

MODULE-II

MULTI VARIABLE CALCULUS - DIFFERENTIATIONIntroduction - Basic Results

1. $\frac{d(k)}{dx} = 0$, where k is a constant

2. $\frac{d(1)}{dx} = 0$

$$(17) \frac{d(a^x)}{dx} = a^x \log a$$

3. $\frac{d(x^n)}{dx} = nx^{n-1}$

$$(18) \frac{d(\tan^{-1}x)}{dx} = \frac{1}{1+x^2}$$

4. $\frac{d(x)}{dx} = 1$

5. $\frac{d(\sqrt{x})}{dx} = \frac{1}{2\sqrt{x}}$

6. $\frac{d(e^x)}{dx} = e^x$

7. $\frac{d(e^{ax})}{dx} = ae^{ax}$

8. $\frac{d(\log x)}{dx} = \frac{1}{x}$

9. $\frac{d(\sin x)}{dx} = \cos x$

10. $\frac{d(\cos x)}{dx} = -\sin x$

11. $\frac{d(\tan x)}{dx} = \sec^2 x$

12. $\frac{d(\cot x)}{dx} = -\operatorname{cosec}^2 x$

13. $\frac{d(\sec x)}{dx} = \sec x \tan x$

14. $\frac{d(\operatorname{cosec} x)}{dx} = -\operatorname{cosec} x \cot x$

15 Product rule: $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$

i.e; $\frac{d}{dx}(\text{I}^{\text{st}} \text{ function} \times \text{II}^{\text{nd}} \text{ function}) = \text{I}^{\text{st}} \text{ function} \times \frac{d}{dx}(\text{II}^{\text{nd}} \text{ function})$
 $+ \text{II}^{\text{nd}} \text{ function} \times \frac{d}{dx}(\text{I}^{\text{st}} \text{ function}).$

16 Quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$

i.e; $\frac{d}{dx}\left(\frac{Nr}{Dr}\right) = \frac{Dr \cdot \frac{d(Nr)}{dx} - Nr \cdot \frac{d(Dr)}{dx}}{(Dr)^2}$

Partial derivatives of functions of two variables.

If $z = f(x, y)$ is a function of 2 variables x and y , then partial derivative of $f(x, y)$ w.r.t. x is the derivative of $f(x, y)$ w.r.t. x keeping y as a constant and is denoted as $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$ or $f_x(x, y)$.

If $z = f(x, y)$ is a function of 2 variables x and y , then partial derivative of $f(x, y)$ w.r.t. y is the derivative of $f(x, y)$ w.r.t. y keeping x as a constant and is denoted as $\frac{\partial z}{\partial y}$ or $\frac{\partial f}{\partial y}$ or $f_y(x, y)$.

Eg 1. $z = xy$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(xy) = y \cdot \frac{\partial}{\partial x}(x) = y \cdot 1 = \underline{\underline{y}}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(xy) = x \cdot \frac{\partial}{\partial y}(y) = x \cdot 1 = \underline{\underline{x}}$$

2. $z = x y^2$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(xy^2) = y^2 \frac{\partial}{\partial x}(x) = y^2 \cdot 1 = \underline{\underline{y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(xy^2) = x \cdot \frac{\partial}{\partial y}(y^2) = x \cdot 2y = \underline{\underline{2xy}}$$

$$3. z = x + y$$

$$\frac{\partial z}{\partial x} = \frac{\partial (x+y)}{\partial x} = \frac{\partial (x)}{\partial x} + \frac{\partial (y)}{\partial x} = 1+0 = \underline{\underline{1}}$$

$$\frac{\partial z}{\partial y} = \frac{\partial (x+y)}{\partial y} = \frac{\partial (x)}{\partial y} + \frac{\partial (y)}{\partial y} = 0+1 = \underline{\underline{1}}$$

$$4. z = x^3 y^2$$

$$\frac{\partial z}{\partial x} = \frac{\partial (x^3 y^2)}{\partial x} = y^2 \frac{\partial (x^3)}{\partial x} = y^2 \cdot 3x^2 = \underline{\underline{3x^2 y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{\partial (x^3 y^2)}{\partial y} = x^3 \frac{\partial (y^2)}{\partial y} = x^3 \cdot 2y = \underline{\underline{2x^3 y}}$$

Problems

$$1. \text{ If } f(x, y) = x^2 + y^2, \text{ then find } \frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y}$$

$$\text{Ans: } \frac{\partial f}{\partial x} = \frac{\partial (x^2 + y^2)}{\partial x} = \partial x + 0 = \underline{\underline{\partial x}}$$

$$\frac{\partial f}{\partial y} = \frac{\partial (x^2 + y^2)}{\partial y} = 0 + \partial y = \underline{\underline{\partial y}}$$

$$2. \text{ If } z = x + y + 3x^2 + 4, \text{ then find } \frac{\partial z}{\partial x} \text{ and } \frac{\partial z}{\partial y}$$

$$\text{Ans: } \frac{\partial z}{\partial x} = \frac{\partial (x + y + 3x^2 + 4)}{\partial x} = 1 + 0 + 6x + 0 \\ = \underline{\underline{1+6x}}$$

$$\frac{\partial z}{\partial y} = \frac{\partial (x + y + 3x^2 + 4)}{\partial y} = 0 + 1 + 0 + 0 = \underline{\underline{1}}$$

$$3. \text{ If } f(x, y) = 4x^3 y^2. \text{ Find (1) } f_x(x, y) \text{ (2) } f_y(x, y)$$

$$(3) f_x(1, y) \text{ (4) } f_x(x, 1) \text{ (5) } f_y(1, y) \text{ (6) } f_y(x, 1)$$

$$(7) f_x(1, 2) \text{ (8) } f_y(1, 2)$$

$$\text{Ans: (1) } f_x(x, y) = \frac{\partial f}{\partial x} = \frac{\partial (4x^3 y^2)}{\partial x} = 4y^2 \cdot 3x^2 = \underline{\underline{12x^2 y^2}}$$

$$(2) f_y(x, y) = \frac{\partial f}{\partial y} = \frac{\partial (4x^3 y^2)}{\partial y} = 4x^3 \cdot 2y = \underline{\underline{8x^3 y}}$$

$$(1) f_x(1, y) = 12 \times 1^2 \times y^2 = \underline{12y^2}$$

$$(2) f_x(x, 1) = 12 \times x^2 \times 1^2 = \underline{12x^2}$$

$$(3) f_y(1, y) = 8 \times 1^3 \times y = \underline{8y}$$

$$(4) f_y(x, 1) = 8 \times x^3 \times 1 = \underline{8x^3}$$

$$(5) f_z(1, z) = 12 \times 1^2 \times z^2 = 12 \times 4 = \underline{48}$$

$$(6) f_z(1, z) = 8 \times 1^3 \times z = \underline{8z}$$

4. Find $f_x(1, 3)$ and $f_y(1, 3)$ if $f(x, y) = 2x^3y^2 + 8y + 4x$

$$\text{Ans: } f_x(x, y) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(2x^3y^2 + 8y + 4x) \\ = \frac{\partial}{\partial x}(2x^3y^2) + \frac{\partial}{\partial x}(8y) + \frac{\partial}{\partial x}(4x) \\ = 6x^2y^2 + 0 + 4 = 6x^2y^2 + 4$$

$$\therefore f_x(1, 3) = 6 \times 1^2 \times 3^2 + 4 = 54 + 4 = \underline{58}$$

$$f_y(x, y) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(2x^3y^2 + 8y + 4x) \\ = 4x^3y + 0 + 0 = 4x^3y + 0$$

$$\therefore f_y(1, 3) = 4 \times 1^3 \times 3 + 0 = 12 + 0 = \underline{12}$$

5. If $z = e^{3x} \sin y$, find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, $\frac{\partial z}{\partial x}(0, y)$ and $\frac{\partial z}{\partial y}(x, 0)$.

$$\text{Ans: } \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(e^{3x} \sin y) = \sin y \cdot e^{3x} \cdot 3 \\ = \underline{3e^{3x} \sin y}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(e^{3x} \sin y) = \underline{e^{3x} \cos y}$$

$$\frac{\partial z}{\partial x}(0, y) = 3e^0 \sin y = 3 \times 1 \times \sin y = \underline{3 \sin y}$$

$$\frac{\partial z}{\partial y}(x, 0) = e^{3x} \cos 0 = e^{3x} \times 1 = \underline{e^{3x}}$$

6 Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z = x^4 \sin(xy^3)$

Ans:
$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial}{\partial x}(x^4 \sin(xy^3)) \\ &= x^4 \frac{\partial}{\partial x}(\sin(xy^3)) + \sin(xy^3) \frac{\partial}{\partial x}(x^4) \quad [\text{using product rule}] \\ &= x^4 \cos(xy^3)y^3 + \sin(xy^3) \cdot 4x^3 \\ &= \underline{\underline{x^4 y^3 \cos(xy^3) + 4x^3 \sin(xy^3)}} \\ \frac{\partial z}{\partial y} &= \frac{\partial}{\partial y}(x^4 \sin(xy^3)) \\ &= x^4 \cdot \frac{\partial}{\partial y}(\sin(xy^3)) \\ &= x^4 \cdot \cos(xy^3) \cdot x \cdot 3y^2 \\ &= \underline{\underline{3x^5 y^2 \cos(xy^3)}}\end{aligned}$$

7 If $f(x, y) = x \sin(xy)$, find $f_x(x, y)$ and $f_y(x, y)$?

Ans:
$$\begin{aligned}f_x(x, y) &= \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x \sin(xy)) \\ &= x \frac{\partial}{\partial x}(\sin(xy)) + \sin(xy) \frac{\partial}{\partial x}(x) \\ &= x \cos(xy)y + \sin(xy) \cdot 1 \\ &= \underline{\underline{xy \cos(xy) + \sin(xy)}}\end{aligned}$$

$$\begin{aligned}f_y(x, y) &= \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x \sin(xy)) \\ &= x \frac{\partial}{\partial y}(\sin(xy)) \\ &= x \cos(xy)x = \underline{\underline{x^2 \cos(xy)}}\end{aligned}$$

8 If $z = \cos(xy^3)$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

Ans:
$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial}{\partial x}(\cos(xy^3)) \\ &= -\sin(xy^3) \cdot y^3 = \underline{\underline{-y^3 \sin(xy^3)}}\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial y} &= \frac{\partial}{\partial y} (\cos(xy^3)) \\ &= -\sin(xy^3) \cdot x \cdot 3y^2 = \underline{-3xy^2 \sin(xy^3)}\end{aligned}$$

Note

1. The slope of the surface $z = f(x, y)$ in the x -direction at the point (x_0, y_0) is $f_x(x_0, y_0)$
2. The slope of the surface $z = f(x, y)$ in the y -direction at the point (x_0, y_0) is $f_y(x_0, y_0)$
3. The rate of change of z w.r.t. x at the point (x_0, y_0) is $\frac{\partial z}{\partial x}$ at (x_0, y_0) and the rate of change of z w.r.t. y at the point (x_0, y_0) is $\frac{\partial z}{\partial y}$ at (x_0, y_0)

Problems

1. Find the slope of the surface $f(x, y) = x^2y + 5y^3$ at the point $(1, -2)$ in the x -direction and in the y -direction.

