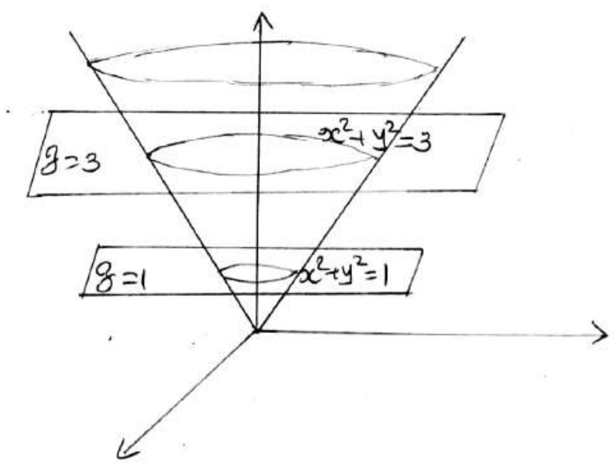
(b) Let σ be a surface with equation $y = g(x,z)$ and let R be its projection on xz plane. If g has continuous first partial derivatives on R and $f(x,y,z)$ is continuous on σ, then

$$\iint_\sigma f(x,y,z)\,ds = \iint_R f[x, g(x,z), z] \sqrt{\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + 1}\; dA. \quad (dx$$

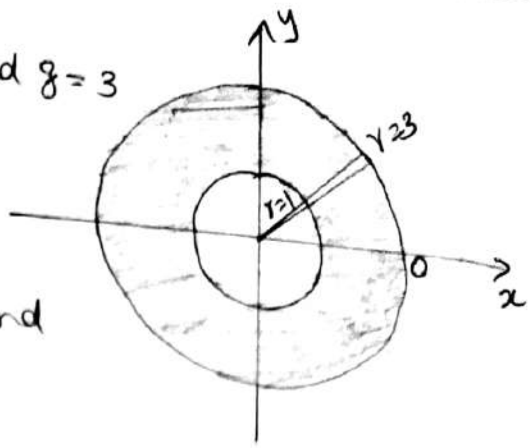
(c) Let σ be a surface with equation $x = g(y,z)$ and let R be its projection on the yz-plane. If g has continuous first partial derivatives on R and $f(x,y,z)$ is continuous on σ, then

$$\iint_\sigma f(x,y,z)\,ds = \iint_R f[g(y,z), y, z] \sqrt{\left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 + 1}\; dA. \quad (dy\,dz$$

Q Evaluate the surface integral $\iint_\sigma z^2\,ds$ where σ is the portion of the cone $z = \sqrt{x^2+y^2}$ between the planes $z=1$ and $z=3$.



no $z=\sqrt{x^2+y^2}$ between the planes $z=1$ and $z=3$



Let R be the orthogonal projection of the surface σ on the xy plane. R is the region between the circles $x^2+y^2=1$ and $x^2+y^2=9$.

$$\therefore \iint_{\sigma} f(x,y,z)\,ds = \iint_{R} f[x,y,z(x,y)]\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}\;dx\,dy \quad \text{—①}$$

$$z=\sqrt{x^2+y^2}$$

$$\frac{\partial z}{\partial x} = \frac{1\times 2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1} = \sqrt{\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}+1} = \sqrt{2}$$

$$① \Rightarrow \iint_{\sigma} z^2\,ds = \iint_{R} (x^2+y^2)\sqrt{2}\,dx\,dy$$

$$= \sqrt{2}\int_{0}^{2\pi}\int_{r=1}^{3} r^2\cdot r\,dr\,d\theta$$

$$= \frac{\sqrt{2}}{4}\int_{0}^{2\pi}\left[r^4\right]_1^3\,d\theta$$

$$= \frac{\sqrt{2}}{4}\times 80\,[\theta]_0^{2\pi}$$

$$= 20\sqrt{2}\times 2\pi$$

$$= 40\sqrt{2}\,\pi$$

Evaluation of flux

The volume of fluid that passes through the surface in one unit of time is flux

If σ is a smooth parametric surface and $\hat{n}$ be the unit normal towards the +ve side of σ at (x, y, z) then flux $\phi = \iint_{\sigma} \vec{F} \cdot \hat{n}\, ds$

