

MODULE - I
LINEAR ALGEBRA

Elementary row operations (elementary row transformations)

Any one of the following operations on a matrix is called an elementary transformation.

- (i) Interchange of two rows. Interchange of i^{th} and j^{th} rows is denoted as $R_i \leftrightarrow R_j$.
- (ii) Multiplication of a row by a non-zero number k . Multiplication of each element of i^{th} row by a non-zero number k is denoted as $R_i \rightarrow kR_i$.
- (iii) Addition of k times the elements of a row to the corresponding elements of another row. Addition of k times the elements of j^{th} row to the corresponding elements of i^{th} row is denoted as $R_i \rightarrow R_i + kR_j$.

Note

If a matrix B is obtained from a matrix A by one or more elementary transformations, then B is said to be equivalent to A , and is denoted as $A \sim B$.

Row Echelon form and rank of a matrix.

A matrix is said to be an echelon matrix if it satisfies the following conditions.

- (i) The first non-zero number in any row is 1.
- (ii) The remaining elements below 1 are 0's.
- (iii) All rows consisting entirely of zeros appear at the bottom of the matrix.

$$\text{Ex: (1)} \quad A = \begin{bmatrix} 1 & -3 & 4 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad (2) \quad B = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note

The number of non-zero rows in the echelon form is the rank of the matrix and rank of a matrix A is denoted as $R(A)$.

Problems

Q: Find the rank of the following matrices by reducing to echelon form.

$$1. \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

$$\text{Ans: } R_2 \rightarrow R_2 - R_1 \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & -6 \\ 0 & -2 & 2 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + 2R_1$$

$$R_2 \rightarrow \frac{R_2}{2} \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -3 \\ 0 & -2 & 2 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + 2R_2 \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & -4 \end{bmatrix}$$

$$R_3 \rightarrow \frac{R_3}{-4} \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}, \text{ which is in}$$

the echelon form and rank = no. of non-zero rows = 3

$$2. \begin{bmatrix} 2 & 4 & 6 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$$

$$\text{Ans: } R_1 \rightarrow \frac{R_1}{2} \sim \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1 \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 3 & 6 & 10 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 3R_1 \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_2 \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \text{ which is in}$$

echelon form and rank = no. of non-zero rows

$$= \underline{\underline{2}}$$

$$3. \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 1 & 5 & 5 & 7 \\ 8 & 1 & 14 & 17 \end{bmatrix}$$

$$\text{Ans: } R_2 \rightarrow R_2 - 2R_1 \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 1 & 5 & 5 & 7 \\ 8 & 1 & 14 & 17 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_1 \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & \frac{2}{3} & 1 \\ 0 & 3 & 2 & 3 \\ 0 & -15 & -10 & -15 \end{bmatrix}$$

$$R_4 \rightarrow R_4 + 15R_2 \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & \frac{2}{3} & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ which is in}$$

echelon form and rank = no. of non-zero rows

$$= \underline{\underline{2}}$$

4. Reduce the matrix $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ to row echelon form and hence find its rank.

$$\text{Ans. } A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2 \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1 \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$$R_2 \rightarrow \frac{R_2}{5} \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & 3/5 & 7/5 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 4R_2 \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & 3/5 & 7/5 \\ 0 & 0 & 33/5 & 22/5 \\ 0 & 0 & 33/5 & 22/5 \end{bmatrix}$$

$$R_3 \rightarrow R_3 \times \frac{5}{33} \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & 3/5 & 7/5 \\ 0 & 0 & 1 & 20/33 \\ 0 & 0 & 33/5 & 22/5 \end{bmatrix}$$

$$R_4 \rightarrow R_4 - \frac{33}{5}R_3 \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & 3/5 & 7/5 \\ 0 & 0 & 1 & 20/33 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ which is}$$

in echelon form and rank = 3

Linear System of Equations

A Linear system of m equations in n unknowns x_1, x_2, \dots, x_n is a set of equations of the form

$$\left. \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{array} \right\} \quad \textcircled{1}$$

The system is called linear because each variable x_j appears in the first power only

If b_1, b_2, \dots, b_m are all zeros then equation $\textcircled{1}$ is called a homogeneous system

If atleast one b_j is not zero, then equation $\textcircled{1}$ is called a non-homogeneous system

A solution of equation $\textcircled{1}$ is a set of numbers x_1, x_2, \dots, x_n that satisfies all the m equations.

If the system $\textcircled{1}$ is homogeneous, it always has atleast the trivial solution $x_1=0, x_2=0, \dots, x_n=0$.

Matrix representation of equation $\textcircled{1}$ is

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

i.e., $AX=B$, where

A is the co-efficient matrix

X is the column matrix of unknowns

B is the column matrix of constants

Augmented matrix

Two matrices A and B writing side by side is called augmented matrix and is denoted as $[AB]$.

i.e; $[AB] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$

The system of equation ① is consistent if it has at least one solution and inconsistent if it has no solution.

Gauss Elimination method

It is a method for solving linear system of equations. Here we first form the augmented matrix $[AB]$ for the system $AX=B$. By converting the augmented matrix $[AB]$ to echelon form, we have to find the rank of $[AB]$ and the rank of A .

Fundamental theorem

- 1) If rank of $A \neq$ rank of $[AB]$, then the system is inconsistent. i.e; The system has no solution.
- 2) If rank of $A =$ rank of $[AB] =$ number of unknowns, the system is consistent and has unique solution.
- 3) If rank of $A =$ rank of $[AB] <$ number of unknowns, the system is consistent and has infinite solution, obtained by giving $(n-r)$ variable an arbitrary value and using backward substitution. (n -no. of variables and r =rank of the matrix)

Problems

1. solve using Gauss elimination method, $3x+2y+z=3$,
 $2x+y+z=0$, $6x+ay+4z=6$.

Ans: $Ax = B \Rightarrow \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 1 \\ 6 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}$

\therefore Augmented matrix, $[AB] = \begin{bmatrix} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{bmatrix}$

$$R_1 \rightarrow \frac{R_1}{3} \sim \begin{bmatrix} 1 & 2/3 & 1/3 & 1 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1 \sim \begin{bmatrix} 1 & 2/3 & 1/3 & 1 \\ 0 & -1/3 & 1/3 & -2 \\ 6 & 2 & 4 & 6 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 6R_1 \sim \begin{bmatrix} 1 & 2/3 & 1/3 & 1 \\ 0 & 1 & -1 & 6 \\ 0 & -2 & 2 & 0 \end{bmatrix}$$

$$R_2 \rightarrow R_2 \times -3 \sim \begin{bmatrix} 1 & 2/3 & 1/3 & 1 \\ 0 & 1 & -1 & 6 \\ 0 & 0 & 0 & 12 \end{bmatrix}$$

\therefore Rank of $[AB]$, $S(AB) = 3$ and rank of A , $S(A) = 2$

$$\therefore S(AB) \neq S(A)$$

so the system is inconsistent and has no solution.

2. solve $4x+y=4$, $5x-3y+z=2$, $-9x+2y-z=5$.

Ans: $[AB] = \begin{bmatrix} 4 & 1 & 0 & 4 \\ 5 & -3 & 1 & 2 \\ -9 & 2 & -1 & 5 \end{bmatrix}$

$$R_1 \rightarrow R_1 - \frac{R_1}{4} \sim \begin{bmatrix} 1 & \frac{y_4}{4} & 0 & 1 \\ 5 & -3 & 1 & 2 \\ -9 & 2 & -1 & 5 \end{bmatrix}$$

$$\begin{array}{l} R_2 \rightarrow R_2 - 5R_1 \\ R_3 \rightarrow R_3 + 9R_1 \end{array} \sim \begin{bmatrix} 1 & \frac{y_4}{4} & 0 & 1 \\ 0 & -\frac{17}{4} & 1 & -3 \\ 0 & \frac{17}{4} & -1 & 14 \end{bmatrix}$$

$$R_2 \rightarrow R_2 \times \frac{-4}{17} \sim \begin{bmatrix} 1 & \frac{y_4}{4} & 0 & 1 \\ 0 & 1 & -\frac{4}{17} & \frac{12}{17} \\ 0 & \frac{17}{4} & -1 & 14 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - \frac{17}{4}R_2 \sim \begin{bmatrix} 1 & \frac{y_4}{4} & 0 & 1 \\ 0 & 1 & -\frac{4}{17} & \frac{12}{17} \\ 0 & 0 & 0 & 11 \end{bmatrix}$$

\therefore Rank of $[AB]$, $S(AB) = 3$ and
rank of A , $S(A) = 2$

$\therefore S(AB) \neq S(A)$
 \therefore the system is inconsistent and has no solution.

H.W. 3. Show that the system of equations are inconsistent;
 $2x+6y=-11$, $6x+20y-6z=-3$, $6y-18z=-1$
 $2x+6y=-11$, $x+2y-3z=-4$, $-x-4y+9z=18$.