Ans: Slope in the x -direction = $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2y + 5y^3)$
 $= 2xy + 0 = 2xy$

$$\frac{\partial f}{\partial x} \text{ at the point } (1, -2) = 2 \times 1 \times -2 = \underline{-4}$$

$$\begin{aligned}\text{Slope in the } y\text{-direction} &= \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2y + 5y^3) \\ &= x^2 + 15y^2\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial y} \text{ at the point } (1, -2) &= 1^2 + 15(-2)^2 \\ &= 1 + 15 \times 4 = 1 + 60 = \underline{61}\end{aligned}$$

U.G

2. Find the slope of $z = f(x, y)$ in the x -direction and in the y -direction at the point $(4, 0)$ if $z = xe^{-y} + 5y$

Ans: Slope in the x -direction = $f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(xe^{-y} + 5y)$
 $= e^{-y} \cdot 1 + 0 = e^{-y}$

$$\therefore f_x(4,0) = e^{-0} = \underline{\underline{1}}$$

Slope in the y -direction = $f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(xe^{-y} + 5y)$
 $= xe^{-y} \cdot (-1) + 5$
 $= -xe^{-y} + 5$

$$f_y(4,0) = -4e^{-0} + 5
= -4 \times 1 + 5 = -4 + 5 = \underline{\underline{1}}$$

3. $z = \sqrt{3x+2y}$. Find the rate of change of z w.r.t. x and y at the point $(2,5)$.

Ans: Rate of change of z w.r.t. x = $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(\sqrt{3x+2y})$
 $= \frac{1}{2\sqrt{3x+2y}} \cdot 3 = \frac{3}{2\sqrt{3x+2y}}$

$$\frac{\partial z}{\partial x} \text{ at } (2,5) = \frac{3}{2\sqrt{6+10}} = \frac{3}{2\sqrt{16}} = \frac{3}{2 \times 4} = \underline{\underline{\frac{3}{8}}}$$

Rate of change of z w.r.t. y = $\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(\sqrt{3x+2y})$
 $= \frac{1}{2\sqrt{3x+2y}} \cdot 2 = \frac{1}{\sqrt{3x+2y}}$

$$\frac{\partial z}{\partial y} \text{ at } (2,5) = \frac{1}{\sqrt{6+10}} = \frac{1}{\sqrt{16}} = \underline{\underline{\frac{1}{4}}}$$

4. Let $z = \sin(y^2 - 4x)$. Find the rate of change of z w.r.t. x and y at the point $(3,1)$.

Ans: Rate of change of z w.r.t. x = $\frac{\partial z}{\partial x}$
 $= \frac{\partial}{\partial x}(\sin(y^2 - 4x))$

$$= \cos(y^2 - 4x) \times -4 = -4 \cos(y^2 - 4x)$$

$$\frac{\partial z}{\partial x} \text{ at } (3, 1) = -4 \cos(1 - 12)$$

$$= -4 \cos(-11) = \underline{-4 \cos 11}$$

$$\text{Rate of change of } z \text{ w.r.t. } y = \frac{\partial z}{\partial y}$$

$$= \frac{\partial}{\partial y} (\sin(y^2 - 4x))$$

$$= \cos(y^2 - 4x) \cdot 2y$$

$$= 2y \cos(y^2 - 4x)$$

$$\frac{\partial z}{\partial y} \text{ at } (3, 1) = 2 \times 1 \times \cos(1 - 12)$$

$$= 2 \cos(-11)$$

$$= \underline{2 \cos 11}$$

5. $f(x, y) = xy^2$. Find slope in the x -direction and y -direction at the point $(2, 3)$.

$$\text{Ans: Slope in the } x\text{-direction} = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} (xy^2)$$

$$= y^2 \times 1 = y^2$$

$$\frac{\partial f}{\partial x} \text{ at } (2, 3) = \underline{9}$$

$$\text{Slope in the } y\text{-direction} = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (xy^2)$$

$$= x \times 2y = \underline{2xy}$$

$$\frac{\partial f}{\partial y} \text{ at } (2, 3) = 2 \times 2 \times 3 = \underline{12}$$

6. Let $z = (x+y)^{-1}$. Find the rate of change of z w.r.t. x and y at the point $(-1, 4)$.

$$\text{Ans: Rate of change of } z \text{ w.r.t. } x = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x} (x+y)^{-1}$$

$$= -1(x+y)^{-2} = \frac{-1}{(x+y)^2}$$

$$\frac{\partial z}{\partial x} \text{ at } (-1, 4) = \frac{-1}{(-1+4)^2} = \frac{-1}{3^2} = \underline{\underline{\frac{-1}{9}}}$$

$$\text{Rate of change of } z \text{ w.r.t. } y = \frac{\partial z}{\partial y} = \frac{\partial}{\partial y} (x+y)^{-1}$$

$$= -1(x+y)^{-2} = \frac{-1}{(x+y)^2}$$

$$\frac{\partial z}{\partial y} \text{ at } (-1, 4) = \frac{-1}{(-1+4)^2} = \frac{-1}{3^2} = \underline{\underline{-\frac{1}{9}}}$$

7. If $D = \sqrt{x^2+y^2}$. Find the rate of change of D w.r.t. x at $x=3$ and $y=4$.

Ans: Rate of change of D w.r.t. $x = \frac{\partial D}{\partial x}$

$$= \frac{\partial}{\partial x} (\sqrt{x^2+y^2})$$

$$= \frac{1}{2\sqrt{x^2+y^2}} \cdot \frac{\partial x}{\partial x}$$

$$= \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial D}{\partial x} \text{ at } (3, 4) = \frac{3}{\sqrt{3^2+4^2}} = \frac{3}{\sqrt{9+16}}$$

$$= \frac{3}{\sqrt{25}} = \underline{\underline{\frac{3}{5}}}$$

Partial derivatives of functions with more than 2 variables

Let $w = f(x, y, z)$ be a function of three variables, then the partial derivative of w w.r.t. x is calculated by holding y and z as constants and differentiating w.r.t. x and it is denoted as $\frac{\partial w}{\partial x}$ or $\frac{\partial f}{\partial x}$ or f_x or $f_x(x, y, z)$. Similarly we can calculate $\frac{\partial w}{\partial y}$ or $f_y(x, y, z)$ and $\frac{\partial w}{\partial z}$ or $f_z(x, y, z)$.

1. Q. If $f(x, y, z) = x^3y^2z^4 + 2xy + z$, find $f_x(x, y, z)$, $f_y(x, y, z)$ and $f_z(x, y, z)$.

$$\text{Ans: } f_x(x, y, z) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^3y^2z^4 + 2xy + z) \\ = y^2z^4 \times 3x^2 + 2y \times 1 + 0 \\ = \underline{\underline{3x^2y^2z^4 + 2y}}$$

$$f_y(x, y, z) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^3y^2z^4 + 2xy + z) \\ = x^3z^4 \times 2y + 2x \times 1 + 0 \\ = \underline{\underline{2x^3y^2z^4 + 2x}}$$

$$f_z(x, y, z) = \frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(x^3y^2z^4 + 2xy + z) \\ = x^3y^2 \times 4z^3 + 0 + 1 \\ = \underline{\underline{4x^3y^2z^3 + 1}}$$

2. Let $f(x, y, z) = x^2y^4z^3 + xy + z^2 + 1$. Find $f_x(x, y, z)$, $f_y(x, y, z)$, $f_z(x, y, z)$, $f_x(1, y, z)$, $f_y(1, z)$ and $f_z(1, 2, 3)$.

$$\text{Ans: } f_x(x, y, z) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2y^4z^3 + xy + z^2 + 1) \\ = y^4z^3 \times 2x + y + 0 + 0 \\ = \underline{\underline{2xy^4z^3 + y}}$$

$$f_y(x, y, z) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2y^4z^3 + xy + z^2 + 1) \\ = x^2z^3 \times 4y^3 + x + 0 + 0 \\ = \underline{\underline{4x^2y^3z^3 + x}}$$

$$f_z(x, y, z) = \frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(x^2y^4z^3 + xy + z^2 + 1) \\ = x^2y^4 \times 3z^2 + 0 + 2z + 0 = \underline{\underline{3x^2y^4z^2 + 2z}}$$

$$f_x(1, y, z) = 2 \times 1 \times y^4 z^3 + y = \underline{\underline{2y^4 z^3 + y}}$$

$$f_y(1, 2, z) = 4 \times 1^2 \times 2^3 z^3 + 1 = \underline{\underline{32z^3 + 1}}$$

$$\begin{aligned} f_z(1, 2, 3) &= 3 \times 1^2 \times 2^4 \times 3^2 + 2 \times 3 \\ &= 3 \times 16 \times 9 + 6 = 432 + 6 = \underline{\underline{438}} \end{aligned}$$

3. If $f(\rho, \theta, \phi) = \rho^2 \cos \phi \sin \theta$. Find $f_\rho(\rho, \theta, \phi)$, $f_\theta(\rho, \theta, \phi)$ and $f_\phi(\rho, \theta, \phi)$.