If $\vec{r} = \vec{r}(u, v)$ then $\phi = \iint_{\sigma} \vec{F} \cdot \hat{n}\, ds = \iint_{R} \vec{F} \cdot \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right) dA$

where R is the region in uv - plane

Q. Determine the flux of the vector field $\vec{F} = x\hat{i} + y\hat{j}$ across the outward oriented sphere $x^2 + y^2 + z^2 = a^2$

Ans. Converting to spherical coordinates

$x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$ $z = \rho \cos\phi$ we have

$x^2 + y^2 + z^2 = a^2 \Rightarrow \rho^2 = a^2 \Rightarrow \rho = a$

$\vec{r} = x\hat{i} + y\hat{j} + zk$

$\vec{r} = a\sin\phi \cos\theta\, \hat{i} + a\sin\phi \sin\theta\, \hat{j} + a\cos\phi\, k$

$\frac{\partial \vec{r}}{\partial \phi} = a\cos\theta \cos\phi\, \hat{i} + a\sin\theta \cos\phi\, \hat{j} - a\sin\phi\, k$

$\frac{\partial \vec{r}}{\partial \theta} = -a\sin\phi \sin\theta\, \hat{i} + a\sin\phi \cos\theta\, \hat{j} + 0k$

$$\frac{\partial \vec{r}}{\partial \phi} \times \frac{\partial \vec{r}}{\partial \theta} = \begin{vmatrix} i & j & k \\ a\cos\theta\cos\phi & a\sin\theta\cos\phi & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$

$= i\left[a^2 \sin^2\phi \cos\theta\right] + j\left[a^2 \sin^2\phi \sin\theta\right] + k\left[a^2 \cos^2\theta \cos\phi \sin\phi + a^2 \sin^2\theta \cos\phi \sin\phi\right]$

$= a^2 \sin^2\phi \cos\theta\, \hat{i} + a^2 \sin^2\phi \sin\theta\, \hat{j} + a^2 \sin\phi \cos\phi\, k$

$\vec{F} = x\hat{i} + y\hat{j}$

$= a\sin\phi \cos\theta\, \hat{i} + a\sin\phi \sin\theta\, \hat{j}$

$$\vec{F}\cdot\left(\frac{\partial r}{\partial\phi}\times\frac{\partial r}{\partial\theta}\right) = a^3\sin^3\phi\cos^2\theta + a^3\sin^3\phi\sin^2\theta$$

$$= a^3\sin^3\phi$$

$$\text{flux} = \iint_R \vec{F}\cdot\left(\frac{\partial r}{\partial\phi}\times\frac{\partial r}{\partial\theta}\right)dA$$

$$= \int_0^{\pi}\int_0^{2\pi} a^3\sin^3\phi \; d\theta\, d\phi$$

$$= 2\pi a^3\int_0^{\pi}\sin^3\phi\, d\phi$$

$$= \frac{8\pi a^3}{3}$$

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Gauss Divergence Theorem

Let $\vec{F} = fi + gj + hk$ be a vector field with the functions f, g, h having continuous first order partial derivatives at all points in a solid region H having volume V with closed and bounded surface S having outward orientation. Then

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_H \nabla \cdot \vec{F}\, dV$$ where $\hat{n}$ is outward unit normal to the surface S.

The Gauss Divergence theorem states that the outward flux of a vector field across a closed surface is equal to the triple integral of the divergence over the region enclosed by the surface.

Q Use the divergence theorem to find the outward flux of the vector field $\vec{F}(x, y, z) = z\vec{k}$ across the sphere $x^2 + y^2 + z^2 = a^2$.

