MODULE - 4

NUMERICAL METHODS I

Solution of algebraic and transcendental equations

An equation consists of trigonometric functions, inverse trigonometric functions, exponential functions, logarithmic functions, hyperbolic functions or inverse hyperbolic functions is called transcendental equations

Eg: $\sin x + x = 1$

$1 + \log x + e^{x} = 0$

1. Newton Raphson method or (Newton's iteration method)

Newton Raphson method is commonly used for finding the root of the equation $f(x) = 0$.

Let $x = x_0$ be the approximate root

Then $x_1 = x_0 - \left[\frac{f(x_0)}{f'(x_0)}\right]$

$$x_2 = x_1 - \left[\frac{f(x_1)}{f'(x_1)}\right]$$

$$x_3 = x_2 - \left[\frac{f(x_2)}{f'(x_2)}\right] \text{ and so on.}$$

In general $$\boxed{x_{n+1} = x_n - \left[\frac{f(x_n)}{f'(x_n)}\right]}$$

Note

If x_0 is not given, then find two values a and b such that $f(a)$ is negative and $f(b)$ is positive.

Then $x_0 = \frac{a+b}{2}$

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Problems

1. Using Newton Raphson method find the real root near 2 of the equation $x^4 - 12x + 7 = 0$

Ans: $f(x) = x^4 - 12x + 7$

$f'(x) = 4x^3 - 12$

Here $x_0 = 2$

$$\therefore x_1 = x_0 - \left[\frac{f(x_0)}{f'(x_0)}\right] = 2 - \left[\frac{f(2)}{f'(2)}\right]$$

$$= 2 - \left[\frac{(2^4 - 24 + 7)}{4 \times 2^3 - 12}\right]$$

$$= \underline{\underline{2.05}}$$

$$x_2 = x_1 - \left[\frac{f(x_1)}{f'(x_1)}\right] = 2.05 - \left[\frac{f(2.05)}{f'(2.05)}\right]$$

$$= 2.05 - \left[\frac{((2.05)^4 - 12 \times 2.05 + 7)}{4 \times (2.05)^3 - 12}\right]$$

$$= \underline{\underline{2.04728}}$$

$$x_3 = x_2 - \left[\frac{f(x_2)}{f'(x_2)}\right] = 2.04728 - \left[\frac{((2.04728)^4 - 12 \times 2.04728 + 7)}{4 \times (2.04728)^3 - 12}\right]$$

$$= \underline{\underline{2.04726}}$$

Since $x_2 \cong x_3$ (upto 4 decimal digits), the solution of given equation is $\underline{\underline{x = 2.0473}}$

2. Using Newton Raphson method compute the cube root of 7 which is nearer to 2

Ans. Let $x = \sqrt[3]{7} \Rightarrow x^3 = 7 \Rightarrow x^3 - 7 = 0$

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$\therefore f(x) = x^3 - 7$

$f'(x) = 3x^2$

Here $x_0 = 2$

$$\therefore x_1 = x_0 - \left[\frac{f(x_0)}{f'(x_0)}\right] = 2 - \left[\frac{f(2)}{f'(2)}\right]$$

$$= 2 - \frac{(2^3 - 7)}{3\times 2^2} = \underline{\underline{1.916666}}$$

$$x_2 = x_1 - \left[\frac{f(x_1)}{f'(x_1)}\right] = 1.916666 - \left[\frac{f(1.916666)}{f'(1.916666)}\right]$$

$$= 1.916666 - \left[\frac{((1.916666)^3 - 7)}{3(1.916666)^2}\right]$$

$$= \underline{\underline{1.91295}}$$

$$x_3 = x_2 - \left[\frac{f(x_2)}{f'(x_2)}\right] = 1.91295 - \left[\frac{((1.91295)^3 - 7)}{3(1.91295)^2}\right]$$

$$= \underline{\underline{1.912934}}$$

Since $x_2 \cong x_3$, the solution is $\underline{\underline{x = 1.9129}}$

3. Using Newton Raphson method derive a formula to find $\sqrt[3]{N}$ where N is a real number, hence evaluate $\sqrt[3]{35}$ correct to three decimal places.