$$\begin{aligned} \text{Ans: } f_\rho(\rho, \theta, \phi) &= \frac{\partial f}{\partial \rho} = \frac{\partial}{\partial \rho} (\rho^2 \cos \phi \sin \theta) \\ &= \cos \phi \sin \theta \cdot 2\rho \\ &= \underline{\underline{2\rho \cos \phi \sin \theta}} \end{aligned}$$

$$\begin{aligned} f_\theta(\rho, \theta, \phi) &= \frac{\partial f}{\partial \theta} = \frac{\partial}{\partial \theta} (\rho^2 \cos \phi \sin \theta) \\ &= \underline{\underline{\rho^2 \cos \phi \cos \theta}} \end{aligned}$$

$$\begin{aligned} f_\phi(\rho, \theta, \phi) &= \frac{\partial f}{\partial \phi} = \frac{\partial}{\partial \phi} (\rho^2 \cos \phi \sin \theta) \\ &= \underline{\underline{\rho^2 \sin \theta \cdot -\sin \phi}} \\ &= \underline{\underline{-\rho^2 \sin \theta \sin \phi}} \end{aligned}$$

Higher order partial derivatives

Suppose that f is a function of 2 variables x and y , since the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are also functions of x and y , these functions may themselves have partial derivatives. This gives rise to four possible second order derivatives of f which are defined by,

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = f_{xx}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = f_{yy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = f_{xy}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = f_{yx}$$

Note

1. $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ or i.e; $f_{xy} = f_{yx}$

2. Third order, fourth order and higher order partial derivatives can be obtained by successive differentiation

$$\frac{\partial^3 f}{\partial x^3} = \frac{\partial}{\partial x} \left(\frac{\partial^2 f}{\partial x^2} \right) = f_{xxx}$$

$$\frac{\partial^4 f}{\partial y^4} = \frac{\partial}{\partial y} \left(\frac{\partial^3 f}{\partial y^3} \right) = f_{yyyy}$$

$$\frac{\partial^3 f}{\partial y^2 \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial^2 f}{\partial y \partial x} \right) = f_{yyx}$$

$$\frac{\partial^4 f}{\partial y^2 \partial x^2} = \frac{\partial}{\partial y} \left(\frac{\partial^3 f}{\partial y \partial x^2} \right) = f_{yxyx}$$

Problems

1. Find the second order partial derivatives of $f(x, y) = x^2y^3 + 4x^3y$.

Ans: $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} (x^2y^3 + 4x^3y) = y^3 \cdot 2x + 4y \cdot 3x^2$
 $= 2x y^3 + 12x^2 y$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (x^2y^3 + 4x^3y) = x^2 \cdot 3y^2 + 4x^3 \cdot 1$$
 $= 3x^2y^2 + 4x^3$

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} (2xy^3 + 12x^2y) \\&= 2y^3 \cancel{x} + 12y \times 2x \\&= \underline{\underline{2y^3 + 24xy}}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial y} (3x^2y^2 + 4x^3) \\&= 3x^2 \cdot 2y + 0 = \underline{\underline{6x^2y}}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial x} (3x^2y^2 + 4x^3) \\&= 3y^2 \cdot 2x + 12x^2 = \underline{\underline{6xy^2 + 12x^2}}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial y} (2xy^3 + 12x^2y) \\&= 2x \cdot 3y^2 + 12x^2 \cancel{x} \\&= \underline{\underline{6xy^2 + 12x^2y}}\end{aligned}$$

2. Let $f(x, y) = y^2 e^x + y$. Find f_{xyy}

$$\text{Ans: } f_{xyy} = \frac{\partial^3 f}{\partial x \partial y^2} = \frac{\partial}{\partial x} \left(\frac{\partial^2 f}{\partial y^2} \right)$$

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y} (y^2 e^x + y) = e^x \cdot 2y + 1 \\&= 2e^x y + 1\end{aligned}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial y} (2e^x y + 1) = 2e^x$$

$$\therefore f_{xyy} = \frac{\partial}{\partial x} \left(\frac{\partial^2 f}{\partial y^2} \right) = \frac{\partial}{\partial x} (2e^x) = \underline{\underline{2e^x}}$$

3. Find $\frac{\partial^2 z}{\partial x^2}$ if $z = \sqrt{x} \cdot \cos y$

$$\text{Ans: } \frac{\partial z}{\partial x} = \frac{\partial}{\partial x} (\sqrt{x} \cos y) = \cos y \cdot \frac{1}{2\sqrt{x}} = \frac{\cos y}{2\sqrt{x}}$$

$$\begin{aligned}
 \frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) \\
 &= \frac{\partial}{\partial x} \left(\frac{\cos y}{x^2} \right) = \frac{\partial}{\partial x} \left(\frac{\cos y}{x^{1/2}} \right) \\
 &= \frac{\cos y}{x^2} \cdot \frac{\partial}{\partial x} (x^{-1/2}) = \frac{\cos y}{x^2} \cdot \frac{-1}{2} x^{-\frac{1}{2}-1} \\
 &= \frac{-\cos y}{4} \cdot x^{-\frac{3}{2}} = \underline{\underline{\frac{-\cos y}{4x^{3/2}}}}
 \end{aligned}$$

11.10
4. Find the second order partial derivatives of

$$(a) z = 4x^3 - 2y + 7x^4y^5$$

$$(b) f(x, y) = 4x^2 - 8xy^4 + 7y^4 - 3$$

5. If $f(x, y) = \sqrt{x^2+y^2}$. Find $\frac{\partial^2 f}{\partial x \partial y}$

Ans: $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right)$

$$\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{x^2+y^2}} \cdot \partial y = \frac{y}{\sqrt{x^2+y^2}}$$

$$\begin{aligned}
 \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x} \left(\frac{y}{\sqrt{x^2+y^2}} \right) = \frac{\partial}{\partial x} \left(\frac{y}{(x^2+y^2)^{1/2}} \right) \\
 &= y \cdot \frac{\partial}{\partial x} \left((x^2+y^2)^{-1/2} \right) \\
 &= y \cdot \frac{-1}{2} (x^2+y^2)^{-\frac{1}{2}-1} \cdot \partial x \\
 &= \frac{-2xy}{2} (x^2+y^2)^{-\frac{3}{2}} \\
 &= \underline{\underline{\frac{-xy}{(x^2+y^2)^{3/2}}}}
 \end{aligned}$$

6. Find $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ if $f(x, y) = \ln(4x - 7y)$

$$\text{Ans: } \frac{\partial f}{\partial x} = \frac{1}{4x - 7y} \cdot 4 = \frac{4}{4x - 7y}$$

$$\frac{\partial f}{\partial y} = \frac{1}{4x - 7y} \cdot (-7) = \frac{-7}{4x - 7y}$$

$$\begin{aligned}\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial x} \left(\frac{-7}{4x - 7y} \right) \\ &= -7 \cdot \frac{\partial}{\partial x} \left((4x - 7y)^{-1} \right) \\ &= -7 \cdot -1 (4x - 7y)^{-1-1} \cdot 4 \\ &= 28 (4x - 7y)^{-2} \\ &= \underline{\underline{\frac{28}{(4x - 7y)^2}}}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial y} \left(\frac{4}{4x - 7y} \right) \\ &= 4 \frac{\partial}{\partial y} (4x - 7y)^{-1} \\ &= 4 \cdot -1 (4x - 7y)^{-1-1} \cdot -7 \\ &= 28 (4x - 7y)^{-2} = \underline{\underline{\frac{28}{(4x - 7y)^2}}}\end{aligned}$$

Note

$$1) \frac{d}{dx}(a^x) = a^x \log a$$

$$2) \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$3) \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$$

UQ
7. If $z = x^y$, then find $\frac{\partial^2 z}{\partial x \partial y}$

Ans: $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right)$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} (x^y) = x^y \log x$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} (x^y \log x)$$

$$= x^y \cdot \frac{\partial}{\partial x} (\log x) + \log x \frac{\partial}{\partial x} (x^y) \quad (\text{By product rule})$$

$$= x^y \cdot \frac{1}{x} + \log x \cdot y x^{y-1}$$

$$= x^y \cdot x^{-1} + y x^{y-1} \log x$$

$$= x^{y-1} + y x^{y-1} \log x$$

$$= \underline{\underline{x^{y-1} (1 + y \log x)}}$$

UQ
8. Prove that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ where $f = x^y y$

Ans: $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} (x^y y) = y \cdot \partial x = \partial x y$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (x^y y) = x^y \cdot 1 = x^y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial x} (x^y) = \partial x$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial y} (\partial x y) = \partial x$$

$$\therefore \frac{\partial^2 f}{\partial x \partial y} = \underline{\underline{\frac{\partial^2 f}{\partial y \partial x}}}$$

11. Show that the function $z = e^{-t} \sin\left(\frac{x}{c}\right)$ satisfy the heat equation $\frac{\partial z}{\partial t} = c^2 \cdot \frac{\partial^2 z}{\partial x^2}$.

$$\text{Ans: } \frac{\partial z}{\partial x} = \frac{\partial}{\partial x} \left(e^{-t} \sin\left(\frac{x}{c}\right) \right)$$

$$= e^{-t} \cdot \cos\left(\frac{x}{c}\right) \cdot \frac{1}{c} = \frac{e^{-t}}{c} \cdot \cos\left(\frac{x}{c}\right)$$

$$\frac{\partial z}{\partial t} = \frac{\partial}{\partial t} \left(e^{-t} \sin\left(\frac{x}{c}\right) \right)$$

$$= \sin\left(\frac{x}{c}\right) \cdot e^{-t-1} = -e^{-t} \sin\left(\frac{x}{c}\right) \quad \text{--- (1)}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{e^{-t}}{c} \cos\left(\frac{x}{c}\right) \right)$$

$$= \frac{e^{-t}}{c} \cdot -\sin\left(\frac{x}{c}\right) \cdot \frac{1}{c}$$

$$= -\frac{e^{-t}}{c^2} \sin\left(\frac{x}{c}\right)$$

$$\therefore c^2 \cdot \frac{\partial^2 z}{\partial x^2} = c^2 \cdot -\frac{e^{-t}}{c^2} \sin\left(\frac{x}{c}\right) = -e^{-t} \sin\left(\frac{x}{c}\right) \quad \text{--- (2)}$$

From (1) and (2), $\frac{\partial z}{\partial t} = c^2 \cdot \frac{\partial^2 z}{\partial x^2}$