4. Solve $x+y+z=6$, $x+2y-3z=-4$, $-x-4y+9z=18$.

Ans: $[AB] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & -3 & -4 \\ -1 & -4 & 9 & 18 \end{bmatrix}$

$$R_2 \rightarrow R_2 - R_1 \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -4 & -10 \\ 0 & -3 & 10 & 24 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + 3R_2 \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -4 & -10 \\ 0 & 0 & -2 & -6 \end{bmatrix}$$

$$R_3 \rightarrow \frac{R_3}{-2} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -4 & -10 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

\therefore Rank of $[AB]$, $S(AB) = 3$ and

rank of A , $S(A) = 3$

$$\therefore S(AB) = S(A)$$

so the system is consistent

Here rank = 3 = number of unknowns

\therefore the system has unique solution

$$\text{From echelon form } x+y+z=6$$

$$y-4z=-10$$

$$z=3$$

$$\therefore y = -10 + 4z = -10 + 12 = 2$$

$$x = 6 - y - z = 6 - 2 - 3 = 6 - 5 = 1$$

$$\therefore \underline{x=1, y=2, z=3}$$

5 Solve $3x+3y+2z=1$, $x+2y=4$, $10y+3z=-2$ and

$2x-2y-z=5$ by Gauss elimination method.

$$\text{Ans: } [AB] = \begin{bmatrix} 3 & 3 & 2 & 1 \\ 1 & 2 & 0 & 4 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2 \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 3 & 3 & 2 & 1 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 3R_1 \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{bmatrix}$$

$$R_4 \rightarrow R_4 - 2R_1 \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 10 & 3 & -2 \\ 0 & -7 & -1 & -3 \end{bmatrix}$$

$$R_2 \rightarrow \frac{R_2}{-3} \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -\frac{2}{3} & \frac{11}{3} \\ 0 & 10 & 3 & -2 \\ 0 & -7 & -1 & -3 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 10R_2 \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -\frac{2}{3} & \frac{11}{3} \\ 0 & 0 & \frac{29}{3} & -116/3 \\ 0 & 0 & -17/3 & 68/3 \end{bmatrix}$$

$$R_3 \rightarrow R_3 \times \frac{3}{29} \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -\frac{2}{3} & \frac{11}{3} \\ 0 & 0 & 1 & -116/29 \\ 0 & 0 & -17/3 & 68/3 \end{bmatrix}$$

$$R_4 \rightarrow R_4 + \frac{17}{3}R_3 \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -\frac{2}{3} & \frac{11}{3} \\ 0 & 0 & 1 & -116/29 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

\therefore Rank of $[AB] = 3 =$ Rank of $A =$ no. of unknowns

\therefore The system is consistent and has unique solution.

From echelon form, $x + 2y = 4$

$$y - \frac{2}{3}z = \frac{11}{3}$$

$$z = \frac{-116}{29} = -4$$

$$y - \frac{2}{3}z = \frac{11}{3} \Rightarrow y = \frac{11}{3} + \frac{2}{3}z$$

$$= \frac{11}{3} + \frac{2}{3}(-4) = \frac{11}{3} - \frac{8}{3} = \frac{3}{3} = 1$$

$$x + 2y = 4 \Rightarrow x = 4 - 2y = 4 - 2(1) \\ = 4 - 2 = 2$$

$$\therefore \underline{\underline{x = 2, y = 1, z = -4}}$$

6. Solve the system of equations $x_1 - x_2 + x_3 = 0$,
 $-x_1 + x_2 - x_3 = 0$, $10x_2 + 25x_3 = 90$, $20x_1 + 10x_2 = 80$.

$$Ans: [AB] = \begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{bmatrix}$$

$$R_2 \rightarrow R_2 + R_1 \sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 20R_1 \sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{bmatrix}$$

$$R_2 \leftrightarrow R_4 \sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 30 & -20 & 80 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \rightarrow \frac{R_2}{30} \sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{8}{3} \\ 0 & 10 & 25 & 90 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 10R_1 \sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{8}{3} \\ 0 & 0 & \frac{95}{3} & \frac{190}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_3 \rightarrow R_3 \times \frac{3}{95} \sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{8}{3} \\ 0 & 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank of $[AB] = 3 = \text{Rank of } A = \text{no. of unknowns}$
 \therefore The system is consistent and has unique solution

From echelon form, $x - y + z = 0$

$$y - \frac{2}{3}z = \frac{8}{3}$$

$$z = 2$$

$$y - \frac{2}{3}z = \frac{8}{3} \Rightarrow y = \frac{8}{3} + \frac{2}{3}z = \frac{8}{3} + \frac{2}{3}(2) = \frac{8}{3} + \frac{4}{3} = \frac{12}{3} = 4$$

$$x - y + z = 0 \Rightarrow x = y - z = 4 - 2 = 2$$

$$\therefore \underline{x_1 = 2, \quad x_2 = 4, \quad x_3 = 2}$$

- H.W 7. Solve $x+2y-z=3$, $3x-y+2z=1$, $2x-2y+3z=2$, $x-y+z=-1$.

Ans: $x = -1$, $y = 4$, $z = 4$

- H.W 8. Solve $x+2y+3z=0$, $3x+4y+4z=0$, $7x+10y+12z=0$

Ans: $x = 0$, $y = 0$, $z = 0$

- H.W 9. Using Gauss elimination method, solve the equations
 $x+2y+3z-4w=10$, $2x+3y-3z-w=1$,
 $2x-y+2z+3w=7$, $3x+2y-4z+3w=2$.

Ans: $x = 1$, $y = 2$, $z = 2$ and $w = 1$.

10. Solve $-2y-2z=8$, $3x+4y-5z=8$.

Ans: $[AB] = \begin{bmatrix} 0 & -2 & -2 & 8 \\ 3 & 4 & -5 & 8 \end{bmatrix}$

$$R_1 \rightarrow \frac{R_1}{-2} \sim \begin{bmatrix} 0 & 1 & 1 & -4 \\ 3 & 4 & -5 & 8 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 4R_1 \sim \begin{bmatrix} 0 & 1 & 1 & -4 \\ 3 & 0 & -9 & 24 \end{bmatrix}$$

$$R_2 \rightarrow \frac{R_2}{3} \sim \begin{bmatrix} 0 & 1 & 1 & -4 \\ 1 & 0 & -3 & 8 \end{bmatrix}$$

Rank of $[AB] = 2 = \text{Rank of } A < 3$, the no. of unknowns.

\therefore The system is consistent and has infinite solutions.

From echelon form $y+z=-4$

$$x-3z=8$$

Here $n=3$ and $r=2$

$\therefore n-r = 3-2 = 1$ variable should assign an arbitrary value.

Put $z=t$

$$\therefore x-3z=8 \Rightarrow x=8+3z \\ = 8+3t$$

$$y+z=-4 \Rightarrow y=-4-z = -4-t$$

$$\therefore \underline{x=8+3t, \quad y=-4-t, \quad z=t}$$

ii. solve $y+z-2w=0, \quad 2x-3y-3z+6w=2,$

$$4x+y+z-2w=4.$$

Ans: $[AB] = \begin{bmatrix} 0 & 1 & 1 & -2 & 0 \\ 2 & -3 & -3 & 6 & 2 \\ 4 & 1 & 1 & -2 & 4 \end{bmatrix}$

$$R_2 \rightarrow R_2 + 3R_1 \sim \begin{bmatrix} 0 & 1 & 1 & -2 & 0 \\ 2 & 0 & 0 & 0 & 2 \\ 4 & 0 & 0 & 0 & 4 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_1 \sim \begin{bmatrix} 0 & 1 & 1 & -2 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 & 4 \end{bmatrix}$$

$$R_2 \rightarrow \frac{R_2}{2} \sim \begin{bmatrix} 0 & 1 & 1 & -2 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 & 4 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 4R_2 \sim \begin{bmatrix} 0 & 1 & 1 & -2 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank of $[AB] = 2 = \text{rank of } A < 4$, the no of unknowns

\therefore The system is consistent and has infinite solutions

From echelon form $y+z-2w=0$
 $x=1$

Here $n=4$ and $r=2$

$$\therefore n-r = 4-2 = 2$$

\therefore 2 variables should assign arbitrary values

Put $w=a$ and $x=b$

$$\therefore y+z-aw=0 \Rightarrow y = -z+aw = -b+aa$$

$$\therefore \underline{\underline{x=1, y=2a-b, z=b \text{ and } w=a}}$$

H.W.

12. solve $x_1 + 3x_2 + 2x_3 = 0$, $2x_1 - x_2 + 3x_3 = 0$,

$$3x_1 - 5x_2 + 4x_3 = 0, x_1 + 17x_2 + 4x_3 = 0$$

Ans: $x_1 = \frac{-11t}{7}$, $x_2 = \frac{-t}{7}$, $x_3 = t$.

13 Find the values of μ for which the system of equations $x+y+z=1$, $x+ay+3z=\mu$ and

$x+5y+9z=\mu^2$ will be consistent. For each value of μ obtained, find the solution of the system.

Ans: $[AB] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & \mu \\ 1 & 5 & 9 & \mu^2 \end{bmatrix}$

$$\begin{aligned} R_2 &\rightarrow R_2 - R_1 \\ R_3 &\rightarrow R_3 - R_1 \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & \mu-1 \\ 0 & 4 & 8 & \mu^2-1 \end{bmatrix} \end{aligned}$$

$$R_3 \rightarrow R_3 - 4R_2 \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & \mu-1 \\ 0 & 0 & 0 & \mu^2-4\mu+3 \end{bmatrix}$$

This system will be consistent if $S(A) = S(AB)$

$$\text{Here } S(A) = 2$$

$\therefore S(AB)$ should also be 2. This is possible.

only if $\mu^2 - 4\mu + 3 = 0$

$$\Rightarrow (\mu - 3)(\mu - 1) = 0$$

$$\Rightarrow \mu - 3 = 0, \mu - 1 = 0 \Rightarrow \underline{\underline{\mu = 3, 1}}$$

When $\mu = 3$

$$[AB] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore S(A) = S(AB) = 2 < 3$, the number of unknowns

\therefore system has infinite no. of solutions

we have, $x + y + z = 1$

$$y + 2z = 2$$

Here $n=3$ and $r=2$. $\therefore n-r=3-2=1$ variable

Should assign arbitrary value.