Ans: By divergence theorem, outward flux

$$\phi = \iint_\sigma \vec{F} \cdot \hat{n}\, ds = \iiint_H \nabla \cdot \vec{F}\, dV$$

$$= \iiint \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot z\vec{k}\, dV$$

$$= \iiint dV$$

$=$ volume of the sphere

$$= \frac{4}{3}\pi a^3$$

Q If S is the surface of the sphere $x^2 + y^2 + z^2 = 1$ evaluate

$$\iint_S (xi + 2yj + 3zk) \cdot \hat{n}\, ds$$

Ans: $\vec{F} = xi + 2yj + 3zk$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(3z) = 1+2+3 = 6$$

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dV$$

$$= 6 \iiint dV$$

$$= 6 \times \text{Volume}$$

$$= 6 \times \frac{4\pi}{3} = \underline{\underline{8\pi}}$$

Q Evaluate $\iint_S \vec{F} \cdot \hat{n}\, ds$ where $\vec{F} = axi + byj + czk$ and S the surface of the sphere $x^2+y^2+z^2 = 1$

Ans: $\nabla \cdot \vec{F} = a+b+c.$

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V (a+b+c)\, dV$$

$$= (a+b+c) \times \iiint dV$$

$$= (a+b+c) \times \frac{4\pi}{3}$$

$$= \underline{\underline{\frac{4\pi}{3}(a+b+c)}}$$

Q Using divergence theorem evaluate $\iint_S \vec{F} \cdot \hat{n}\, ds$ where $\vec{F} = x^3 i + y^3 j + z^3 k$. σ is the surface of the cylindrical solid bounded by $x^2+y^2 = 4$ $z=0$ and $z=4$.

Ans: $\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3)$

$$= 3x^2 + 3y^2 + 3z^2$$

$$= 3(x^2+y^2+z^2)$$

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dV$$

$$=3\iiint (x^2+y^2+z^2)\,dV$$

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$$= 3\int_0^4 \int_0^{2\pi} \int_{r=0}^{2} (r^2+z^2)\, r\, dr\, d\theta\, dz$$

$$= 3\int_0^4 \int_0^{2\pi} \int_0^{2} (r^3+z^2 r)\, dr\, d\theta\, dz$$

$$= 3\int_0^4 \int_0^{2\pi} \left[\frac{r^4}{4} + \frac{z^2 r^2}{2}\right]_0^2 d\theta\, dz$$

$$= 3\int_0^4 \int_0^{2\pi} (4+2z^2)\, d\theta\, dz$$

$$= 3\int_0^4 (4+2z^2)\left[\theta\right]_0^{2\pi} dz$$

$$= 6\pi \int_0^4 (4+2z^2)\, dz$$

$$= 6\pi\left[4z + \frac{2z^3}{3}\right]_0^4$$

$$= 6\pi\left[16 + \frac{128}{3}\right]$$

$$= 2\pi[176]$$

$$= 352\pi$$

Q using divergence theorem evaluate $\iint_S \vec{F}\cdot\hat{n}\, ds$ where $\vec{F} = (x^2+y)\,i + z^2 j + (e^y - z)\,k$ and S is the surface of the rectangular solid bounded by the coordinate planes and planes $x=3$, $y=1$, $z=3$.

Ans: $\text{div}\,\vec{F} = \frac{\partial}{\partial x}(x^2+y) + \frac{\partial}{\partial y}(z^2) + \frac{\partial}{\partial z}(e^y - z)$

$$= 2x - 1$$

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot\vec{F}\, dV$$

$$= \int_0^3 \int_0^1 \int_{z=0}^3 (2x-1)\, dz\, dy\, dx$$

$$= \int_0^3 \int_0^1 (2x-1)[z]_0^3\, dy\, dx$$

$$= 3\int_0^3 \int_0^1 (2x-1)\, dy\, dx$$

$$= 3\int_0^3 (2x-1)[y]_0^1\, dx$$

$$= 3\int_0^3 (2x-1)\, dx$$

$$= 3\left[x^2 - x\right]_0^3$$

$$= 3[6] = \underline{\underline{18}}$$

Q Use divergence theorem to find the outward flux of the vector field $\vec{F} = (2x+y^2)\hat{i} + xy\hat{j} + (xy-2z)k$ across the surface σ of the tetrahedron bounded by $x+y+z=2$ and the coordinate planes.