12. Show that the function $f(x, y) = 2 \tan^{-1}\left(\frac{y}{x}\right)$ satisfy

$$f_{xx} + f_{yy} = 0$$

$$\text{Ans: } f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \left(2 \tan^{-1}\left(\frac{y}{x}\right) \right)$$

$$= 2 \cdot \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot y \cdot \frac{-1}{x^2} = \frac{-2y}{\left(1 + \frac{y^2}{x^2}\right)x^2}$$

$$= \frac{-2y}{\left(\frac{x^2 + y^2}{x^2}\right)x^2}$$

$$= \frac{-2y}{x^2 + y^2}$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \left(\alpha \tan^{-1}\left(\frac{y}{x}\right) \right)$$

$$= \alpha \cdot \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} \cdot 1 = \frac{\alpha^2}{\left(1 + \frac{y^2}{x^2}\right)x}$$

$$= \frac{\alpha^2}{\left(\frac{x^2 + y^2}{x^2}\right)x} = \frac{\alpha^2 x^2}{(x^2 + y^2)x} = \frac{\alpha^2 x}{x^2 + y^2}$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{-\alpha y}{x^2 + y^2} \right)$$

$$= -\alpha y \cdot \frac{\partial}{\partial x} \left((x^2 + y^2)^{-1} \right)$$

$$= -\alpha y \cdot -1 (x^2 + y^2)^{-1-1} \cdot 2x$$

$$= 4xy (x^2 + y^2)^{-2} = \frac{4xy}{(x^2 + y^2)^2} \quad \text{--- ①}$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial y} \left(\frac{\alpha x}{x^2 + y^2} \right)$$

$$= \alpha x \cdot \frac{\partial}{\partial y} \left((x^2 + y^2)^{-1} \right)$$

$$= \alpha x \cdot -1 (x^2 + y^2)^{-1-1} \cdot 2y$$

$$= -4xy (x^2 + y^2)^{-2} = \frac{-4xy}{(x^2 + y^2)^2} \quad \text{--- ②}$$

$$\textcircled{1} + \textcircled{2} \Rightarrow f_{xx} + f_{yy} = \frac{4xy}{(x^2 + y^2)^2} + \frac{-4xy}{(x^2 + y^2)^2} = 0$$

13. Show that the function $u(x, t) = \sin(x - ct)$ is a solution of one dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \cdot \frac{\partial^2 u}{\partial x^2}$$

Ans: $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} (\sin(x - ct)) = \cos(x - ct)$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t} (\sin(x - ct)) = \cos(x - ct) \cdot -c$$

$$= -c \cos(x - ct)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial x} (\cos(x-ct)) = -\sin(x-ct)$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial t} \right) = \frac{\partial}{\partial t} (-c \cdot \cos(x-ct))$$

$$= -c \cdot -\sin(x-ct) \cdot -c$$

$$= -c^2 \sin(x-ct) \quad \text{--- } ①$$

$$L \frac{\partial^2 u}{\partial x^2} = c^2 \cdot -\sin(x-ct)$$

$$= -c^2 \sin(x-ct) \quad \text{--- } ②$$

From ① and ②, $\frac{\partial^2 u}{\partial t^2} = L^2 \frac{\partial^2 u}{\partial x^2}$

Given $f = e^x \sin y$, show that the function satisfies the Laplace equation $f_{xx} + f_{yy} = 0$.

The chain rule

Chain rule for derivatives.

1. If $z = f(x, y)$, where $x = x(t)$ and $y = y(t)$, then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

2. If $w = f(x, y, z)$ where $x = x(t)$, $y = y(t)$ and $z = z(t)$,

$$\text{then } \frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}$$

Problems

1. Find $\frac{dz}{dt}$ using chain rule, $z = 3x^9y^3$, $x = t^4$, $y = t^3$

Ans: $\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$

$$z = 3x^9y^3$$

$$\therefore \frac{\partial z}{\partial x} = 6xy^3, \quad \frac{\partial z}{\partial y} = 9x^9y^2$$

$$x = t^4 \Rightarrow \frac{dx}{dt} = 4t^3$$

$$y = t^3 \Rightarrow \frac{dy}{dt} = 3t^2$$

$$\begin{aligned} \therefore \frac{dz}{dt} &= 6xy^3 \cdot 4t^3 + 9x^9y^2 \cdot 3t^2 \\ &= 24x^9y^3t^3 + 27x^9y^2t^2 \end{aligned}$$

$$\begin{aligned} &= 24(t^4)(t^3)^3t^3 + 27(t^4)^2(t^3)^2t^2 \\ &= 24t^{16} + 27t^{16} = \underline{\underline{51t^{16}}} \end{aligned}$$

Q8

2. Find the derivative of $w = x^2 + y^2$ w.r.t. t along the path $x = at^2$, $y = 2at$.

Ans: By chain rule $\frac{d\omega}{dt} = \frac{\partial\omega}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial\omega}{\partial y} \cdot \frac{dy}{dt}$

$$\omega = x^2 + y^2$$

$$\therefore \frac{\partial\omega}{\partial x} = 2x, \quad \frac{\partial\omega}{\partial y} = 2y$$

$$x = at^2 \Rightarrow \frac{dx}{dt} = 2at$$

$$y = 2at \Rightarrow \frac{dy}{dt} = 2a$$

$$\therefore \frac{d\omega}{dt} = 2x \cdot 2at + 2y \cdot 2a$$
$$= 4xat + 4ay$$

$$= 4at^2 \cdot at + 4a \cdot 2at = \underline{\underline{4a^3t^3 + 8a^2t}}$$

[OR] Another method

$$\omega = x^2 + y^2, \quad x = at^2 \quad y = 2at$$

$$\therefore \omega = (at^2)^2 + (2at)^2$$
$$= a^2t^4 + 4a^2t^2$$

$$\therefore \frac{d\omega}{dt} = a^2 \cdot 4t^3 + 4a^2 \cdot 2t = \underline{\underline{4a^3t^3 + 8a^2t}}$$

3. Suppose that $\omega = xy + yz$, $y = \sin x$, $z = e^x$. Use chain rule.

to find $\frac{d\omega}{dx}$

Ans: $\frac{d\omega}{dx} = \frac{\partial\omega}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial\omega}{\partial y} \cdot \frac{dy}{dx} + \frac{\partial\omega}{\partial z} \cdot \frac{dz}{dx}$

$$\omega = xy + yz$$

$$\therefore \frac{\partial\omega}{\partial x} = y, \quad \frac{\partial\omega}{\partial y} = x + z, \quad \frac{\partial\omega}{\partial z} = 0 + y = y$$

$$y = \sin x \Rightarrow \frac{dy}{dx} = \cos x$$

$$z = e^x \Rightarrow \frac{dz}{dx} = e^x$$

$$\therefore \frac{d\omega}{dx} = y \cdot 1 + (x+z)\cos x + y \cdot e^x$$
$$= \sin x + (x+e^x)\cos x + e^x \sin x$$
$$= \underline{\underline{\sin x + (x+e^x)\cos x + e^x \sin x}}$$

4. If $\omega = \sqrt{x^2 + y^2 + z^2}$, $x = \cos\theta$, $y = \sin\theta$, $z = \tan\theta$. Find $\frac{d\omega}{d\theta}$ when $\theta = \frac{\pi}{4}$.

Ans: $\frac{d\omega}{d\theta} = \frac{\partial\omega}{\partial x} \cdot \frac{dx}{d\theta} + \frac{\partial\omega}{\partial y} \cdot \frac{dy}{d\theta} + \frac{\partial\omega}{\partial z} \cdot \frac{dz}{d\theta} \quad \dots \text{--- } ①$

$$\omega = \sqrt{x^2 + y^2 + z^2}$$

$$\therefore \frac{\partial\omega}{\partial x} = \frac{1}{2\sqrt{x^2 + y^2 + z^2}} \cdot 2x = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial\omega}{\partial y} = \frac{1}{2\sqrt{x^2 + y^2 + z^2}} \cdot 2y = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial\omega}{\partial z} = \frac{1}{2\sqrt{x^2 + y^2 + z^2}} \cdot 2z = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$x = \cos\theta \Rightarrow \frac{dx}{d\theta} = -\sin\theta$$

$$y = \sin\theta \Rightarrow \frac{dy}{d\theta} = \cos\theta$$

$$z = \tan\theta \Rightarrow \frac{dz}{d\theta} = \sec^2\theta$$

$$\therefore ① \Rightarrow \frac{d\omega}{d\theta} = \frac{x}{\sqrt{x^2 + y^2 + z^2}} \cdot -\sin\theta + \frac{y}{\sqrt{x^2 + y^2 + z^2}} \cdot \cos\theta + \frac{z}{\sqrt{x^2 + y^2 + z^2}} \cdot \sec^2\theta$$

$$= \frac{1}{\sqrt{x^2 + y^2 + z^2}} (-x\sin\theta + y\cos\theta + z\sec^2\theta)$$

$$= \frac{1}{\sqrt{\cos^2\theta + \sin^2\theta + \tan^2\theta}} (-\cos\theta\sin\theta + \sin\theta\cos\theta + \tan\theta\sec^2\theta)$$

$$= \frac{\tan\theta\sec^2\theta}{\sqrt{1 + \tan^2\theta}}$$

$$\text{When } \theta = \frac{\pi}{4}, \frac{d\omega}{d\theta} = \frac{\tan\frac{\pi}{4} \cdot \sec^2\frac{\pi}{4}}{\sqrt{1 + \tan^2\frac{\pi}{4}}} = \frac{1 \cdot (\sqrt{2})^2}{\sqrt{1+1}} = \frac{(\sqrt{2})^2}{\sqrt{2}} = \underline{\underline{\sqrt{2}}}$$

Q.5 Find $\frac{dz}{dt}$ if $z = x^2y$, $x = t^2$, $y = t^3$ using chain rule

Ans: $\frac{dz}{dt} = 7t^6$.

Q.6 Using chain rule find $\frac{dz}{dt}$ if $z = 3\cos x - 5\sin(xy)$,
 $x = \frac{1}{t}$, $y = 4t$.