Put $z=t$ $\therefore y = 2 - 2z = 2 - 2t$

$$\begin{aligned} x &= 1 - y - z = 1 - (2 - 2t) - t \\ &= 1 - 2 + 2t - t \\ &= -1 + t = t - 1 \end{aligned}$$

$\therefore \underline{\underline{x = t - 1, y = 2 - 2t, z = t}}$

when $\mu = 1$

$$[AB] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore S(A) = S(AB) = 2 < 3$, the number of unknowns

\therefore system has infinite no. of solutions

we have, $x + y + z = 1$

$$y + 2z = 0$$

Here $n=3$ and $r=2$. $\therefore n-r=3-2=1$ variable should assign arbitrary value

$$\text{Put } z=t \quad \therefore y = -az = -at$$

$$x = 1 - y - z = 1 - -at - t$$

$$= 1 + at - t = 1 + t$$

$$\therefore \underline{\underline{x = 1 + t, \quad y = -at, \quad z = t}}$$

14. Find the values of a and b for which the systems of equations $x+y+az=2$, $2x-y+3z=10$, $5x-y+az=b$, has (1) no solution (2) unique solution (3) infinite number of solutions.

Ans: $[AB] = \begin{bmatrix} 1 & 1 & a & 2 \\ 2 & -1 & 3 & 10 \\ 5 & -1 & a & b \end{bmatrix}$

$$R_2 \rightarrow R_2 - 2R_1 \sim \begin{bmatrix} 1 & 1 & a & 2 \\ 0 & -3 & -1 & 6 \\ 5 & -1 & a & b-10 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 5R_1 \sim \begin{bmatrix} 1 & 1 & a & 2 \\ 0 & 1 & a-5 & -2 \\ 0 & -6 & a-10 & b-10 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + 6R_2 \sim \begin{bmatrix} 1 & 1 & a & 2 \\ 0 & 1 & a-5 & -2 \\ 0 & 0 & a-8 & b-22 \end{bmatrix}$$

(1) The system has no solution when $S(A) \neq S(AB)$

$\therefore a-8=0$ and $b-22 \neq 0$ for no solution

$$\Rightarrow \underline{\underline{a=8 \text{ and } b \neq 22}}$$

(2) The system has unique solution when $S(A) = S(AB)$
= no. of unknowns

Here no. of unknowns = 3

$$\therefore S(A) = S(AB) = 3$$

$\therefore a-8 \neq 0$ and $b-2a$ can be any value

$$\Rightarrow \underline{a \neq 8 \text{ and } b \text{ can be any value}}$$

(3) The system has infinite number of solutions when

$$S(A) = S(AB) < \text{no. of unknowns}$$

i.e; $S(A)$ and $S(AB)$ should be less than 3

$$\therefore a-8=0 \text{ and } b-2a=0$$

$$\Rightarrow \underline{a=8 \text{ and } b=2a}$$

- H.W 15. Find the values of a and b for which the system of equations $x+2y+3z=6$, $x+3y+5z=9$, $2x+5y+az=b$ has (1) no solution (2) unique solution and (3) infinite no. of solutions.

Ans (1) $a=8$ $b \neq 15$

(2) $a \neq 8$, b can be any value

(3) $a=8$, $b=15$.

Eigen values and eigen vectors

The problem of determining the eigen values and eigen vectors of a matrix is called an eigen value problem. Let A be a square matrix of order n , I corresponding identity matrix, λ a scalar, then $[A - \lambda I]$ is called the characteristic matrix.

The determinant $|A - \lambda I|$ is called the characteristic determinant of A . By developing $|A - \lambda I|$ we obtain a polynomial of n^{th} degree in λ and this is called characteristic polynomial of A . $|A - \lambda I| = 0$ is called the characteristic equation of A and its roots are called eigen values or characteristic roots or latent roots of A .

The set of all eigen values of A is called the spectrum of A . The largest of the absolute value of eigen values of A is called the spectral radius of A .

Let the roots of the characteristic equation be $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$. Corresponding to each of these eigen values the systems of equation $[A - \lambda I]x = 0$ have a non-trivial solution. These solutions are known as eigen vectors corresponding to

$$\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n.$$

Q: Find the eigen values and the corresponding eigen vectors for the following matrices:

$$1) A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

$$\text{Ans: Here } A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \therefore I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned}[A - \lambda I] &= \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix}\end{aligned}$$

$$\begin{aligned}|A - \lambda I| &= \begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = (-5-\lambda)(-2-\lambda) - 4 \\ &= 10 + 5\lambda + 2\lambda + \lambda^2 - 4 \\ &= \lambda^2 + 7\lambda + 6\end{aligned}$$

$$\therefore |A - \lambda I| = 0 \Rightarrow \lambda^2 + 7\lambda + 6 = 0$$

$$\Rightarrow \lambda = -6, -1$$

Eigen values are $\lambda = -6, -1$

$$\text{Now, } [A - \lambda I]x = 0 \Rightarrow \begin{bmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow$$

when $\lambda = -6$

$$① \Rightarrow \begin{bmatrix} -5-(-6) & 2 \\ 2 & -2-(-6) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \cdot \text{ Rank of } \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = 1$$

$$\Rightarrow x_1 + 2x_2 = 0$$

$$2x_1 + 4x_2 = 0$$

considering the first equation, $x_1 + 2x_2 = 0$

$$\Rightarrow x_1 = -2x_2$$

$$\Rightarrow \frac{x_1}{-2} = \frac{x_2}{1}$$

$\therefore \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = -6$

when $\lambda = -1$

$$\textcircled{1} \Rightarrow \begin{bmatrix} -5 & -1 & 2 \\ 2 & -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} . \quad \text{Rank of } \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} = 1$$

$$\Rightarrow -4x_1 + 2x_2 = 0$$

$$2x_1 - 1x_2 = 0$$

considering the first equation, $-4x_1 + 2x_2 = 0$

$$\Rightarrow -4x_1 = -2x_2$$

$$\Rightarrow \frac{x_1}{-2} = \frac{x_2}{-4}$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{2}$$

$\therefore \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = -1$

H.W.

a. $\begin{bmatrix} 6 & 3 \\ 4 & 7 \end{bmatrix}$

Ans: $\lambda = 10, 3$

when $\lambda = 10$, eigen vector = $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

when $\lambda = 3$, eigen vector = $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Note

1. If given matrix A is a 2×2 matrix,

i.e.; if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then

$$\begin{aligned}|A - \lambda I| &= \lambda^2 - (\text{sum of diagonal elements of } A)\lambda + |A| \\ &= \lambda^2 - (a_{11} + a_{22})\lambda + |A|.\end{aligned}$$

2. If given matrix A is a 3×3 matrix,

i.e.; if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

$$\begin{aligned}|A - \lambda I| &= \lambda^3 - (\text{sum of diagonal elements of } A)\lambda^2 \\ &\quad + (\text{sum of minors of diagonal elements of } A)\lambda - |A|\end{aligned}$$

i.e.; $|A - \lambda I| = \lambda^3 - (a_{11} + a_{22} + a_{33})\lambda^2 + (M_{11} + M_{22} + M_{33})\lambda - |A|$

where M_{11} = minor of $a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

$$M_{22} = \text{minor of } a_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \quad \text{and}$$

$$M_{33} = \text{minor of } a_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Problems

1. Find the eigen values and eigen vectors of $A = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$

Ans $|A - \lambda I| = 0 \Rightarrow \lambda^2 - (\text{sum of diagonal elements of } A)\lambda + |A| = 0$

$$\Rightarrow \lambda^2 - 0\lambda + 16 = 0$$

$$\Rightarrow \lambda^2 + 16 = 0 \Rightarrow \lambda^2 = -16 \Rightarrow \lambda = \sqrt{-16} = \pm 4i$$

\therefore Eigen values are $\lambda = 4i, -4i$

$$[A - \lambda I]x = 0 \Rightarrow \begin{bmatrix} -\lambda & 4 \\ -4 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \textcircled{1}$$

when $\lambda = 4i$

$$\textcircled{1} \Rightarrow \begin{bmatrix} -4i & 4 \\ -4 & -4i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{Rank of } \begin{bmatrix} -4i & 4 \\ -4 & -4i \end{bmatrix} = 1$$

$$\Rightarrow -4i x_1 + 4 x_2 = 0$$

$$-4 x_1 - 4i x_2 = 0$$

considering first equation, $-4i x_1 + 4 x_2 = 0$

$$\Rightarrow -4i x_1 = -4 x_2$$

$$\Rightarrow \frac{x_1}{-4} = \frac{x_2}{-4i} \Rightarrow \frac{x_1}{1} = \frac{x_2}{i}$$