Ans.
$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(2x+y^2) + \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial z}(xy-2z)$$

$$= \cancel{2f+xj} \quad 2 + x + 0 - 2$$

$$= x$$

$$\text{Flux} = \iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot\vec{F}\, dV$$

$$= \iiint_V x\, dz\, dy\, dx$$

$$= \int_{x=0}^{2} \int_{y=0}^{2-x} \int_{z=0}^{2-x-y} x\, dz\, dy\, dx$$

$$= \int_0^2 \int_0^{2-x} x\,[z]_0^{2-x-y}\, dy\, dx$$

$$= \int_0^2 \int_0^{2-x} x(2-x-y)\, dy\, dx$$

$$= \int_0^2 \int_0^{2-x} (2x - x^2 - xy)\, dy\, dx$$

$$= \int_0^2 \left[\cancel{2(x} \; x(2-x)y - \frac{xy^2}{2} \right]_0^{2-x} dx$$

$$= \int_0^2 \left[x(2-x)^2 - \frac{x}{2}(2-x)^2 \right] dx$$

$$= \int_0^2 \frac{x}{2}(2-x)^2\, dx$$

$$= \frac{1}{2} \int_0^2 x(4 - 4x + x^2)\, dx$$

$$= \frac{1}{2} \int_0^2 (4x - 4x^2 + x^3)\, dx$$

$$= \frac{1}{2} \left[2x^2 - 4\frac{x^3}{3} + \frac{x^4}{4} \right]_0^2$$

$$= \frac{1}{2} \left[8 - \frac{32}{3} + 4 \right]$$

$$= \frac{1}{2} \times \frac{4}{3} \quad = \underline{\underline{\frac{2}{3}}}$$

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SOURCES AND SINKS

A point P in an incompressible flow is said to be a source if the divergence of the field $\vec{F}$ at the point is positive $\nabla \cdot \vec{F}(P) > 0$ and the point P is said to be a sink if the divergence at the point is negative, $\nabla \cdot \vec{F}(P) < 0$

An incompressible flow without sources and sinks can be characterized by the equation $\nabla \cdot \vec{F} = 0$

Q Determine whether the vector fields

$\vec{F}(x,y,z) = (y+z)\hat{i} - xz^3\hat{j} + x^2 \sin y\, k$ is free of source and sinks. If it is not, locate them.

Ans $\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(y+z) + \frac{\partial}{\partial y}(-xz^3) + \frac{\partial}{\partial z}(x^2 \sin y)$

$= 0$

No source and sink. $\vec{F}$ is free of sources and sinks at all points.

Q Determine whether the vector field $\vec{F} = x^3\hat{i} + y^3\hat{j} + 2z^3 k$ is free of source and sinks. If it is not locate them.

Ans. $\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(2z^3)$

$= 3x^2 + 3y^2 + 6z^2$

Vector $\vec{F}$ is free of source and sinks at $(0,0,0)$.

$\operatorname{div} \vec{F} > 0$ except at origin. So $\vec{F}$ has sources at all points except origin and no sinks.

STOKE'S THEOREM

If S be an open surface bounded by a closed curve C and $\vec{F} = F_1 i + F_2 j + F_3 k$ be any vector point function having continuous first order partial derivatives, then $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot \hat{n}\, ds$ where $\hat{n}$ is a unit normal vector at any point of S drawn in the sense in which a right handed screw would advance when rotated in the sense of description of C.