Ans: $\frac{dz}{dt} = \frac{3}{t^2} \sin\left(\frac{1}{t}\right)$

Q.7 If $w = \ln(3x^2 - 2y + 4z^3)$, $x = t^{1/2}$, $y = t^{2/3}$, $z = t^{-1}$. Find $\frac{dw}{dt}$.

Ans: $\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt} \quad \text{--- (1)}$

$w = \ln(3x^2 - 2y + 4z^3)$

$$\frac{\partial w}{\partial x} = \frac{1}{3x^2 - 2y + 4z^3} \cdot 6x = \frac{6x}{3x^2 - 2y + 4z^3}$$

$$\frac{\partial w}{\partial y} = \frac{1}{3x^2 - 2y + 4z^3} \cdot (-2) = \frac{-2}{3x^2 - 2y + 4z^3}$$

$$\frac{\partial w}{\partial z} = \frac{1}{3x^2 - 2y + 4z^3} \cdot 12z^2 = \frac{12z^2}{3x^2 - 2y + 4z^3}$$

$$x = t^{1/2} \Rightarrow \frac{dx}{dt} = \frac{1}{2} t^{\frac{1}{2}-1} = \frac{1}{2} t^{-1/2}$$

$$y = t^{2/3} \Rightarrow \frac{dy}{dt} = \frac{2}{3} t^{\frac{2}{3}-1} = \frac{2}{3} t^{-1/3}$$

$$z = t^{-1} \Rightarrow \frac{dz}{dt} = -1 t^{-1-1} = -t^{-2}$$

$$\therefore (1) \Rightarrow \frac{dw}{dt} = \frac{6x}{3x^2 - 2y + 4z^3} \cdot \frac{1}{2} t^{-1/2} - \frac{2}{3x^2 - 2y + 4z^3} \cdot \frac{2}{3} t^{-1/3} \\ + \frac{12z^2}{3x^2 - 2y + 4z^3} \cdot -t^{-2}$$

$$\begin{aligned}
 &= \frac{1}{3x^2 - 2y + 4z^3} \left(\frac{6xt^{-\frac{1}{2}}}{2} - \frac{4t^{-\frac{1}{3}}}{3} + -12z^2 t^{-2} \right) \\
 &= \frac{1}{3(t^{1/2})^2 - 2t^{2/3} + 4(t^{-1})^3} \left(3t^{1/2} - \frac{4t^{-\frac{1}{3}}}{3} - 12(t^{-1})t^{-2} \right) \\
 &= \frac{1}{3t - 2t^{2/3} + 4t^{-3}} \left(3 - \frac{4}{3}t^{-\frac{1}{3}} - 12t^{-4} \right) \\
 &= \frac{3 - \frac{4}{3}t^{-\frac{1}{3}} - 12t^{-4}}{3t - 2t^{2/3} + 4t^{-3}}
 \end{aligned}$$

Chain rule for partial derivatives

If $z = f(x, y)$ where $x = x(u, v)$ and $y = y(u, v)$ and $z = z(u, v)$ where x and y are functions of u and v . Then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

If $w = f(x, y, z)$ where $x = x(u, v)$, $y = y(u, v)$ and $z = z(u, v)$, then

$$\begin{aligned}
 \frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial u} \\
 \frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial v}
 \end{aligned}$$

Problems

- U.Q 1. Given that $z = e^{xy}$, $x = 2u+v$, $y = \frac{u}{v}$. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ using chain rule.

Ans: $z = e^{xy} \therefore \frac{\partial z}{\partial x} = e^{xy} \cdot y$

$$\frac{\partial z}{\partial y} = e^{xy} \cdot x$$

$$z = 2u+v \Rightarrow \frac{\partial z}{\partial u} = 2 \text{ and } \frac{\partial z}{\partial v} = 1$$

$$y = \frac{u}{v} \Rightarrow \frac{\partial y}{\partial u} = \frac{1}{v} \text{ and } \frac{\partial y}{\partial v} = u \cdot \frac{-1}{v^2} = \frac{-u}{v^2}$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$= e^{xy} y \cdot 2 + e^{xy} x \cdot \frac{1}{v}$$

$$= e^{(2u+v)\frac{u}{v}} \cdot \frac{u}{v} \cdot 2 + e^{(2u+v)\frac{u}{v}} (2u+v) \cdot \frac{1}{v}$$

$$= e^{(2u+v)\frac{u}{v}} \left(\frac{2u}{v} + \frac{(2u+v)}{v} \right)$$

$$= \underline{\underline{\frac{e^{(2u+v)\frac{u}{v}}}{v} \left(4u+v \right)}}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$= e^{xy} y \cdot 1 + e^{xy} x \cdot \frac{-u}{v^2}$$

$$= e^{xy} \left(y - \frac{xu}{v^2} \right)$$

$$= \underline{\underline{e^{(2u+v)\frac{u}{v}} \left(\frac{u}{v} - \frac{(2u+v)u}{v^2} \right)}}$$

$$= e^{(2u+v)\frac{u}{v}} \left(\frac{u}{v} - \left(\frac{2u^2+vu}{v^2} \right) \right)$$

$$= e^{(2u+v)\frac{u}{v}} \left(\frac{u}{v} - \frac{2u^2}{v^2} - \frac{vu}{v^2} \right)$$

$$= e^{(2u+v)\frac{u}{v}} \left(\frac{u}{v} - \frac{2u^2}{v^2} - \frac{u}{v} \right)$$

$$= \underline{\underline{e^{(2u+v)\frac{u}{v}} \left(\frac{-2u^2}{v^2} \right)}}$$

Q. Suppose that $\omega = x^2 + y^2 - z^2$, $x = s \sin \phi \cos \theta$,
 $y = s \sin \phi \sin \theta$, $z = s \cos \phi$. Find $\frac{\partial \omega}{\partial s}$ and $\frac{\partial \omega}{\partial \theta}$?

Ans:
$$\begin{aligned}\frac{\partial \omega}{\partial s} &= \frac{\partial \omega}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial \omega}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial \omega}{\partial z} \cdot \frac{\partial z}{\partial s} \\&= 2x \cdot \sin \phi \cos \theta + 2y \cdot \sin \phi \sin \theta + -2z \cdot \cos \phi \\&= 2s \sin \phi \cos \theta \cdot \sin \phi \cos \theta + 2s \sin \phi \sin \theta \cdot \sin \phi \sin \theta \\&\quad - 2s \cos \phi \cdot \cos \phi \\&= 2s \sin^2 \phi \cos^2 \theta + 2s \sin^2 \phi \sin^2 \theta - 2s \cos^2 \phi \\&= 2s \sin^2 \phi (\cos^2 \theta + \sin^2 \theta) - 2s \cos^2 \phi \\&= 2s \sin^2 \phi (1) - 2s \cos^2 \phi \\&= 2s \sin^2 \phi - 2s \cos^2 \phi \\&= -2s(\cos^2 \phi - \sin^2 \phi) \\&= \underline{-2s \cos 2\phi} \quad (\cos^2 \phi - \sin^2 \phi = \cos 2\phi)\end{aligned}$$

$$\begin{aligned}\frac{\partial \omega}{\partial \theta} &= \frac{\partial \omega}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial \omega}{\partial y} \cdot \frac{\partial y}{\partial \theta} + \frac{\partial \omega}{\partial z} \cdot \frac{\partial z}{\partial \theta} \\&= 2x \cdot s \sin \phi \cdot \sin \theta + 2y \cdot s \sin \phi \cos \theta + -2z \cdot 0 \\&= -2s \sin \phi \cos \theta \cdot s \sin \phi \sin \theta + 2s \sin \phi \sin \theta \cdot s \sin \phi \cos \theta \\&= -2s^2 \sin^2 \phi \sin \theta \cos \theta + 2s^2 \sin^2 \phi \sin \theta \cos \theta \\&= \underline{0}\end{aligned}$$

U.Q
 3. Let $\omega = 4x^2 + 4y^2 + z^2$, where $x = s \sin \phi \cos \theta$,
 $y = s \sin \phi \sin \theta$, $z = s \cos \phi$. Find $\frac{\partial \omega}{\partial s}$ using
 chain rule.

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial s}$$

$$= sx \cdot \sin\phi \cos\theta + sy \cdot \sin\phi \sin\theta + sz \cos\phi$$

$$= 8s \sin\phi \cos\theta \cdot \sin\phi \cos\theta + 8s \sin\phi \sin\theta \cdot \sin\phi \sin\theta + 8s \cos\phi \cos\theta$$

$$= 8s \sin^2\phi \cos^2\theta + 8s \sin^2\phi \sin^2\theta + 8s \cos^2\phi$$

$$= 8s \sin^2\phi (\cos^2\theta + \sin^2\theta) + 8s \cos^2\phi$$

$$= 8s \sin^2\phi + 8s \cos^2\phi$$

$$= \underline{8s(4\sin^2\phi + \cos^2\phi)}$$

H.W 4. using chain rule find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ if $z = x^2 - y \tan x$,

$$x = \frac{u}{v}, \quad y = uv$$

Ans: $\frac{\partial z}{\partial u} = \frac{\partial u}{v^2} - u \sec^2\left(\frac{u}{v}\right) - v \tan\left(\frac{u}{v}\right)$

$$\frac{\partial z}{\partial v} = \frac{-2u^2}{v^3} + \frac{u^2}{v} \sec^2\left(\frac{u}{v}\right) - u \tan\left(\frac{u}{v}\right)$$