Ans: Let $x = \sqrt[3]{N} \Rightarrow x^3 = N \Rightarrow x^3 - N = 0$

$\therefore f(x) = x^3 - N$

$f'(x) = 3x^2$

$$x_{n+1} = x_n - \left[\frac{f(x_n)}{f'(x_n)}\right] = x_n - \left[\frac{(x_n^3 - N)}{3x_n^2}\right]$$

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$$= \frac{3x_n^3 - x_n^3 + N}{3x_n^2} = \frac{2x_n^3 + N}{3x_n^2}$$

When $N = 35$, $f(x) = x^3 - 35$ and $f'(x) = 3x^2$

Here x_0 is not given

$\therefore$ Consider $f(0) = 0 - 35 = -35$

$f(1) = 1 - 35 = -34$

$f(2) = 8 - 35 = -27$

$f(3) = 27 - 35 = -8$

$f(4) = 64 - 35 = 29$

$f(3)$ is negative and $f(4)$ is positive.

So $x_0 = \frac{3+4}{2} = \frac{7}{2} = 3.5$

we have $x_{n+1} = \frac{2x_n^3 + N}{3x_n^2}$

$$\therefore x_1 = \frac{2x_0^3 + N}{3x_0^2} = \frac{2(3.5)^3 + 35}{3(3.5)^2}$$

$$= 3.285714$$

$$x_2 = \frac{2x_1^3 + N}{3x_1^2} = \frac{2(3.285714)^3 + 35}{3(3.285714)^2}$$

$$= 3.271131$$

$$x_3 = \frac{2x_2^3 + N}{3x_2^2} = \frac{2(3.271131)^3 + 35}{3(3.271131)^2}$$

$$= 3.271066$$

$\therefore x_2 \cong x_3$ and hence $\sqrt[3]{35} = 3.2711$

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2. **Regula Falsi method or (Method of false position)**

Consider the equation $f(x)=0$. Let $f(a)$ and $f(b)$ be of opposite sign. ie; let $f(a)$ be negative and $f(b)$ be positive, then there is a root of $f(x)=0$ lying between a and b.

Now the first approximation is

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}, \text{ where } a<b$$

If $f(x_1)$ is positive and $f(a)$ is negative, then root lies between a and x_1. Replacing b by x_1, the second approximation is

$$x_2 = \frac{a f(x_1) - x_1 f(a)}{f(x_1) - f(a)}$$

This procedure is repeated until we get the desired accuracy.

Problems

1. Solve the equation $x \tan x + 1 = 0$ by Regula falsi method starting with $a=2.5$, $b=3$ correct to 3 decimal places.

Ans: $f(x) = x \tan x + 1$

Given $a=2.5$ and $b=3$

First approximation is $x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$

$$\Rightarrow x_1 = \frac{2.5 f(3) - 3 f(2.5)}{f(3) - f(2.5)}$$

$$= \frac{2.5\left[3 \tan 3 + 1\right] - 3\left[2.5 \tan 2.5 + 1\right]}{(3 \tan 3 + 1) - (2.5 \tan 2.5 + 1)}$$

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$= 2.80125$

$f(x_1) = f(2.80125) = 2.80125 \tan(2.80125) + 1$

$= 8.02 \times 10^{-3} > 0$

$\therefore f(x_1)$ is positive.

$\therefore$ Replace b by x_1

So Now $a = 2.5$ and $b = 2.80125$

$$\therefore x_2 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$= \frac{2.5 f(2.80125) - 2.80125 f(2.5)}{f(2.80125) - f(2.5)}$$

$= 2.79849244$

$f(x_2) = f(2.79849244)$

$= 2.79849244 \tan(2.79849244) + 1$

$= 2.977320 \times 10^{-4} > 0$

$\therefore f(x_2)$ is positive

So Now $a = 2.5$ and $b = x_2 = 2.79849244$

$$\therefore x_3 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$= \frac{2.5 f(2.79849244) - 2.79849244 f(2.5)}{f(2.79849244) - f(2.5)}$$

$= 2.798390$

$\therefore x_2 \cong x_3$ and the solution is $\underline{\underline{x = 2.798}}$

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Interpolation

Interpolation is the process of computing intermediate values of a function from a given set of tabulor functions.