$\therefore \begin{bmatrix} 1 \\ i \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 4i$

when $\lambda = -4i$

$$\textcircled{1} \Rightarrow \begin{bmatrix} 4i & 4 \\ -4 & 4i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{Rank of } \begin{bmatrix} 4i & 4 \\ -4 & 4i \end{bmatrix} = 1$$

$$\Rightarrow 4i x_1 + 4 x_2 = 0$$

$$-4 x_1 + 4i x_2 = 0$$

considering first equation, $4i x_1 + 4 x_2 = 0$

$$\Rightarrow 4i x_1 = -4 x_2$$

$$\Rightarrow \frac{x_1}{-4} = \frac{x_2}{4i}$$

$$\Rightarrow \frac{x_1}{-1} = \frac{x_2}{i}$$

$\therefore \begin{bmatrix} -1 \\ i \end{bmatrix}$ is the eigen vector corresponding to $\lambda = -4i$

H.W
Q. Find the eigen values and eigen vectors of $A = \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & 3 \end{bmatrix}$

Ans: $\lambda = \frac{3}{2}, 3$. When $\lambda = \frac{3}{2}$, eigen vector = $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

When $\lambda = 3$, eigen vector = $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

3. Find the eigen values and eigen vectors of $A = \begin{bmatrix} 4 & 2 & -2 \\ 2 & 5 & 0 \\ -2 & 0 & 3 \end{bmatrix}$

Ans: $|A - \lambda I| = 0 \Rightarrow \lambda^3 - (\text{sum of diagonal elements of } A)\lambda^2 + (\text{Sum of minors of diagonal elements of } A)\lambda - |A|_0$

$$\text{Sum of diagonal elements} = 4+5+3=12$$

$$\text{minor of } a_{11}, m_{11} = \begin{vmatrix} 5 & 0 \\ 0 & 3 \end{vmatrix} = 15$$

$$\text{minor of } a_{22}, m_{22} = \begin{vmatrix} 4 & -2 \\ -2 & 3 \end{vmatrix} = 12 - 4 = 8$$

$$\text{minor of } a_{33}, m_{33} = \begin{vmatrix} 4 & 2 \\ 2 & 5 \end{vmatrix} = 20 - 4 = 16$$

$$\text{Sum of minors} = 15+8+16=39$$

$$|A| = 28$$

$$\therefore |A - \lambda I| = 0 \Rightarrow \lambda^3 - 12\lambda^2 + 39\lambda - 28 = 0$$

$$\therefore \lambda = 1, 7, 4$$

\therefore Eigen values are $\lambda = 1, 7, 4$

$$[A - \lambda I]x = 0 \Rightarrow \begin{bmatrix} 4-\lambda & 2 & -2 \\ 2 & 5-\lambda & 0 \\ -2 & 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{--- (1)}$$

When $\lambda = 1$

$$(1) \Rightarrow \begin{bmatrix} 3 & 2 & -2 \\ 2 & 4 & 0 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \quad \text{Rank of} \begin{bmatrix} 3 & 2 & -2 \\ 2 & 4 & 0 \\ -2 & 0 & 2 \end{bmatrix} = 2$$

$$\Rightarrow 3x_1 + 2x_2 - 2x_3 = 0$$

$$2x_1 + 4x_2 + 0x_3 = 0$$

$$-2x_1 + 0x_2 + 2x_3 = 0$$

considering first two equations,

$$\begin{matrix} x_2 & x_3 & x_1 & x_2 \\ 2 & -2 & 3 & 2 \\ 4 & 0 & 2 & 4 \end{matrix}$$

$$\therefore \frac{x_1}{0-8} = \frac{x_2}{-4-0} = \frac{x_3}{12-4}$$

$$\Rightarrow \frac{x_1}{8} = \frac{x_2}{-4} = \frac{x_3}{8} \implies \frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{2}$$

$\therefore \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 1$

when $\lambda = 7$

$$① \Rightarrow \begin{bmatrix} -3 & 2 & -2 \\ 2 & -2 & 0 \\ -2 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -3x_1 + 2x_2 - 2x_3 = 0$$

$$2x_1 - 2x_2 + 0x_3 = 0$$

$$-2x_1 + 0x_2 - 4x_3 = 0$$

Rank = 2. Considering first two equations,

$$\begin{matrix} x_2 & x_3 & x_1 & x_2 \\ 2 & -2 & -3 & 2 \\ -2 & 0 & 2 & -2 \end{matrix}$$

$$\therefore \frac{x_1}{0-4} = \frac{x_2}{-4-0} = \frac{x_3}{6-4}$$

$$\Rightarrow \frac{x_1}{-4} = \frac{x_2}{-4} = \frac{x_3}{2} \implies \frac{x_1}{-2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$\therefore \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 7$

when $\lambda = 4$

$$① \Rightarrow \begin{bmatrix} 0 & 2 & -2 \\ 2 & 1 & 0 \\ -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 0x_1 + 2x_2 - 2x_3 = 0$$

$$2x_1 + x_2 + 0x_3 = 0$$

$$-2x_1 + 0x_2 - 1x_3 = 0$$

Rank = 2. Considering first two equations,

$$\begin{array}{cccc} x_1 & x_2 & x_1 & x_2 \\ 2 & -2 & 0 & 2 \\ 1 & 0 & 2 & 1 \end{array}$$

$$\therefore \frac{x_1}{0-2} = \frac{x_2}{-4-0} = \frac{x_3}{0-4}$$

$$\Rightarrow \frac{x_1}{2} = \frac{x_2}{-4} = \frac{x_3}{-4} \Rightarrow \frac{x_1}{1} = \frac{x_2}{-2} = \frac{x_3}{-2}$$

$\therefore \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 4$

H.W. 4. Find the eigen values and eigen vectors of

$$A = \begin{bmatrix} 3 & 5 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix}.$$

Ans: $\lambda = 1, 4, 3$

$$\lambda = 1, \text{ eigen vector} = \begin{bmatrix} 7 \\ -4 \\ 2 \end{bmatrix}$$

$$\text{when } \lambda = 4, \text{ eigen vector} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

$$\text{when } \lambda = 3, \text{ eigen vector} = \begin{bmatrix} 27 \\ 0 \\ 0 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

5 Find the eigen values and eigen vectors of $A = \begin{bmatrix} 6 & 5 & 2 \\ 2 & 0 & -8 \\ 5 & 4 & 0 \end{bmatrix}$

Ans: $|A - \lambda I| = 0 \Rightarrow \lambda^3 - (\text{sum of diagonal elements of } A)\lambda^2 + (\text{sum of minors of diagonal elements of } A)\lambda - |A| = 0$.

$$\text{sum of diagonal elements} = 6+0+0=6$$

$$\text{minor of } a_{11}, m_{11} = \begin{vmatrix} 0 & -8 \\ 4 & 0 \end{vmatrix} = 0 - -32 = 32$$

$$\text{minor of } a_{22}, m_{22} = \begin{vmatrix} 6 & 2 \\ 5 & 0 \end{vmatrix} = 0 - 10 = -10$$

$$\text{minor of } a_{33}, m_{33} = \begin{vmatrix} 6 & 5 \\ 2 & 0 \end{vmatrix} = 0 - 10 = -10$$

$$\therefore \text{sum of minors} = 32 - 10 - 10 = 12$$

$$|A| = 8$$

$$\therefore |A - \lambda I| = 0 \Rightarrow \lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$$

$$\therefore \underline{\underline{\lambda = 2, 2, 2}}$$

$$[A - \lambda I] x = 0 \Rightarrow \begin{bmatrix} 6-\lambda & 5 & 2 \\ 2 & -\lambda & -8 \\ 5 & 4 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{--- (1)}$$

$$\text{When } \lambda = 2$$

$$(1) \Rightarrow \begin{bmatrix} 4 & 5 & 2 \\ 2 & -2 & -8 \\ 5 & 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 4x_1 + 5x_2 + 2x_3 = 0$$

$$2x_1 - 2x_2 - 8x_3 = 0$$

$$5x_1 + 4x_2 - 2x_3 = 0$$

$$\text{Rank of } \begin{bmatrix} 4 & 5 & 2 \\ 2 & -2 & -8 \\ 5 & 4 & -2 \end{bmatrix} = 2$$

$$x_2 \quad x_3 \quad x_1 \quad x_2$$

$$5 \quad 2 \quad 4 \quad 5$$

$$-2 \quad -8 \quad 2 \quad -2$$

$$\therefore \frac{x_1}{\begin{vmatrix} 5 & 2 \\ -2 & -8 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 2 & 4 \\ -8 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 4 & 5 \\ 2 & -2 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{-40-4} = \frac{x_2}{4-32} = \frac{x_3}{-8-10}$$

$$\Rightarrow \frac{x_1}{-36} = \frac{x_2}{36} = \frac{x_3}{-18}$$

$$\Rightarrow \frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$\therefore \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda=2$