H.W 5. $w = e^{xyz}$, $x = 3u + v$, $y = 3u - v$, $z = u^2v$. Find $\frac{\partial w}{\partial u}$

and $\frac{\partial w}{\partial v}$

Ans: $\frac{\partial w}{\partial u} = e^{(9u^2-v^2)u^2v} (36u^3v - 2uv^3)$

$$\frac{\partial w}{\partial v} = e^{(9u^2-v^2)u^2v} (9u^4 - 3u^2v^2)$$

H.W 6. Let f be a differentiable function of 3 variables and suppose that $w = f(x-y, y-z, z-x)$. Prove that

$$\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

Ans: Let $w = f(P, Q, R)$ where $P = x-y$, $Q = y-z$, $R = z-x$

$$\frac{\partial \omega}{\partial z} = \frac{\partial \omega}{\partial P} \cdot \frac{\partial P}{\partial z} + \frac{\partial \omega}{\partial Q} \cdot \frac{\partial Q}{\partial z} + \frac{\partial \omega}{\partial R} \cdot \frac{\partial R}{\partial z}$$

$$= \frac{\partial \omega}{\partial P} \cdot 1 + \frac{\partial \omega}{\partial Q} \cdot 0 + \frac{\partial \omega}{\partial R} \cdot -1$$

$$= \frac{\partial \omega}{\partial P} - \frac{\partial \omega}{\partial R} \quad \text{--- } ①$$

$$\frac{\partial \omega}{\partial y} = \frac{\partial \omega}{\partial P} \cdot \frac{\partial P}{\partial y} + \frac{\partial \omega}{\partial Q} \cdot \frac{\partial Q}{\partial y} + \frac{\partial \omega}{\partial R} \cdot \frac{\partial R}{\partial y}$$

$$= \frac{\partial \omega}{\partial P} \cdot -1 + \frac{\partial \omega}{\partial Q} \cdot 1 + \frac{\partial \omega}{\partial R} \cdot 0$$

$$= -\frac{\partial \omega}{\partial P} + \frac{\partial \omega}{\partial Q} \quad \text{--- } ②$$

$$\frac{\partial \omega}{\partial z} = \frac{\partial \omega}{\partial P} \cdot \frac{\partial P}{\partial z} + \frac{\partial \omega}{\partial Q} \cdot \frac{\partial Q}{\partial z} + \frac{\partial \omega}{\partial R} \cdot \frac{\partial R}{\partial z}$$

$$= \frac{\partial \omega}{\partial P} \cdot 0 + \frac{\partial \omega}{\partial Q} \cdot -1 + \frac{\partial \omega}{\partial R} \cdot 1$$

$$= -\frac{\partial \omega}{\partial Q} + \frac{\partial \omega}{\partial R} \quad \text{--- } ③$$

$$\textcircled{1} + \textcircled{2} + \textcircled{3} \Rightarrow \frac{\partial \omega}{\partial x} + \frac{\partial \omega}{\partial y} + \frac{\partial \omega}{\partial z} = \frac{\partial \omega}{\partial P} - \frac{\partial \omega}{\partial R} - \frac{\partial \omega}{\partial P}$$

$$+ \frac{\partial \omega}{\partial Q} - \frac{\partial \omega}{\partial Q} + \frac{\partial \omega}{\partial R}$$

$$= \underline{\underline{0}}$$

Q. If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$, show that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0$.

Ans. Let $u = f(P, Q, R)$ where $P = \frac{x}{y}$, $Q = \frac{y}{z}$, $R = \frac{z}{x}$

$$\begin{aligned} \therefore \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial P} \cdot \frac{\partial P}{\partial x} + \frac{\partial u}{\partial Q} \cdot \frac{\partial Q}{\partial x} + \frac{\partial u}{\partial R} \cdot \frac{\partial R}{\partial x} \\ &= \frac{\partial u}{\partial P} \cdot \frac{1}{y} + \frac{\partial u}{\partial Q} \cdot 0 + \frac{\partial u}{\partial R} \cdot -\frac{z}{x^2} \end{aligned}$$

$$\therefore x \frac{\partial u}{\partial x} = \frac{\partial u}{\partial P} \cdot \frac{x}{y} - \frac{\partial u}{\partial R} \cdot \frac{z}{x^2} \cdot x$$

$$\Rightarrow x \frac{\partial u}{\partial x} = \frac{x}{y} \frac{\partial u}{\partial P} - \frac{z}{x} \frac{\partial u}{\partial R} \quad \text{--- } ①$$

$$\begin{aligned}\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial P} \cdot \frac{\partial P}{\partial y} + \frac{\partial u}{\partial Q} \cdot \frac{\partial Q}{\partial y} + \frac{\partial u}{\partial R} \cdot \frac{\partial R}{\partial y} \\ &= \frac{\partial u}{\partial P} \cdot \frac{-x}{y^2} + \frac{\partial u}{\partial Q} \cdot \frac{1}{z} + \frac{\partial u}{\partial R} \cdot 0\end{aligned}$$

$$\begin{aligned}\therefore y \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial P} \cdot \frac{-x}{y^2} \cdot y + \frac{\partial u}{\partial Q} \cdot \frac{y}{z} \\ &= -\frac{x}{y} \frac{\partial u}{\partial P} + \frac{y}{z} \frac{\partial u}{\partial Q} \quad \text{--- } ②\end{aligned}$$

$$\begin{aligned}\frac{\partial u}{\partial z} &= \frac{\partial u}{\partial P} \cdot \frac{\partial P}{\partial z} + \frac{\partial u}{\partial Q} \cdot \frac{\partial Q}{\partial z} + \frac{\partial u}{\partial R} \cdot \frac{\partial R}{\partial z} \\ &= \frac{\partial u}{\partial P} \cdot 0 + \frac{\partial u}{\partial Q} \cdot \frac{-y}{z^2} + \frac{\partial u}{\partial R} \cdot \frac{1}{x}\end{aligned}$$

$$\begin{aligned}\therefore z \frac{\partial u}{\partial z} &= \frac{\partial u}{\partial Q} \cdot \frac{-y}{z^2} \cdot z + \frac{\partial u}{\partial R} \cdot \frac{1}{x} \cdot z \\ &= -\frac{y}{z} \frac{\partial u}{\partial Q} + \frac{z}{x} \frac{\partial u}{\partial R} \quad \text{--- } ③\end{aligned}$$

$$\begin{aligned}① + ② + ③ \Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} &= \frac{x}{y} \frac{\partial u}{\partial P} - \frac{z}{x} \frac{\partial u}{\partial R} \\ - \frac{x}{y} \frac{\partial u}{\partial P} + \frac{y}{z} \frac{\partial u}{\partial Q} - \frac{y}{z} \frac{\partial u}{\partial Q} + \frac{z}{x} \frac{\partial u}{\partial R} &= 0\end{aligned}$$

8. Let $z = f(x, y)$ where $x = r \cos \theta$, $y = r \sin \theta$. Prove that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta}\right)^2.$$

$$\begin{aligned}\text{Ans: } \frac{\partial z}{\partial x} &= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial x} \\ &= \frac{\partial z}{\partial x} \cdot \cos \theta + \frac{\partial z}{\partial y} \cdot \sin \theta.\end{aligned}$$

$$\begin{aligned}\left(\frac{\partial z}{\partial x}\right)^2 &= \left(\frac{\partial z}{\partial x} \cos \theta + \frac{\partial z}{\partial y} \sin \theta\right)^2 \\ &= \left(\frac{\partial z}{\partial x}\right)^2 \cos^2 \theta + \left(\frac{\partial z}{\partial y}\right)^2 \sin^2 \theta + 2 \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} \cos \theta \sin \theta. \quad \text{--- } 1\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial y} &= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial y} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial y} \\ &= \frac{\partial z}{\partial x} \cdot r \sin \theta + \frac{\partial z}{\partial y} \cdot r \cos \theta\end{aligned}$$

$$\begin{aligned}
 \left(\frac{\partial z}{\partial \theta}\right)^2 &= \left(-r \sin \theta \frac{\partial z}{\partial x} + r \cos \theta \frac{\partial z}{\partial y}\right)^2 \\
 &= r^2 \sin^2 \theta \left(\frac{\partial z}{\partial x}\right)^2 + r^2 \cos^2 \theta \left(\frac{\partial z}{\partial y}\right)^2 - 2 \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} r^2 \sin \theta \cos \theta \\
 \therefore \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta}\right)^2 &= \sin^2 \theta \left(\frac{\partial z}{\partial x}\right)^2 + \cos^2 \theta \left(\frac{\partial z}{\partial y}\right)^2 - 2 \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} \frac{\sin \theta \cos \theta}{r^2} \\
 \textcircled{1} + \textcircled{2} &\Rightarrow \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 \cos^2 \theta + \left(\frac{\partial z}{\partial y}\right)^2 \sin^2 \theta + \alpha^2 \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} \cos \theta \\
 &\quad + \sin^2 \theta \left(\frac{\partial z}{\partial x}\right)^2 + \cos^2 \theta \left(\frac{\partial z}{\partial y}\right)^2 - 2 \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} \frac{\sin \theta \cos \theta}{r^2} \\
 &= \left(\frac{\partial z}{\partial x}\right)^2 (\cos^2 \theta + \sin^2 \theta) + \left(\frac{\partial z}{\partial y}\right)^2 (\sin^2 \theta + \cos^2 \theta) \\
 &= \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 \\
 \therefore \underline{\underline{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}} &= \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta}\right)^2
 \end{aligned}$$

Total differential

If $z = f(x, y)$, then total differential of z is dz
 and is defined as $dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$
 $= \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$

Problems

1. Compute the differential dz of the function

$$z = \tan^{-1}(xy)$$

$$\begin{aligned}
 \text{Ans: } dz &= \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \\
 &= \frac{1}{1+(xy)^2} \cdot y dx + \frac{1}{1+(xy)^2} \cdot x dy \\
 &= \underline{\underline{\frac{y dx + x dy}{1+(xy)^2}}}
 \end{aligned}$$

Local linearity

Local linear approximation of $f(x, y)$ at (x_0, y_0)

is given by

$$L(x, y) = f(x_0, y_0) + (x - x_0) f_x(x_0, y_0) + (y - y_0) f_y(x_0, y_0)$$

and the error in approximation is

$$E(x, y) = f(x, y) - L(x, y).$$

- Q. 1. Find the local linear approximation L to $f(x, y) = \ln(xy)$ at $P(1, 2)$ and compare the errors in approximating f by L at $Q(1.01, 2.01)$ with the distance between P and Q .