Newton's forward interpolation formula

Let $y = f(x)$ be a function which takes the values $f(x_0), f(x_1), \dots f(x_n)$ corresponding to the values $x_0, x_1, \dots x_n$ of the independent variable x. Let the values of x be regularly spaced.

ie; $x_1 - x_0 = h$, $x_2 - x_1 = h$, $x_3 - x_2 = h$ and so on.

The Newton's forward difference interpolation formula is

$$\boxed{\begin{gathered} f(x) = f(x_0) + r\Delta f_0 + \frac{r(r-1)}{2!}\Delta^2 f_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 f_0 + \dots \\ \text{where } r = \frac{x - x_0}{h} \end{gathered}}$$

Here Δ is the forward difference operator and

$$\Delta f_j = f_{j+1} - f_j$$

$$\Delta^2 f_j = \Delta f_{j+1} - \Delta f_j$$

$$\Delta^3 f_j = \Delta^2 f_{j+1} - \Delta^2 f_j \text{ and so on.}$$

Newton's backward interpolation formula

Newton's backward interpolation formula is

$$\boxed{\begin{gathered} f(x) = f(x_n) + r\nabla f_n + \frac{r(r+1)}{2!}\nabla^2 f_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 f_n + \dots \\ \text{where } r = \frac{x - x_n}{h}. \end{gathered}}$$

The operator ∇ is called the backward difference operator

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Problems

1. Compute $\cosh 0.56$ using Newton's interpolation formula from the following table:

x_j	0.5	0.6	0.7	0.8
$\cosh x_j$	1.127626	1.185465	1.255169	1.337435

Ans: Here $y = f(x) = \cosh x$

The difference table is

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
0.5	1.127626			
		$1.185465 - 1.127626 = 0.057839$		
0.6	1.185465		$0.069704 - 0.057839 = 0.011865$	
		$1.255169 - 1.185465 = 0.069704$		$0.012562 - 0.011865 = 0.000697$
0.7	1.255169		$0.082266 - 0.069704 = 0.012562$	
		$1.337435 - 1.255169 = 0.082266$		
0.8	1.337435			

We want to find $\cosh 0.56$

$\therefore$ $x = 0.56$ and the value $x = 0.56$ is very near to the starting value $x_0 = 0.5$.

So we use Newton's forward interpolation formula to compute $\cosh 0.56$

$$\therefore r = \frac{x - x_0}{h} \qquad \left(h = x_1 - x_0 = 0.6 - 0.5 = 0.1 \right)$$

$$= \frac{0.56 - 0.5}{0.1} = 0.6$$

$f(x_0) = 1.127626$, $\Delta f_0 = 0.057839$

$\Delta^2 f_0 = 0.011865$, $\Delta^3 f_0 = 0.000697$

Newton's forward interpolation formula is

$$f(x) = f(x_0) + r\,\Delta f_0 + \frac{r(r-1)}{2!}\Delta^2 f_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 f_0 + \cdots$$

$$\therefore f(0.56) = \cosh 0.56 = 1.127626 + 0.6(0.057839)$$
$$+ \frac{0.6(0.6-1)}{2!}(0.011865) + \frac{0.6(0.6-1)(0.6-2)}{3!}(0.000697)$$

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$= 1.160944$

2. From the following data, find the value of $\tan 45^\circ 15'$

x°	45	46	47	48	49	50
$\tan x^\circ$	1	1.03553	1.07237	1.11061	1.15037	1.19175

Ans: Here $y = f(x) = \tan x$

The difference table is

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	$\Delta^5 f$
45	1					
		0.03553				
46	1.03553		0.00131			
		0.03684		0.00009		
47	1.07237		0.00140		0.00003	
		0.03824		0.00012		−0.00005
48	1.11061		0.00152		−0.00002	
		0.03976		0.0001		
49	1.15037		0.00162			
		0.04138				
50	1.19175					