Note

* If S is in the form $z = g(x,y)$ and oriented up (+ve orientation) counter clockwise
  then $\iint_S \cancel{\vec{F} \cdot \hat{n}}$ $\quad \hat{n} = -\frac{\partial g}{\partial x} i - \frac{\partial g}{\partial y} j + k$

* If S is in the form $z = g(x,y)$ and oriented down (-ve orientation)
  then $\hat{n} = \frac{\partial g}{\partial x} i + \frac{\partial g}{\partial y} j - k$

* If S is in the xy plane $\hat{n} = k$

Problems

1. Use stoke's theorem to evaluate $\int_C e^x dx + 2y\,dy - dz$ where C is the curve $x^2+y^2=4$, $z=2$.

Ans. $\vec{F} = e^x i + 2y j - k$

$$\text{curl}\,\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix} = i[0-0] - j[0-0] + k[0-0]$$

$$= \vec{0}$$

By stokes theorem $\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\text{curl}\,\vec{F}) \cdot \hat{n}\, ds$

$$= \iint_S 0\, ds$$

$$= 0$$

2. Apply Stokes theorem to evaluate $\int_C (x+y)dx + (2x-y)dy + (y+z)dz$ where C is the boundary of the triangle with vertices $(0,0,0)$, $(2,0,$ and $(0,3,0)$

Ans $\vec{F} = (x+y)i + (2x-y)j + (y+z)k.$

$$\text{curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-y & y+z \end{vmatrix} = i[1-0] - j[0-0] + k[2-1] = i + k$$

$\hat{n} = k$

$\text{curl } \vec{F} \cdot \hat{n} = (i+k) \cdot k = 1$

By Stokes theorem $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \hat{n}\, ds$

$= \iint_S 1\, ds$

$= \iint_R dA$

$=$ Area of the triangle

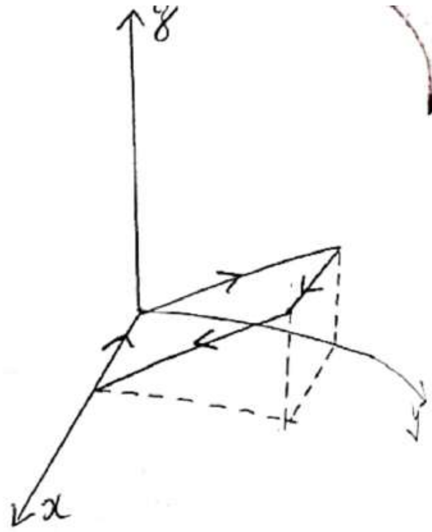
$= \frac{1}{2} \times 2 \times 3 = 3$

3. Use Stokes theorem to evaluate $\oint_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = xz\,i + 4x^2y^2 j + yx\,k$. C is the rectangle $0 \le x \le 1$, $0 \le$ in the plane $z = y$.

Ans: $\vec{F} = xz\,i + 4x^2y^2 j + yx\,k$

$$\text{curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & 4x^2y^2 & yx \end{vmatrix} = i[x-0] - j[y-x] + k[8x$$

$= x\,i + (x-y)j + 8xy^2 k$



$z = y$

$\frac{\partial z}{\partial x} = 0 \quad , \quad \frac{\partial z}{\partial y} = 1$

~~Counter~~ clockwise orientation $\therefore$

$\hat{n} = \frac{\partial z}{\partial x} i + \frac{\partial z}{\partial y} j - k$

$= 0i + j - k$

$\text{curl } \vec{F} \cdot \hat{n} = \left[x i + (x-y) j + 8xy^2 k \right] \cdot \left[0i + j - k \right]$

$= 0 + (x-y) - 8xy^2$

$= x - y - 8xy^2$

By stokes theorem $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \hat{n} \, ds$

$= \iint_S (x - y - 8xy^2) \, ds$

$= \int_0^1 \int_0^3 (x - y - 8xy^2) \, dy \, dx$

$= \int_0^1 \left[xy - \frac{y^2}{2} - 8x\frac{y^3}{3} \right]_0^3 dx$

$= \int_0^1 \left(3x - \frac{9}{2} - 72x \right) dx$

$= \int_0^1 \left(-69x - 9/2 \right) dx$

$= \left[-\frac{69x^2}{2} - \frac{9}{2}x \right]_0^1$

$= -\frac{69}{2} - \frac{9}{2} = \underline{-39}$

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