Ans: Local linear approximation of $f(x, y)$ at (x_0, y_0) is

$$L(x, y) = f(x_0, y_0) + (x - x_0) f_x(x_0, y_0) + (y - y_0) f_y(x_0, y_0)$$

$$L(x, y) = f(x_0, y_0) + (x - x_0) f_x(x_0, y_0) + (y - y_0) f_y(x_0, y_0)$$

$$\text{Here } f(x, y) = \ln(xy) \text{ and } (x_0, y_0) = (1, 2)$$

$$\therefore L(x, y) = f(1, 2) + (x - 1) f_x(1, 2) + (y - 2) f_y(1, 2)$$

$$\begin{aligned} &= \ln(2) + (x - 1) \cdot 1 + (y - 2) \cdot \frac{1}{2} \\ &= \ln(2) + (x - 1) \cdot 1 + (y - 2) \cdot \frac{1}{2} \end{aligned}$$

Error in approximation is

$$E(x, y) = f(x, y) - L(x, y)$$

\therefore Error in approximation at

$Q(1.01, 2.01)$ is

$$E(1.01, 2.01) = f(1.01, 2.01) - L(1.01, 2.01)$$

$$f(1.01, 2.01) = \ln(1.01 \times 2.01)$$

$$= \ln(2.0301) = 0.708085$$

$$\begin{aligned} f(x, y) &= \ln(xy) \\ f_x(x, y) &= \frac{1}{xy} \cdot y = \frac{1}{x} \\ \therefore f_x(1, 2) &= \frac{1}{1} = 1 \end{aligned}$$

$$f_y(x, y) = \frac{1}{xy} \cdot x = \frac{1}{y}$$

$$f_y(1, 2) = \frac{1}{2}$$

$$\begin{aligned}
 L(1.01, 2.01) &= \ln(2) + (1.01 - 1) + (2.01 - 2) \cdot \frac{1}{2} \\
 &= 0.693147 + 0.01 + 5 \times 10^{-3} \\
 &= 0.708147
 \end{aligned}$$

$$\begin{aligned}
 \therefore ① \Rightarrow E(1.01, 2.01) &= 0.708085 - 0.708147 \\
 &= -6.2 \times 10^{-5} \\
 &= -0.000062 \quad \text{ie; } E = \underline{\underline{0.000062}}
 \end{aligned}$$

Distance between $P(1, 2)$ and $Q(1.01, 2.01)$

$$\begin{aligned}
 &= \sqrt{(1.01 - 1)^2 + (2.01 - 2)^2} \\
 &= \sqrt{1 \times 10^{-4} + 1 \times 10^{-4}} = \sqrt{2 \times 10^{-4}} = 0.014142
 \end{aligned}$$

∴ error in approximation is very much less than the distance between the points $P(1, 2)$ and $Q(1.01, 2.01)$

- Q. Let $L(x, y)$ denote the local linear approximation to $f(x, y) = \sqrt{x^2+y^2}$ at the point $(3, 4)$. Compare the error in approximating $f(3.04, 3.98)$ by $L(3.04, 3.98)$ with the distance between the points $(3, 4)$ and $(3.04, 3.98)$.

$$L(x, y) = f(x_0, y_0) + (x - x_0) f_x(x_0, y_0) + (y - y_0) f_y(x_0, y_0) \quad \text{--- } ①$$

$$\text{Ans: } L(x, y) = f(x_0, y_0) + (x - x_0) f_x(x_0, y_0) + (y - y_0) f_y(x_0, y_0)$$

$$\text{Here } f(x, y) = \sqrt{x^2+y^2} \text{ and } (x_0, y_0) = (3, 4)$$

$$\therefore f_x(x, y) = \frac{\partial f}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x = \frac{x}{\sqrt{x^2+y^2}}$$

$$f_x(x_0, y_0) = f_x(3, 4) = \frac{3}{\sqrt{9+16}} = \frac{3}{\sqrt{25}} = \frac{3}{5}$$

$$f_y(x, y) = \frac{\partial f}{\partial y} = \frac{1}{2\sqrt{x^2+y^2}} \cdot 2y = \frac{y}{\sqrt{x^2+y^2}}$$

$$f_y(x_0, y_0) = f_y(3, 4) = \frac{4}{\sqrt{9+16}} = \frac{4}{\sqrt{25}} = \frac{4}{5}$$

$$\text{Also } f(x_0, y_0) = \sqrt{3^2+4^2} = \sqrt{9+16} = \sqrt{25} = 5$$

$$\therefore \textcircled{1} \Rightarrow L(x, y) = 5 + (x-3) \cdot \frac{3}{5} + (y-4) \cdot \frac{4}{5} \quad \text{--- \textcircled{2}}$$

Error in approximation, $E(x, y) = f(x, y) - L(x, y)$

$$E(3.04, 3.98) = f(3.04, 3.98) - L(3.04, 3.98) \quad \text{--- \textcircled{3}}$$

$$\text{Now, } f(3.04, 3.98) = \sqrt{(3.04)^2 + (3.98)^2}$$

$$= \sqrt{9.2416 + 15.8404}$$

$$= \sqrt{25.082} = 5.00819$$

$$L(3.04, 3.98) = 5 + (3.04-3) \frac{3}{5} + (3.98-4) \frac{4}{5} \quad (\text{from } \textcircled{2})$$

$$= 5 + 0.024 + -0.016 = 5.008$$

$$\therefore \textcircled{3} \Rightarrow E(3.04, 3.98) = 5.00819 - 5.008$$

$$= \underline{\underline{0.00019}}$$

Distance between $P(3, 4)$ and $Q(3.04, 3.98)$

$$= \sqrt{(3.04-3)^2 + (3.98-4)^2}$$

$$= \sqrt{0.0016 + 0.0004}$$

$$= \sqrt{0.002} = \underline{\underline{0.0447}}$$

\therefore Error in approximation is very much less than the distance between the points $P(3, 4)$ and $Q(3.04, 3.98)$

3. Find the local linear approximation $L(x, y)$ to $f(x, y)$ at the point $P(4, 3)$ where $f(x, y) = \frac{1}{\sqrt{x^2+y^2}}$. Compare the error in approximating f by L at the point $Q(3.92, 3.01)$ with the distance between P and Q .

Ans: $E = 0.2023342 - 0.20232 = 0.0000142$ and Distance, $D = 0.0866$

Maxima and Minima (or Relative extrema) of functions of 2 variables.

Critical point

A point (x_0, y_0) in the domain of a function $f(x, y)$ is called a critical point if $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$.

The second partials test

Let f be a function of 2 variables with continuous second order partial derivatives at a critical point (x_0, y_0) and let $D = f_{xx} f_{yy} - (f_{xy})^2$

- (1) If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$, then (x_0, y_0) is a minimum point and f has a relative minimum at (x_0, y_0) .
- (2) If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$, then (x_0, y_0) is a maximum point and f has a relative maximum at (x_0, y_0) .

- (3) If $D(x_0, y_0) < 0$, then (x_0, y_0) is a saddle point.
- (4) If $D(x_0, y_0) = 0$, then no conclusion can be drawn.

Problems

1. Locate all relative extrema and saddle points of

$$f(x, y) = 3x^2 - 2xy + y^2 - 8y$$

Ans: $f_x = 6x - 2y$, $f_y = -2x + 2y - 8$

$$f_{xx} = 6, \quad f_{yy} = 2, \quad f_{xy} = -2$$

At the critical point, $f_x = 0$ and $f_y = 0$

$$f_x = 0 \Rightarrow 6x - 2y = 0$$

$$f_y = 0 \Rightarrow -2x + 2y - 8 = 0 \Rightarrow -2x + 2y = 8$$
$$\therefore x = 2, y = 6$$

$\therefore (2, 6)$ is the critical point.

Now $D = f_{xx} f_{yy} - (f_{xy})^2$
 $= 6 \times 2 - (-2)^2 = 12 - 4 = 8$

$$\therefore D(x_0, y_0) = D(2, 6) = 8$$

$$\text{Also } f_{xx}(x_0, y_0) = f_{xx}(2, 6) = 6$$

$$\therefore D(x_0, y_0) > 0 \text{ and } f_{xx}(x_0, y_0) > 0$$

$\therefore (x_0, y_0) = (2, 6)$ is a minimum point

Minimum value = $f(x_0, y_0) = f(2, 6)$
 $= 3 \cdot (2)^3 - 2(2)(6) + (6)^2 - 8(6)$
 $= 12 - 24 + 36 - 48 = \underline{\underline{-24}}$

Q. Find relative extrema and saddle point of

$$f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4$$

Ans: $f_x = y - 2x - 2, f_y = x - 2y - 2$

$$f_{xx} = -2, f_{yy} = -2, f_{xy} = 1$$

$$f_x = 0 \Rightarrow y - 2x - 2 = 0 \Rightarrow -2x + y = 2$$

$$f_y = 0 \Rightarrow x - 2y - 2 = 0 \Rightarrow x - 2y = 2$$

$$\therefore x = -2, y = -2$$

$\therefore (-2, -2)$ is the critical point

Now, $D = f_{xx} f_{yy} - (f_{xy})^2$
 $= -2 \times -2 - (1)^2 = 4 - 1 = 3$

$$\therefore D(x_0, y_0) = D(-2, -2) = 3$$

$$\text{Also } f_{xx}(x_0, y_0) = f_{xx}(-2, -2) = -2$$

$$\therefore D(x_0, y_0) > 0 \text{ and } f_{xx}(x_0, y_0) < 0$$

$\therefore (x_0, y_0) = (-2, -2)$ is a maximum point

$$\text{Maximum value} = f(x_0, y_0) = f(-2, -2)$$

$$= (-2)(-2) - (-2)^2 - 2(-2) - 2(-2) + 4$$

$$= 4 - 4 - 4 + 4 + 4 + 4 = \underline{\underline{8}}$$

3. Locate all relative extrema and saddle point if any
of $f(x, y) = y^2 + xy + 4y + x^2 + 3$.

Ans: $f_x = y + 2x, f_y = 2y + x + 4$

$$f_{xx} = 0, f_{yy} = 2, f_{xy} = 1$$

$$f_x = 0 \Rightarrow y + 2x = 0 \Rightarrow y = -2x$$

$$f_y = 0 \Rightarrow 2y + x + 4 = 0 \Rightarrow x + 2y = -4$$

$$\therefore x = 0, y = -2$$

$\therefore (0, -2)$ is the critical point

$$\text{Now, } D = f_{xx} f_{yy} - (f_{xy})^2$$

$$= 0 \times 2 - (1)^2 = -1$$

$$\therefore D(x_0, y_0) = D(0, -2) = -1 < 0$$

$\therefore (x_0, y_0) = (0, -2)$ is a saddle point

4. Locate all relative extrema and saddle point if any
of $f(x, y) = x^2 + xy - 2y - 3x + 1$

Ans: Critical point $= (2, -1)$

$(x_0, y_0) = (2, -1)$ is a saddle point.