We want to find $\tan 45^\circ 15'$

$\therefore x = 45^\circ 15'$ and this value is very near to the starting value $x_0 = 45$. So we use Newton's forward interpolation formula

Here $r = \dfrac{x - x_0}{h} = \dfrac{45^\circ 15' - 45}{1^\circ}$ $\quad (h = x_1 - x_0 = 46 - 45 = 1)$

$$= \frac{15'}{1^\circ} = \frac{15'}{60'} = 0.25$$

$f(x_0) = 1$, $\Delta f_0 = 0.03553$, $\Delta^2 f_0 = 0.00131$

$\Delta^3 f_0 = 0.00009$, $\Delta^4 f_0 = 0.00003$, $\Delta^5 f_0 = -0.00005$

$$f(x) = f(x_0) + r\Delta f_0 + \frac{r(r-1)}{2!}\Delta^2 f_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 f_0$$

$$+ \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 f_0 + \frac{r(r-1)(r-2)(r-3)(r-4)}{5!}\Delta^5 f_0$$

$$+ \cdots$$

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$\therefore \tan 45^\circ 15' = 1 + (0.25 \times 0.03553) + \left(\frac{0.25(0.25-1)}{2!} \times 0.00131\right)$

$+ \frac{0.25(0.25-1)(0.25-2)}{3!} \times 0.00009$

$+ \frac{0.25(0.25-1)(0.25-2)(0.25-3)}{4!} \times 0.00003$

$+ \frac{0.25(0.25-1)(0.25-2)(0.25-3)(0.25-4)}{5!} \times -0.00005$

$= 1.00876$

3. From the data given below, find the number of students whose weight is between 60 and 70

weight :	0-40	40-60	60-80	80-100	100-120
no. of students :	250	120	100	70	50

Ans: The difference table is

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
Below 40	250				
		120			
Below 60	370		-20		
		100		-10	
Below 80	470		-30		20
		70		10	
Below 100	540		-20		
		50			
Below 120	590				

First let us calculate the no. of students whose weight is less than 70. For that we use Newton's forward interpolation formula.

Here $x = 70$, $x_0 = 40$, $f(x_0) = 250$, $\Delta f_0 = 120$

$\Delta^2 f_0 = -20$, $\Delta^3 f_0 = -10$, $\Delta^4 f_0 = 20$

$h = x_1 - x_0 = 60 - 40 = 20$

$r = \frac{x - x_0}{h} = \frac{70 - 40}{20} = \frac{30}{20} = 1.5$

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$$f(x) = f(x_0) + r\Delta f_0 + \frac{r(r-1)}{2!}\Delta^2 f_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 f_0$$
$$+ \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 f_0 + \cdots$$
$$= 250 + 1.5\times 120 + \frac{1.5(1.5-1)}{2!}\times -20 + \frac{1.5(1.5-1)(1.5-2)}{3!}\times -10$$
$$+ \frac{1.5(1.5-1)(1.5-2)(1.5-3)}{4!}\times 20 = \underline{\underline{424}}$$

From the given data, we have, no. of students whose weight is less than 60 = 370

$\therefore$ No. of students whose weight is between 60 and 70

$$= 424 - 370 = \underline{\underline{54}}$$

4. A second degree polynomial passes through the points $(1,-1)$, $(2,-1)$, $(3,1)$, $(4,5)$. Find the Polynomial.

Ans: The difference table is

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
1	-1			
		0		
			2	
2	-1			0
		2		
			2	
3	1			
		4		
4	5			

To find the polynomial, put $x = x$.