6. Find the eigen values and eigen vectors of

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Ans: $|A - \lambda I| = 0 \Rightarrow \lambda^3 - (\text{sum of diagonal elements of } A) \lambda^2 + (\text{sum of minors of diagonal elements of } A) \lambda - |A| = 0$

$$\text{Sum of diagonal elements} = -2 + 1 + 0 = -1$$

$$m_{11} = \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} = 0 - 12 = -12$$

$$m_{22} = \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} = 0 - 3 = -3$$

$$m_{33} = \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix} = -2 - 4 = -6$$

$$\text{Sum of minors} = -12 + -3 + -6 = -21$$

$$|A| = 45$$

$$\therefore |A - \lambda I| = 0 \Rightarrow \lambda^3 - \lambda^2 - 21\lambda - 45 = 0$$

$$\Rightarrow \lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

$$\therefore \underline{\underline{\lambda = 5, -3, -3}}$$

$$[A - \lambda I]x = 0 \Rightarrow \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{--- (1)}$$

when $\lambda = 5$

$$(1) \Rightarrow \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -7x_1 + 2x_2 - 3x_3 = 0$$

$$2x_1 - 4x_2 - 6x_3 = 0$$

$$-x_1 - 2x_2 - 5x_3 = 0$$

$$\text{Rank of } \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} = 2$$

Considering first two equations,

$$\begin{array}{cccc} 2 & 2 & x_1 & x_2 \\ 2 & -3 & -7 & 2 \\ -4 & -6 & 2 & -4 \end{array}$$

$$\frac{x_1}{\begin{vmatrix} 2 & -3 \\ -4 & -6 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} -3 & -7 \\ -6 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -7 & 2 \\ 2 & -4 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{-12 - 12} = \frac{x_2}{-6 - 4} = \frac{x_3}{28 - 4}$$

$$\Rightarrow \frac{x_1}{-24} = \frac{x_2}{-48} = \frac{x_3}{24}$$

$$\Rightarrow \frac{x_1}{-1} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$\therefore \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 5$

when $\lambda = -3$

$$(1) \Rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 + 2x_2 - 3x_3 = 0$$

$$2x_1 + 4x_2 - 6x_3 = 0$$

$$-x_1 - 2x_2 + 3x_3 = 0$$

Rank of $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} = 1$

rank, $r=1$ and no. of unknowns $n=3$

$$\therefore n-r = 3-1 = 2$$

so 2 variables have to assign arbitrary values

consider the first equation $x_1 + 2x_2 - 3x_3 = 0$

Put $x_3 = a, x_2 = b$

$$\therefore x_1 = 3x_3 - 2x_2 = 3a - 2b$$

when $a=0$ and $b=1$

$$x_1 = -2, x_2 = 1, x_3 = 0$$

when $a=1$ and $b=0$

$$x_1 = 3, x_2 = 0, x_3 = 1$$

$\therefore \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ are the eigen vectors corresponding to $\lambda = -3$

7. Find the eigen values and eigen vectors of

$$A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$$

Ans: $|A-\lambda I| = 0 \Rightarrow \lambda^3 - (\text{sum of diagonal elements of } A)\lambda^2 + (\text{sum of minors of diagonal elements of } A)\lambda - |A|=0$

Sum of diagonal elements = $5+4-4 = 5$

$$m_{11} = \begin{vmatrix} 4 & 2 \\ -6 & -4 \end{vmatrix} = -16 - (-12) = -16 + 12 = -4$$

$$m_{22} = \begin{vmatrix} 5 & -6 \\ 3 & -4 \end{vmatrix} = -20 - -18 = -20 + 18 = -2$$

$$m_{33} = \begin{vmatrix} 5 & -6 \\ -1 & 4 \end{vmatrix} = 20 - 6 = 14$$

$$\text{Sum of minors} = -4 + -2 + 14 = 8$$

$$|A| = 4$$

$$\therefore |A - \lambda I| = 0 \Rightarrow \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$\therefore \lambda = \underline{\underline{1, 2, 2}}$$

$$[A - \lambda I]x = 0 \Rightarrow \begin{bmatrix} 5-\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & -4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{--- (1)}$$

when $\lambda = 1$

$$(1) \Rightarrow \begin{bmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \Rightarrow 4x_1 - 6x_2 - 6x_3 &= 0 \\ -x_1 + 3x_2 + 2x_3 &= 0 \quad \text{Rank} = 2 \\ 3x_1 - 6x_2 - 5x_3 &= 0 \end{aligned}$$

$$\begin{array}{cccc} x_2 & x_3 & x_1 & x_2 \\ -6 & -6 & 4 & -6 \\ 3 & 2 & -1 & 3 \end{array}$$

$$\frac{x_1}{\begin{vmatrix} x_2 \\ -6 & -6 \\ 3 & 2 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} x_2 \\ -6 & 4 \\ 2 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 4 & -6 \\ -1 & 3 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{-12 - -18} = \frac{x_2}{6 - 8} = \frac{x_3}{12 - 6}$$

$$\Rightarrow \frac{x_1}{-12} = \frac{x_2}{-2} = \frac{x_3}{6} \Rightarrow \frac{x_1}{3} = \frac{x_2}{-1} = \frac{x_3}{3}$$

$\therefore \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$ is the eigen vector corresponding to $\lambda=1$

when $\lambda=2$

$$\textcircled{1} \Rightarrow \begin{bmatrix} 3 & -6 & -6 \\ -1 & 2 & 2 \\ 3 & -6 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 3x_1 - 6x_2 - 6x_3 = 0$$

$$-x_1 + 2x_2 + 2x_3 = 0$$

$$3x_1 - 6x_2 - 6x_3 = 0$$

Rank = 1 and no. of unknowns, $n=3$

$$\therefore n - r = 3 - 1 = 2$$

so 2 variables should assign arbitrary values

Consider the first equation $3x_1 - 6x_2 - 6x_3 = 0$

Put $x_3 = a$ and $x_2 = b$

$$\therefore 3x_1 = 6x_2 + 6x_3 = 6b + 6a$$

$$\Rightarrow x_1 = \frac{6b+6a}{3} = \frac{3(2a+2b)}{3} = 2a+2b$$

when $a=0$ and $b=1$

$$x_1 = 2, x_2 = 1, x_3 = 0$$

when $a=1$ and $b=0$

$$x_1 = 2, x_2 = 0, x_3 = 1$$

$\therefore \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ are the eigen vectors corresponding to $\lambda = 2$

Properties of eigen values.

1. The sum of eigen values of a matrix is the sum of diagonal elements of the matrix.
2. The eigen values of a square matrix A and its transpose A^T are the same.
3. The product of the eigen values of A is equal to $|A|$.
4. If λ is an eigen value of a non-singular square matrix A , then $\frac{1}{\lambda}$ is an eigen value of A^{-1} .
5. If λ is an eigen value of a non singular square matrix A then $\frac{|A|}{\lambda}$ is an eigen value of $\text{adj } A$.
6. If λ is an eigen value of A , then $A - kI$ has the eigen value $\lambda - k$.
7. If λ is an eigen value of A , then kA has the eigen value $k\lambda$.
8. If λ is an eigen value of A , then A^m has the eigen value λ^m .

Problems

1. If the eigen values of the matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ are 1, 2 and 3. Find the eigen values of A^5 and A^{-1} without using characteristic equation.

Ans: Eigen values of A are 1, 2 and 3

$$\therefore \text{Eigen values of } A^5 \text{ are } \lambda^5 = 1^5, 2^5 \text{ and } 3^5 \\ = 1, 32 \text{ and } 243$$

Eigen values of A^{-1} are $\lambda^{-1} = \frac{1}{\lambda}$

\therefore Eigen values of $A^{-1} = \frac{1}{1}, \frac{1}{2}, \frac{1}{3}$

$$= 1, \frac{1}{2}, \frac{1}{3}$$

Q. If α is an eigen value of $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$,

without using characteristic equation, find the other eigen values of the matrix A . Also find the eigen values of $A^3, A^T, A^{-1}, 5A, A-3I$ and $\text{adj } A$.

Ans: Given $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Let λ_1, λ_2 and λ_3 be the eigen values of A .

Let $\lambda_1 = \alpha$

$$\lambda_1 + \lambda_2 + \lambda_3 = \text{sum of diagonal elements}$$
$$= 3+5+3 = 11$$

$$\therefore \lambda_2 + \lambda_3 = 11 - \lambda_1 = 11 - \alpha = 9$$

$$\Rightarrow \lambda_2 + \lambda_3 = 9 \quad \text{--- (1)}$$

$$\text{Also } \lambda_1 \times \lambda_2 \times \lambda_3 = |A|$$

$$\Rightarrow \lambda_2 \lambda_3 = 36 \quad (\text{Here } |A| = 36)$$

$$\Rightarrow \lambda_2 \lambda_3 = \frac{36}{\alpha} = 18$$

$$\Rightarrow \lambda_2 \lambda_3 = 18$$

$$\Rightarrow \lambda_2 = \frac{18}{\lambda_3}$$

$$\therefore (1) \Rightarrow \frac{18}{\lambda_3} + \lambda_3 = 9 \Rightarrow \frac{18 + \lambda_3^2}{\lambda_3} = 9$$

$$\Rightarrow 18 + \lambda_3^2 = 9\lambda_3$$

$$\Rightarrow \lambda_3^2 - 9\lambda_3 + 18 = 0$$

$$\rightarrow \lambda_3 = 6, 3$$

$$\therefore \lambda_2 = \frac{18}{\lambda_3} = \frac{18}{6} \text{ and } \frac{18}{3} \\ = 3 \text{ and } 6$$

$$\therefore \underline{\lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 6}$$

Eigen values of $A^3 = \lambda_1^3, \lambda_2^3, \lambda_3^3$
 $= 2^3, 3^3, 6^3 = \underline{\underline{8, 27, 216}}$

Eigen values of A^T = eigen values of $A = \underline{\underline{2, 3, 6}}$

Eigen values of $A^{-1} = \frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$
 $= \frac{1}{2}, \frac{1}{3}, \frac{1}{6}$

Eigen values of $5A = 5\lambda_1, 5\lambda_2, 5\lambda_3$
 $= \underline{\underline{10, 15, 30}}$

Eigen values of $A - 3I = \lambda_1 - 3, \lambda_2 - 3, \lambda_3 - 3$
 $= 2 - 3, 3 - 3, 6 - 3 = \underline{\underline{-1, 0, 3}}$

Eigen values of $\text{adj } A = \frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}, \frac{|A|}{\lambda_3}$
 $= \frac{36}{2}, \frac{36}{3}, \frac{36}{6}$
 $= \underline{\underline{18, 12, 6}}$

Diagonalisation of a matrix

Let A be an $n \times n$ square matrix. If A has a basis of eigen vectors, then the matrix $D = X^{-1}AX$ is a diagonal matrix with eigen values of A as the entries on the main diagonal and X is the matrix with the basis of eigen vectors as columns.