Ques. 5. Locate all relative extrema of $4xy - x^4 - y^4$

$$f(x, y) = 4xy - x^4 - y^4$$

$$f_x = 4y - 4x^3, \quad f_y = 4x - 4y^3$$

$$f_{xx} = -12x^2, \quad f_{yy} = -12y^2, \quad f_{xy} = 4$$

$$f_x = 0 \Rightarrow 4y - 4x^3 = 0 \Rightarrow 4y = 4x^3 \Rightarrow y = \frac{4x^3}{4} = x^3$$

$\text{ie; } y = x^3$

$$f_y = 0 \Rightarrow 4x - 4y^3 = 0 \Rightarrow 4(x - y^3) = 0$$

$$\Rightarrow x - y^3 = 0$$

$$\Rightarrow x - (x^3)^3 = 0$$

$$\Rightarrow x - x^9 = 0$$

$$\Rightarrow x(1 - x^8) = 0$$

$$\Rightarrow x = 0, \quad 1 - x^8 = 0$$

$$\Rightarrow x = 0, \text{ and } x^8 = 1$$

$$\Rightarrow x = 0 \text{ and } x = 1, -1$$

$$y = x^3 \Rightarrow y = 0^3, 1^3, (-1)^3$$

$$= 0, 1, -1$$

$\therefore (0, 0), (1, 1)$ and $(-1, -1)$ are the critical points.

$$\text{Now, } D = f_{xx} f_{yy} - (f_{xy})^2$$

$$= (-12x^2)(-12y^2) - (4)^2$$

$$= 144x^2y^2 - 16$$

$$\text{At } (0, 0), \quad D = 144 \times 0 - 16 = -16 < 0$$

$\therefore \underline{(0, 0) \text{ is a saddle point}}$

$$\text{At } (1, 1), \quad D = 144 \times 1^2 \times 1^2 - 16 = 144 - 16 = 128 > 0$$

$$\therefore f_{xx}(1, 1) = -12(1^2) = -12 < 0$$

$\therefore (1, 1)$ is a maximum point

$$\text{Maximum value} = f(1, 1) = 4x^4 - 1^4 - 1^4 \\ = 4 - 1 - 1 = \underline{\underline{2}}$$

$$\text{At } (-1, -1), D = 144 \times (-1)^2 \times (-1)^2 - 16 \\ = 144 - 16 = 128 > 0$$

$$\therefore f_{xx}(-1, -1) = -12(-1)^2 = -12 < 0$$

$\therefore (-1, -1)$ is also a maximum point

$$\text{Maximum value} = f(-1, -1) = 4x^4 - 1^4 - (-1)^4 - (-1)^4 \\ = 4 - 1 - 1 = \underline{\underline{2}}$$

Q. 6. Locate all relative extrema of $f(x, y) = xy + \frac{8}{x} + \frac{8}{y}$

$$\text{Ans. } f_x = y - \frac{8}{x^2}, \quad f_y = x - \frac{8}{y^2}$$

$$f_{xx} = \frac{\partial}{\partial x} \left(y - \frac{8}{x^2} \right) \\ = \frac{\partial}{\partial x} \left(y - 8x^{-2} \right) \\ = 0 - -8 \times 2x^{-3} = 16x^{-3} = \frac{16}{x^3}$$

$$f_{yy} = \frac{\partial}{\partial y} \left(x - \frac{8}{y^2} \right) \\ = \frac{\partial}{\partial y} \left(x - 8y^{-2} \right) = 0 - 8 \times -2y^{-3} = 16y^{-3} = \frac{16}{y^3}$$

$$f_{xy} = 1$$

$$f_x = 0 \Rightarrow y - \frac{8}{x^2} = 0 \Rightarrow y = \frac{8}{x^2}$$

$$f_y = 0 \Rightarrow x - \frac{8}{y^2} = 0 \Rightarrow x - \frac{8}{(\frac{8}{x^2})^2} = 0$$

$$\Rightarrow x - \frac{8}{(\frac{64}{x^4})} = 0$$

$$\Rightarrow x - \frac{8x^4}{64} = 0$$

$$\Rightarrow x - \frac{x^4}{8} = 0$$

$$\Rightarrow x \left(1 - \frac{x^3}{8}\right) = 0$$

$$\Rightarrow x = 0, \quad 1 - \frac{x^3}{8} = 0$$

$$\Rightarrow x = 0, \quad 1 = \frac{x^3}{8}$$

$$\Rightarrow x = 0, \quad x^3 = 8$$

$$\Rightarrow x = 0, \quad x = \sqrt[3]{8} = 2$$

when $x = 0, \quad y = \frac{8}{x^3} \Rightarrow y$ is undefined

$$\text{when } x = 2, \quad y = \frac{8}{2^3} = \frac{8}{8} = 1$$

$\therefore (2, 1)$ is the critical point.

$$\text{Now, } D = f_{xx} f_{yy} - (f_{xy})^2$$

$$= \frac{16}{x^3} \cdot \frac{16}{y^3} - (1)^2 = \frac{256}{x^3 y^3} - 1$$

$$\text{At } (2, 1), \quad D = \frac{256}{2^3 \cdot 2^3} - 1 = \frac{256}{64} - 1 = 4 - 1 = 3 > 0$$

$$\text{Now } f_{xx}(2, 1) = \frac{16}{x^3} = \frac{16}{2^3} = \frac{16}{8} = 2 > 0$$

$\therefore (2, 1)$ is a minimum point

$$\begin{aligned} \text{Minimum value} &= f(2, 1) = 2 \times 2 + \frac{8}{2} + \frac{8}{2} \\ &= 4 + 4 + 4 = \underline{\underline{12}} \end{aligned}$$

Absolute extrema on closed and bounded sets

If $f(x,y)$ is continuous on a closed and bounded set R , then f has both an absolute maximum and an absolute minimum on R . These absolute extrema can occur either on the boundary of R or in the interior of R , but if an absolute extremum occurs at the interior, then it occurs at a critical point.

Method to find the absolute extrema of a continuous function of two variables on a closed and bounded set R .

- 1) Find the critical points of f that lie in the interior of R .
- 2) Find all boundary points at which the absolute extrema can occur.
- 3) Evaluate $f(x,y)$ at the points obtained in the preceding steps. The largest of these values is the absolute maximum and the smallest the absolute minimum.

1. Find the absolute maximum and minimum values of $f(x, y) = 3xy - 6x - 3y + 7$ on the closed triangular region R with vertices $(0, 0)$, $(3, 0)$ and $(0, 5)$

Solution

$$f(x, y) = 3xy - 6x - 3y + 7$$

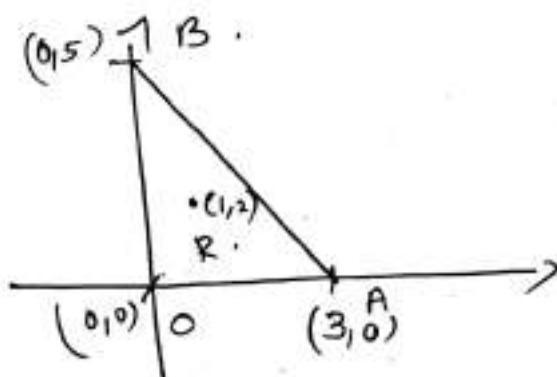
$$f_x = 3y - 6$$

$$f_y = 3x - 3$$

$$f_x = 0 \Rightarrow 3y - 6 = 0 \\ y = 2$$

$$f_y = 0 \Rightarrow 3x - 3 = 0 \\ x = 1$$

Critical point = $(1, 2)$



$(1, 2)$ is the critical point in the interior of R

The boundary of R consists of 3 line segments

Along OA, $y = 0$.

$$u(x) = f(x, 0) = -6x + 7.$$

$u'(x) = -6$ is nonzero $\forall x$.

$u'(x) = -6$, $0 \leq x \leq 3$, is nonzero for all x .

Extreme values of $u(x)$ occur at the endpoints $x=0$ and $x=3$

$(0, 0)$ $(3, 0)$.

Along OB

$$x = 0$$

$$f(0, y) = -3y + 7.$$

$$v(y) = -3y + 7 \quad 0 \leq y \leq 5.$$

$v'(y) = -3$ is non zero for all y .

∴ Extreme values occur at the end points $y=0$ and $y=5$

$(0, 0)$ $(0, 5)$

Along AB

$(3, 0)$ $(0, 5)$

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1}$$

$$\frac{x-3}{0-3} = \frac{y-0}{5-0} \Rightarrow \frac{x-3}{-3} = \frac{y}{5}$$

~~0-3~~

~~x+3=5y~~

$$\Rightarrow y = -\frac{5}{3}x + 5$$

$$\begin{aligned}
 w &= f(x, y) \\
 &= f\left(x, -\frac{5}{3}x+5\right) \\
 &= 3x^2 \times \left(-\frac{5}{3}x+5\right) - 3\left(-\frac{5}{3}x+5\right) - 6x + 7 \\
 &= 3x\left(-\frac{5}{3}x+5\right) - 3\left(-\frac{5}{3}x+5\right) - 6x + 7 \\
 &= -5x^2 + 15x + 5 - 6x + 7 \\
 &= -5x^2 + 14x + 8 \quad 0 \leq x \leq 3
 \end{aligned}$$

$$\begin{aligned}
 w' &= -10x + 14 \\
 w'(x) = 0 &\Rightarrow -10x + 14 = 0 \\
 &\Rightarrow 10x = 14 \\
 &\Rightarrow x = \frac{14}{10} = \frac{7}{5}
 \end{aligned}$$

$$\begin{aligned}
 y &= -\frac{5}{3} \times \frac{7}{5} + 5 \\
 &= -\frac{35}{15} + 5 \\
 &= \frac{8}{3}
 \end{aligned}$$

$$\left(\frac{7}{5}, \frac{8}{3}\right)$$

(x, y)	$(0, 0)$	$(3, 0)$	$(0, 5)$	$\left(\frac{7}{5}, \frac{8}{3}\right)$	$(1, 2)$
$f(x, y)$	-7	-11	-8	$\frac{9}{5}$	1

Absolute minimum = -11

Absolute maximum = 1