By Newton's forward interpolation formula

$$f(x) = f(x_0) + r\Delta f_0 + \frac{r(r-1)}{2!}\Delta^2 f_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 f_0 + \cdots$$

Here $x_0 = 1$, $\Delta f_0 = 0$, $\Delta^2 f_0 = 2$, $\Delta^3 f_0 = 0$

$f(x_0) = -1$, $h = x_1 - x_0 = 2-1 = 1$

$$\therefore r = \frac{x - x_0}{h} = \frac{x-1}{1} = x-1$$

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$$\therefore f(x) = -1 + 0 + \frac{(x-1)(x-1-1)}{2!} \times 2 + 0$$

$$= -1 + \frac{(x-1)(x-2)}{2} \times 2$$

$$= -1 + x^2 - 2x - x + 2 = \underline{\underline{x^2 - 3x + 1}}$$

5. The following data gives the melting point of an alloy of lead and zinc, where t is the temperature in degrees and p is the percentage of lead in alloy

p:	40	50	60	70	80	90
t:	180	204	226	250	276	304

Find the melting point of alloy containing 84% lead

Ans: The difference table is

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	$\Delta^5 f$
40	180					
		24				
50	204		-2			
		22		4		
60	226		2		-4	
		24		0		4
70	250		2		0	
		26		0		
80	276		2			
		28				
90	304					

Here $x = 84$, which is near to the end of the table So we use Newton's backward interpolation formula

Here $x_n = 90$, $f(x_n) = 304$, $h = x_1 - x_0 = 50 - 40 = 10$

$$\therefore r = \frac{x - x_n}{h} = \frac{84 - 90}{10} = -0.6$$

$\nabla f_n = 28$, $\nabla^2 f_n = 2$, $\nabla^3 f_n = 0$, $\nabla^4 f_n = 0$, $\nabla^5 f_n = 4$

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Newton's Backward interpolation formula is

$$f(x) = f(x_n) + r \nabla f_n + \frac{r(r+1)}{2!} \nabla^2 f_n + \frac{r(r+1)(r+2)}{3!} \nabla^3 f_n$$

$$+ \frac{r(r+1)(r+2)(r+3)}{4!} \nabla^4 f_n + \frac{r(r+1)(r+2)(r+3)(r+4)}{5!} \nabla^5 f_n + \ldots$$

$$\therefore f(84) = 304 + -0.6 \times 28 + \frac{-0.6(-0.6+1)}{2!} \times 2$$

$$+ 0 + 0 + \frac{-0.6(-0.6+1)(-0.6+2)(-0.6+3)(-0.6+4)}{5!} \times 4$$

$$= \underline{\underline{286.86}}$$

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Lagrange's interpolation formula.

Let $y = f(x)$ be a function which assume the values $f(x_0), f(x_1), f(x_2), \ldots, f(x_n)$ corresponding to the values $x = x_0, x_1, x_2, \ldots x_n$ where the values of x are not equispaced. Then the general Lagrange interpolation polynomial is

$$P_n(x) = \frac{(x-x_1)(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)\ldots(x_0-x_n)} \cdot f(x_0) +$$

$$\frac{(x-x_0)(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\ldots(x_1-x_n)} \cdot f(x_1) +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)\ldots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\ldots(x_2-x_n)} \cdot f(x_2) +$$

$$\ldots\ldots\ldots\ldots +$$

$$\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)(x_n-x_3)\ldots(x_n-x_{n-1})} \cdot f(x_n)$$

Problems

1\. Find $f(9.2)$ from the values given below by Lagrange's interpolation formula.

x	8	9	9.5	11
$f(x)$	2.197225	2.251292	2.397895	2.079442

Ans: $x_0 = 8$, $x_1 = 9$, $x_2 = 9.5$, $x_3 = 11$

$f(x_0) = 2.197225$, $f(x_1) = 2.251292$

$f(x_2) = 2.397895$ $f(x_3) = 2.079442$

We want to find $f(9.2)$. $\therefore x = 9.2$

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Since x_0, x_1, x_2, x_3 values are given, we can evaluate $P_3(x)$

$$P_3(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) +$$

$$\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1) +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) +$$

$$\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3).$$

ie; $f(9.2) = P_3(9.2) = \dfrac{(9.2-9)(9.2-9.5)(9.2-11)}{(8-9)(8-9.5)(8-11)} \times 2.1972$

$$+ \frac{(9.2-8)(9.2-9.5)(9.2-11)}{(9-8)(9-9.5)(9-11)} \times 2.251292 +$$

$$\frac{(9.2-8)(9.2-9)(9.2-11)}{(9.5-8)(9.5-9)(9.5-11)} \times 2.397895 +$$

$$\frac{(9.2-8)(9.2-9)(9.2-9.5)}{(11-8)(11-9)(11-9.5)} \times 2.079442$$

$$= \underline{\underline{2.313032}}$$

2. Compute $\log 9.2$ from $\log 9.0 = 2.1972$, $\log 9.5 = 2.2513$, and $\log 11 = 2.3979$ using Lagrange's quadratic interpolation.