Steps of diagonalisation process

Let A be a square matrix of order 3.

- 1) Find the eigen values of A . Let it be $\lambda_1, \lambda_2, \lambda_3$
- 2) Find the eigen vectors of A . They must be independent. Let it be x_1, x_2, x_3 .
- 3) Form the matrix $X = [x_1 \ x_2 \ x_3]$
- 4) Find $D = X^{-1}AX$

D should be equal to $\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

Note

If $D = X^{-1}AX$, then $D^m = X^{-1}A^mX$

$$\Rightarrow A^m = X D^m X^{-1}$$

- 1) Diagonalise $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Ans: $|A - \lambda I| = 0 \Rightarrow \lambda^3 - (\text{sum of diagonal elements of } A)\lambda^2 + (\text{sum of minors of diagonal elements of } A)\lambda - |A| = 0$

sum of diagonal elements of $A = 3+3+3 = 9$

$$m_{11} = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 9 - 1 = 8$$

$$m_{22} = \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} = 9 - 1 = 8$$

$$m_{33} = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 9 - 1 = 8$$

sum of minors of diagonal elements = $8 + 8 + 8 = 24$

$$|A| = 20$$

$$\therefore |A - \lambda I| = 0 \Rightarrow \lambda^3 - 9\lambda^2 + 24\lambda - 20 = 0$$

$$\because \lambda = 2, 2, 5$$

$$[A - \lambda I] x = 0 \Rightarrow \begin{bmatrix} 3-\lambda & -1 & 1 \\ -1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \textcircled{1}$$

when $\lambda = 2$

$$\textcircled{1} \Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 - x_2 + x_3 = 0$$

$$-x_1 + x_2 - x_3 = 0 \quad \text{Rank} = 1$$

$$x_1 - x_2 + x_3 = 0$$

$$x_1 - x_2 + x_3 = 0$$

$\therefore n - r = 3 - 1 = 2$. So a variable should be assigned arbitrary value

Let $x_3 = a$ and $x_2 = b$

$$\therefore x_1 = x_2 - x_3 = b - a$$

$$\text{when } a=0, b=1$$

$$x_1 = 1, x_2 = 1, x_3 = 0$$

$$\left| \begin{array}{l} \text{when } a=1, b=0 \\ x_1 = -1, x_2 = 0, x_3 = 1 \end{array} \right.$$

$\therefore \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are the eigen vectors corresponding to $\lambda = 2$

when $\lambda = 5$

$$① \Rightarrow \begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -2x_1 - x_2 + x_3 = 0$$

$$-x_1 - 2x_2 - x_3 = 0 \quad \text{Rank} = 2$$

$$x_1 - 2x_2 - 2x_3 = 0$$

$$\begin{array}{cccc} x_2 & x_3 & x_1 & x_2 \\ -1 & 1 & -2 & -1 \\ -2 & -1 & -1 & -2 \end{array} \quad (\text{considering first two equations})$$

$$\frac{x_1}{1 - -2} = \frac{x_2}{-1 - 2} = \frac{x_3}{-1 - 1}$$

$$\Rightarrow \frac{x_1}{3} = \frac{x_2}{-3} = \frac{x_3}{3} \rightarrow \frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$\therefore \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 5$

$$\therefore x_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ and } x_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\text{so } x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$D = X^{-1} A X = \begin{bmatrix} \gamma_3 & \frac{2}{3} \gamma_3 & \gamma_3 \\ -\gamma_3 & \gamma_3 & \frac{2}{3} \gamma_3 \\ \gamma_3 & -\gamma_3 & \gamma_3 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

g. Diagonalise the matrix $A = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix}$ and hence find A^2 .

Ans: $|A - \lambda I| = 0 \Rightarrow \lambda^3 - (\text{sum of diagonal elements of } A) \lambda^2 + (\text{sum of minors of diagonal elements of } A) \lambda - |A| = 0$

Sum of diagonal elements = $1 + 0 + -1 = 0$

$$m_{11} = \begin{vmatrix} 0 & 2 \\ 2 & -1 \end{vmatrix} = 0 - 4 = -4$$

$$m_{22} = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = -1 - 0 = -1$$

$$m_{33} = \begin{vmatrix} 1 & -2 \\ -2 & 0 \end{vmatrix} = 0 - 4 = -4$$

Sum of minors = $-4 + -1 + -4 = -9$

$$|A| = 0$$

$$|A - \lambda I| = 0 \Rightarrow \lambda^3 - 0\lambda^2 - 9\lambda - 0 = 0$$

$$\therefore \underline{\lambda = 0, 3, -3}$$

$$[A - \lambda I]x = 0 \Rightarrow \begin{bmatrix} 1-\lambda & -2 & 0 \\ -2 & -\lambda & 2 \\ 0 & 2 & -1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{--- (1)}$$

when $\lambda = 0$

$$① \rightarrow \begin{bmatrix} 1 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 - 2x_2 + 0x_3 = 0$$

$$-2x_1 + 0x_2 + 2x_3 = 0 \quad \text{Rank} = 2$$

$$0x_1 + 2x_2 - x_3 = 0$$

considering first two equations

$$\begin{array}{cccc} x_2 & x_3 & x_1 & x_2 \\ -2 & 0 & 1 & -2 \\ 0 & 2 & -2 & 0 \end{array}$$

$$\frac{x_1}{-4-0} = \frac{x_2}{0-2} = \frac{x_3}{0-4}$$

$$\frac{x_1}{-4} = \frac{x_2}{-2} = \frac{x_3}{-4} \Rightarrow \frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{2}$$

$\therefore \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ is the eigen vector

when $\lambda = 3$

$$① \rightarrow \begin{bmatrix} -2 & -2 & 0 \\ -2 & -3 & 2 \\ 0 & 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -2x_1 - 2x_2 + 0x_3 = 0$$

$$-2x_1 - 3x_2 + 2x_3 = 0$$

$$0x_1 + 2x_2 - 4x_3 = 0$$

considering first two equations,

$$\begin{array}{cccc} x_2 & x_3 & x_1 & x_2 \\ -2 & 0 & -2 & -2 \\ -3 & 2 & -2 & -3 \end{array}$$

$$\frac{x_1}{-4-0} = \frac{x_2}{0-4} = \frac{x_3}{6-4} \Rightarrow \frac{x_1}{-4} = \frac{x_2}{4} = \frac{x_3}{2}$$

$$\Rightarrow \frac{x_1}{-2} = \frac{x_2}{2} = \frac{x_3}{1}$$

$\therefore \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$ is the eigen vector

when $\lambda = -3$

$$① \Rightarrow \begin{bmatrix} 4 & -2 & 0 \\ -2 & 3 & 2 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 4x_1 - 2x_2 + 0x_3 = 0$$

$$-2x_1 + 3x_2 + 2x_3 = 0 \quad \text{Rank} = 2$$

$$0x_1 + 2x_2 + 2x_3 = 0$$

$$\begin{array}{cccc} x_2 & x_3 & x_1 & x_2 \\ -2 & 0 & 4 & -2 \\ 3 & 2 & -2 & 3 \end{array}$$

$$\frac{x_1}{-4-0} = \frac{x_2}{0-8} = \frac{x_3}{12-4} \Rightarrow \frac{x_1}{-4} = \frac{x_2}{-8} = \frac{x_3}{8}$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{-2}$$

$\therefore \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ is the eigen vector.

$$\therefore x_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, x_2 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \text{ and } x_3 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

$$\text{so } X = \begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{bmatrix}$$

$$D = x^{-1}Ax = \begin{bmatrix} -\frac{2}{9} & \frac{1}{9} & \frac{2}{9} \\ -\frac{2}{9} & \frac{2}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{2}{9} & -\frac{2}{9} \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Now $A^3 = x D^2 x^{-1}$

$$= \begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & -2 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} -\frac{2}{9} & \frac{1}{9} & \frac{2}{9} \\ -\frac{2}{9} & \frac{2}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{2}{9} & -\frac{2}{9} \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -2 & -4 \\ -2 & 8 & -2 \\ -4 & -2 & 5 \end{bmatrix}$$

3. Diagonalise the matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ and hence find A^4 .

Ans: $|A - \lambda I| = 0 \Rightarrow \lambda^3 - (\text{sum of diagonal elements of } A)\lambda + (\text{sum of minors of diagonal elements of } A)\lambda - |A| = 0$
 sum of diagonal elements = $2+2+2 = 6$

$$m_{11} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 - 0 = 4$$

$$m_{22} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3 \quad : m_{33} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 - 0 = 4$$

sum of minors = 4 + 3 + 4 = 11

$$|A| = 6$$

$$\therefore |A - \lambda I| = 0 \Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\therefore \lambda = 1, 2, 3$$

$$[A - \lambda I] \mathbf{x} = 0 \Rightarrow \begin{bmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{--- (1)}$$