Ans: $x_0 = 9$, $x_1 = 9.5$, $x_2 = 11$

$f(x_0) = 2.1972$, $f(x_1) = 2.2513$, $f(x_2) = 2.3979$

we want to find $\log 9.2$. $\therefore x = 9.2$

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Since x_0, x_1, x_2 are given, we can evaluate $P_2(x)$

$$P_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) +$$

$$\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

$$\therefore \log 9.2 = P_2(9.2)$$

$$= \frac{(9.2-9.5)(9.2-11)}{(9-9.5)(9-11)} \times 2.1972 +$$

$$\frac{(9.2-9)(9.2-11)}{(9.5-9)(9.5-11)} \times 2.2513 +$$

$$\frac{(9.2-9)(9.2-9.5)}{(11-9)(11-9.5)} \times 2.3979$$

$$= \underline{\underline{2.2192}}$$

3. Given $f(2)=5$, $f(2.5)=6$. Find the linear interpolating polynomial using Lagrange's formula and also find $f(2.2)$.

Ans: $x_0 = 2$, $x_1 = 2.5$

$f(x_0) = 5$, $f(x_1) = 6$

First, we want to find <u>Lagrange's polynomial</u>

Since x_0, x_1 are given we can evaluate $P_1(x)$. To find the polynomial take x as x itself.

$$\therefore P_1(x) = \frac{(x-x_1)}{(x_0-x_1)} f(x_0) + \frac{(x-x_0)}{(x_1-x_0)} f(x_1)$$

$$= \frac{(x-2.5)}{(2-2.5)} \times 5 + \frac{(x-2)}{(2.5-2)} \times 6$$

$$= \frac{(x-2.5)}{-0.5} \times 5 + \frac{(x-2)}{0.5} \times 6$$

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$$= \frac{5x - 12.5}{-0.5} + \frac{6x - 12}{0.5}$$

$$= \frac{-5x + 12.5 + 6x - 12}{0.5} = \frac{x + 0.5}{0.5}$$

$$= 2x + 1$$

Now $f(2.2) = P_1(2.2) = (2 \times 2.2) + 1$

$$= 4.4 + 1 = 5.4$$

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Divided Differences

Let $f(x_0), f(x_1), f(x_2) \ldots f(x_n)$ be the values of the function $f(x)$ corresponding to the arguments $x_0, x_1, x_2, \ldots x_n$.

The first divided difference for the arguments x_0, x_1 is denoted by $f(x_0, x_1)$ or $\underset{x_1}{\Delta} f(x_0)$ or $[x_0, x_1]$ and is defined as

$$f(x_0, x_1) = \underset{x_1}{\Delta} f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Similarly $f(x_1, x_2) = \underset{x_2}{\Delta} f(x_1) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

and $f(x_2, x_3) = \underset{x_3}{\Delta} f(x_2) = \dfrac{f(x_3) - f(x_2)}{x_3 - x_2}$ etc.

The second divided difference for the arguments x_0, x_1, x_2 is defined as

$$f(x_0, x_1, x_2) = \underset{x_1, x_2}{\Delta^2} f(x_0) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

The third divided difference for the arguments x_0, x_1, x_2, x_3 is defined as

$$f(x_0, x_1, x_2, x_3) = \underset{x_1, x_2, x_3}{\Delta^3} f(x_0) = \frac{f(x_1, x_2, x_3) - f(x_0, x_1, x_2)}{x_3 - x_0}$$

and so on.