When $\lambda = 1$

$$(1) \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 + 0x_2 + x_3 = 0$$

$$0x_1 + x_2 + 0x_3 = 0$$

Rank = 2

$$x_1 + 0x_2 + x_3 = 0$$

Considering first two equations,

$$\begin{array}{cccc} x_1 & x_2 & x_1 & x_2 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}$$

$$\frac{x_1}{0-1} = \frac{x_2}{0-0} = \frac{x_3}{1-0} \Rightarrow \frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$$

$\therefore \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 1$

When $\lambda = 2$

$$(1) \Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 0x_1 + 0x_2 + x_3 = 0$$

$$0x_1 + 0x_2 + 0x_3 = 0 \quad \text{Rank} = 2$$

$$x_1 + 0x_2 + 0x_3 = 0$$

considering first and third equations,

$$\begin{matrix} x_2 & x_3 & x_1 & x_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{matrix}$$

$$\frac{x_1}{0-0} = \frac{x_2}{1-0} = \frac{x_3}{0-0} \Rightarrow \frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{0}$$

$\therefore \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is the eigen vector corresponding to $\lambda=2$

when $\lambda=3$

$$\textcircled{1} \Rightarrow \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -x_1 + 0x_2 + x_3 = 0$$

$$0x_1 - x_2 + 0x_3 = 0 \quad \text{Rank} = 2$$

$$x_1 + 0x_2 - x_3 = 0$$

considering first two equations,

$$\begin{matrix} x_2 & x_3 & x_1 & x_2 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & -1 \end{matrix}$$

$$\frac{x_1}{0-1} = \frac{x_2}{0-0} = \frac{x_3}{1-0} \Rightarrow \frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$$

$\therefore \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda=3$

$$\therefore x_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } x_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{so } X = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$D = X^{-1} A X = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\text{now, } A^4 = X D^4 X^{-1}$$

$$= \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 41 & 0 & 40 \\ 0 & 16 & 0 \\ 40 & 0 & 41 \end{bmatrix}$$

H.W

4 Diagonalise $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

$$\text{Ans: } \lambda = 0, 3, 15$$

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, x_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \text{ and } x_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$D = X^{-1} A X = \begin{bmatrix} \frac{1}{9} & \frac{2}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{1}{9} & -\frac{2}{9} \\ \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{bmatrix} \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

Quadratic form

A homogeneous polynomial of 2nd degree in any number of variables is called quadratic form

Eg: $az_1^2 + 2bz_1z_2 + cz_2^2 \rightarrow$ quadratic form in 2 variables

$az_1^2 + bz_2^2 + cz_3^2 + 2dz_1z_2 + 2ez_2z_3 + 2fz_1z_3 \rightarrow$ quadratic form in 3 variables

Every quadratic form can be written as

$Q = x^T Ax$, where A is a symmetric matrix and is called the matrix of the quadratic form.

Principal axes theorem

The substitution $x = xy$ transforms a quadratic form $Q = x^T Ax$ to the principal axis form or canonical form $Q = y^T Dy = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2$, where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigen values (not necessarily distinct) of A and X is an orthogonal matrix.

$$X = \begin{bmatrix} \frac{x_1}{\|x_1\|} & \frac{x_2}{\|x_2\|} & \dots & \frac{x_n}{\|x_n\|} \end{bmatrix} \text{ where } x_1, x_2, \dots, x_n$$

are the eigen vectors corresponding to $\lambda_1, \lambda_2, \dots, \lambda_n$.

Problems

I write the quadratic form of the following matrices.

1. $A = \begin{bmatrix} 3 & 4 \\ 4 & 2 \end{bmatrix}$

Ans:

$$\begin{bmatrix} z_1 & z_2 \\ z_1 & 3 & 4 \\ z_2 & 4 & 2 \end{bmatrix}$$

$$\therefore Q = 3z_1^2 + 4z_1z_2 + 4z_2^2 + 2z_2^2$$

$$= 3z_1^2 + 8z_1z_2 + 2z_2^2$$

$$A = \begin{bmatrix} 2 & 4 & 5 \\ 4 & 3 & 1 \\ 5 & 1 & 1 \end{bmatrix}$$

Ans:

$$\begin{array}{c} x_1 \quad x_2 \quad x_3 \\ \begin{bmatrix} x_1 & 2 & 4 & 5 \\ x_2 & 4 & 3 & 1 \\ x_3 & 5 & 1 & 1 \end{bmatrix} \end{array}$$

\therefore Quadratic form $Q = 2x_1^2 + 4x_1x_2 + 5x_1x_3 + 4x_2x_1 + 3x_2^2 + 1x_2x_3 + 5x_3x_1 + 1x_3x_2 + 1x_3^2$

$$\Rightarrow Q = \underline{\underline{2x_1^2 + 3x_2^2 + x_3^2 + 8x_1x_2 + 10x_1x_3 + 2x_2x_3}}$$

II write down the matrices of the following quadratic forms.

1. $2x^2 + 3y^2 + 6xy$

Ans: $A = \begin{bmatrix} x & y \\ x & 2 & 3 \\ y & 3 & 3 \end{bmatrix}$ i.e; $\underline{\underline{A = \begin{bmatrix} 2 & 3 \\ 3 & 3 \end{bmatrix}}}$

2. $2x^2 + 5y^2 - 6z^2 - 2xy - yz + 8xz$

Ans: $A = \begin{bmatrix} x & y & z \\ x & 2 & -1 & 4 \\ y & -1 & 5 & -\frac{1}{2} \\ z & 4 & -\frac{1}{2} & -6 \end{bmatrix}$ i.e; $\underline{\underline{A = \begin{bmatrix} 2 & -1 & 4 \\ -1 & 5 & -\frac{1}{2} \\ 4 & -\frac{1}{2} & -6 \end{bmatrix}}}$

3. Find a matrix C s.t. $Q = x^T C x$, where

$$Q = -3x_1^2 + 4x_1x_2 - x_2^2 + 2x_1x_3 - 5x_3^2$$

Ans: $C = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_1 & -3 & 2 & 1 \\ x_2 & 2 & -1 & 0 \\ x_3 & 1 & 0 & -5 \end{bmatrix}$ i.e; $\underline{\underline{C = \begin{bmatrix} -3 & 2 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & -5 \end{bmatrix}}}$

4. what kind of conic section or pair of straight line is given by the quadratic form
 $Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$? Transform it into Principal axis or canonical form.

Ans. Matrix of the quadratic form, $A = \begin{bmatrix} x_1 & x_2 \\ x_1 & 17 & -15 \\ x_2 & -15 & 17 \end{bmatrix}$

i.e., $A = \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix}$

Characteristic equation $|A - \lambda I| = 0$

$$\Rightarrow \lambda^2 - (\text{sum of diagonal elements of } A)\lambda + |A| = 0$$

$$\Rightarrow \lambda^2 - 34\lambda + 64 = 0$$

$$\therefore \lambda = 32, 2$$

Principal axis form or canonical form is

$$Q = \lambda_1 y_1^2 + \lambda_2 y_2^2$$

$$\Rightarrow Q = 32y_1^2 + 2y_2^2$$

To find the conic section, given $Q = 128$

$$\therefore 32y_1^2 + 2y_2^2 = 128$$

$$\Rightarrow \frac{32}{128} y_1^2 + \frac{2}{128} y_2^2 = 1$$

$$\Rightarrow \frac{y_1^2}{4} + \frac{y_2^2}{64} = 1$$

$$\Rightarrow \frac{y_1^2}{2^2} + \frac{y_2^2}{8^2} = 1, \text{ which is the equation of an ellipse.}$$

5. write the canonical form of the quadratic form

$Q = 3x_1^2 + 2x_1x_2 + 3x_2^2 = 0$. which conic section this quadratic form represents. Also represent $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ in terms of $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = y$.

$$Q = 3x_1^2 + 2x_1x_2 + 3x_2^2$$

$$\therefore A = \begin{bmatrix} x_1 & x_2 \\ 3 & 11 \\ x_2 & 11 & 3 \end{bmatrix}$$

$$\text{i.e., } A = \begin{bmatrix} 3 & 11 \\ 11 & 3 \end{bmatrix}$$

$$|A - \lambda I| = 0 \Rightarrow \lambda^2 - (\text{sum of diagonal elements of } A)\lambda + |A| = 0$$

$$\Rightarrow \lambda^2 - 6\lambda - 112 = 0$$

$$\therefore \lambda = 14, -8$$

∴ Principal axis form or canonical form is

$$Q = \lambda_1 y_1^2 + \lambda_2 y_2^2$$

$$= 14y_1^2 - 8y_2^2$$

To find the conic section, given $Q = 0$

$$\therefore 14y_1^2 - 8y_2^2 = 0$$

$$\Rightarrow 14y_1^2 = 8y_2^2$$

$$\Rightarrow y_1^2 = \frac{8y_2^2}{14}$$

$$\Rightarrow y_1 = \sqrt{\frac{8y_2^2}{14}} = \pm \sqrt{\frac{8}{14}} y_2$$

$\therefore y_1 = \sqrt{\frac{8}{14}} y_2$ and $y_1 = -\sqrt{\frac{8}{14}} y_2$, which represents
a pair of straight lines.

To express x in terms of y we have to compute eigenvectors.

$$[A - \lambda I]x = 0 \Rightarrow \begin{bmatrix} 3-\lambda & 11 \\ 11 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{--- (1)}$$

When $\lambda = 14$

$$(1) \Rightarrow \begin{bmatrix} -11 & 11 \\ 11 & -11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore -11x_1 + 11x_2 = 0 \Rightarrow -11x_1 = -11x_2 \Rightarrow \frac{x_1}{-11} = \frac{x_2}{-11} \Rightarrow \frac{x_1}{1} = \frac{x_2}{1}$$

$\therefore \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda=14$

when $\lambda = -8$

$$① \Rightarrow \begin{bmatrix} 11 & 11 \\ 11 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore 11x_1 + 11x_2 = 0 \Rightarrow 11x_1 = -11x_2$$

$$\Rightarrow \frac{x_1}{11} = \frac{x_2}{11} \Rightarrow \frac{x_1}{-1} = \frac{x_2}{1}$$

$\therefore \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda=-8$

$$\therefore x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\|x_1\| = \sqrt{1^2 + 1^2} = \sqrt{2} \text{ and } \|x_2\| = \sqrt{(-1)^2 + 1^2} = \sqrt{1+1} = \sqrt{2}$$

$$\frac{x_1}{\|x_1\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \text{ and } \frac{x_2}{\|x_2\|} = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\therefore X = \begin{bmatrix} \frac{x_1}{\|x_1\|} & \frac{x_2}{\|x_2\|} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$x = Xy \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}}y_1 - \frac{1}{\sqrt{2}}y_2 \\ \frac{1}{\sqrt{2}}y_1 + \frac{1}{\sqrt{2}}y_2 \end{bmatrix}$$

$$\therefore x_1 = \frac{1}{\sqrt{2}}y_1 - \frac{1}{\sqrt{2}}y_2 \quad \text{and}$$

$$x_2 = \frac{1}{\sqrt{2}}y_1 + \frac{1}{\sqrt{2}}y_2$$

6. what conic section the quadratic form $Q = -11x_1^2 + 84x_1x_2 + 24x_2^2 = 156$ represents. Represent x in terms of y .

$$Q = -11x_1^2 + 84x_1x_2 + 24x_2^2$$

$$\therefore A = \begin{bmatrix} -11 & 42 \\ 42 & 24 \end{bmatrix}$$

$$|A - \lambda I| = 0 \Rightarrow \lambda^2 - (\text{sum of diagonal elements of } A)\lambda + |A| = 0$$

$$\Rightarrow \lambda^2 - 13\lambda - 2048 = 0$$

$$\therefore \lambda = 52, -39$$

$$\text{Canonical form, } Q = \underline{\underline{52y_1^2 - 39y_2^2}}$$

$$\text{Given } Q = 156 \quad \therefore 52y_1^2 - 39y_2^2 = 156$$

$$\Rightarrow \frac{52y_1^2}{156} - \frac{39y_2^2}{156} = 1$$

$$\Rightarrow \frac{y_1^2}{3} - \frac{y_2^2}{4} = 1, \text{ which represents a hyperbola}$$

To express x in terms of y we have to find eigen vectors.

$$[A - \lambda I]x = 0 \Rightarrow \begin{bmatrix} -11-\lambda & 42 \\ 42 & 24-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{--- (1)}$$

$$\text{when } \lambda = 52$$

$$(1) \Rightarrow \begin{bmatrix} -63 & 42 \\ 42 & -28 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore -63x_1 + 42x_2 = 0 \Rightarrow -63x_1 = -42x_2$$

$$\Rightarrow \frac{x_1}{-42} = \frac{x_2}{-63}$$

$$\Rightarrow \frac{x_1}{2} = \frac{x_2}{3}$$

$\therefore \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 52$.

when $\lambda = -39$

$$① \Rightarrow \begin{bmatrix} 28 & 42 \\ 42 & 63 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore 28x_1 + 42x_2 = 0 \Rightarrow 28x_1 = -42x_2 \\ \Rightarrow \frac{x_1}{-42} = \frac{x_2}{28} \\ \Rightarrow \frac{x_1}{-3} = \frac{x_2}{2}$$

$\therefore \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = -39$

$$\therefore x_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \text{ and } x_2 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$\|x_1\| = \sqrt{2^2 + 3^2} \\ = \sqrt{4 + 9} \\ = \sqrt{13} \\ \|x_2\| = \sqrt{(-3)^2 + 2^2} \\ = \sqrt{9 + 4} \\ = \sqrt{13}$$

$$\frac{x_1}{\|x_1\|} = \begin{bmatrix} \frac{2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{bmatrix} \quad \frac{x_2}{\|x_2\|} = \begin{bmatrix} \frac{-3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{bmatrix}$$

$$\therefore X = \begin{bmatrix} \frac{2}{\sqrt{13}} & \frac{-3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{bmatrix}$$

$$x = Xy$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{13}} & \frac{-3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ = \begin{bmatrix} \frac{2}{\sqrt{13}} y_1 - \frac{3}{\sqrt{13}} y_2 \\ \frac{3}{\sqrt{13}} y_1 + \frac{2}{\sqrt{13}} y_2 \end{bmatrix}$$

$$\therefore x_1 = \frac{2}{\sqrt{13}} y_1 - \frac{3}{\sqrt{13}} y_2 \quad \text{and} \quad x_2 = \frac{3}{\sqrt{13}} y_1 + \frac{2}{\sqrt{13}} y_2$$

Index and signature of the quadratic form

The number of positive terms in the canonical form is called the index of the quadratic form.

signature of the quadratic form = number of positive terms - number of negative terms.

Consider a quadratic form $Q = \mathbf{x}^T \mathbf{A} \mathbf{x}$ with n variables x_1, x_2, \dots, x_n . Then rank of \mathbf{A} is equal to the number of non-zero terms in the canonical form.

Let $\text{rank } (\mathbf{A}) = r$ and index = P . The quadratic form is called

(1) positive definite if $r=n, P=r$

(2) negative definite if $r=n, P=0$

(3) positive semidefinite if $r < n, P=r$

(4) negative semidefinite if $r < n, P=0$

(5) indefinite if canonical form contains both positive and negative terms.

Problems

1. Write the matrix associated with the quadratic form $8x_1^2 + 7x_2^2 + 3x_3^2 - 12x_1x_2 - 8x_2x_3 + 4x_3x_1$. By finding eigen values, determine the nature of the quadratic form.

Ans: Matrix $A = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_1 & 8 & -6 & 2 \\ x_2 & -6 & 7 & -4 \\ x_3 & 2 & -4 & 3 \end{bmatrix}$

$$|A - \lambda I| = 0 \Rightarrow \lambda^3 - (\text{sum of diagonal elements of } A)\lambda^2 + (\text{sum of minors of diagonal elements of } A)\lambda - |A| = 0$$

$$\text{Sum of diagonal elements} = 8 + 7 + 3 = 18$$

$$m_{11} = \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} = 21 - 16 = 5$$

$$m_{22} = \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix} = 24 - 36 = -20$$

$$m_{33} = \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix} = 56 - 36 = 20$$

$$\therefore \text{Sum of minors} = 5 + (-20) + 20 = 5$$

$$|A| = 0$$

$$\therefore |A - \lambda I| = 0 \Rightarrow \lambda^3 - 18\lambda^2 + 45\lambda = 0 = 0$$

$$\therefore \lambda = 0, 3, 15$$

$$\text{Canonical form is } Q = 0y_1^2 + 3y_2^2 + 15y_3^2 \\ = 3y_2^2 + 15y_3^2$$

Index = number of positive terms in canonical form

$$\therefore p = 2$$

rank of A = number of non-zero terms in canonical form

$$\therefore r = 2$$

$$\text{Here } n = 3$$

$$\therefore r < n \text{ and } p = r$$

So Q is positive semidefinite.

- a. Find the nature, rank and signature of the quadratic form $3x^2 + 5y^2 + 3z^2 - 2xy + 2xz - 2yz$

$$A = \begin{bmatrix} x & y & z \\ x & -1 & 1 \\ y & -1 & 5 \\ z & 1 & -1 \end{bmatrix}$$

$|A - \lambda I| = 0 \rightarrow \lambda^3 - (\text{sum of diagonal elements of } A)\lambda^2 + (\text{sum of minors of diagonal elements of } A)\lambda - |A| = 0$

sum of diagonal elements = $3+5+3=11$

$$m_{11} = \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} = 15 - 1 = 14$$

$$m_{22} = \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} = 9 - 1 = 8$$

$$m_{33} = \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix} = 15 - 1 = 14$$

$$|A| = 36$$

$$\therefore |A - \lambda I| = 0 \Rightarrow \lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

$$\therefore \lambda = 2, 3, 6$$

∴ canonical form is $Q = 2y_1^2 + 3y_2^2 + 6y_3^2$

Index = no. of positive terms in canonical form

$$\therefore P = 3$$

Signature = number of positive terms - number of negative terms

$$= 3 - 0 = 3$$

rank, r = no. of non-zero terms in canonical form

$$= 3$$

$$\text{Here } n = 3$$

$$\therefore r = n \text{ and } P = r$$

so Q is positive definite.