Newton's divided difference formula.

$$f(x) = f(x_0) + (x - x_0) \underset{x_1}{\Delta} f(x_0) + (x - x_0)(x - x_1) \underset{x_1, x_2}{\Delta^2} f(x_0)$$

$$+ (x - x_0)(x - x_1)(x - x_2) \underset{x_1, x_2, x_3}{\Delta^3} f(x_0) + \ldots +$$

$$(x - x_0)(x - x_1) \ldots (x - x_{n-1}) \underset{x_1, x_2, \ldots x_n}{\Delta^n} f(x_0)$$

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1. Use Newton's divided difference formula to find $f(7)$, if $f(3)=24$, $f(5)=120$, $f(8)=504$, $f(9)=720$ and $f(12)=1716$.

Ans: The divided difference table is as follows

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$
3	24		
		$\underset{5}{\Delta} f(3) = \frac{120-24}{5-3} = 48$	
			$\underset{5,8}{\Delta^2} f(3) = \frac{128-48}{8-3} = 16$
5	120		
		$\underset{8}{\Delta} f(5) = \frac{504-120}{8-5} = 128$	
			$\underset{8,9}{\Delta^2} f(5) = \frac{216-128}{9-5} = 22$
8	504		
		$\underset{9}{\Delta} f(8) = \frac{720-504}{9-8} = 216$	
			$\underset{9,12}{\Delta^2} f(8) = \frac{332-216}{12-8} = 29$
9	720		
		$\underset{12}{\Delta} f(9) = \frac{1716-720}{12-9} = 332$	
12	1716		

$\Delta^3 f(x)$	$\Delta^4 f(x)$
$\underset{5,8,9}{\Delta^3} f(3) = \frac{22-16}{9-3} = 1$	
	$\underset{5,8,9,12}{\Delta^4} f(3) = \frac{1-1}{12-3} = 0$
$\underset{8,9,12}{\Delta^3} f(5) = \frac{29-22}{12-5} = 1$	

Newton's divided difference formula is

$$f(x) = f(3) + (x-3)\underset{5}{\Delta} f(3) + (x-3)(x-5)\underset{5,8}{\Delta^2} f(3) + (x-3)(x-5)(x-8)\underset{5,8,9}{\Delta^3} f(3) + (x-3)(x-5)(x-8)(x-9)\underset{5,8,9,12}{\Delta^4} f(3)$$

$+ \; - - -$

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Putting $x=7$, we get

$$f(7) = 24 + (7-3)\,48 + (7-3)(7-5)\,16 + (7-3)(7-5)(7-8)\,1$$
$$+ (7-3)(7-5)(7-8)(7-9)\,0$$

$$= 24 + 192 + 128 + -8 + 0$$

$$= \underline{336}$$

2 Use Newton's divided difference formula to find $f(x)$ from the following data:

x:	0	1	2	4	5	6
$f(x)$:	1	14	15	5	6	19

Ans: The divided difference table is as follows

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	1	$\underset{1}{\Delta} f(0) = \frac{14-1}{1-0} = 13$	$\underset{1,2}{\Delta^2} f(0) = \frac{1-13}{2-0} = -6$	$\underset{1,2,4}{\Delta^3} f(0) = \frac{-2-(-6)}{4-0} = 1$
1	14	$\underset{2}{\Delta} f(1) = \frac{15-14}{2-1} = 1$	$\underset{2,4}{\Delta^2} f(1) = \frac{-5-1}{4-1} = -2$	$\underset{2,4,5}{\Delta^3} f(1) = \frac{2-(-2)}{5-1} = 1$
2	15	$\underset{4}{\Delta} f(2) = \frac{5-15}{4-2} = -5$	$\underset{4,5}{\Delta^2} f(2) = \frac{1-(-5)}{5-2} = 2$	$\underset{4,5,6}{\Delta^3} f(2) = \frac{6-2}{6-2} = 1$
4	5	$\underset{5}{\Delta} f(4) = \frac{6-5}{5-4} = 1$	$\underset{5,6}{\Delta^2} f(4) = \frac{13-1}{6-4} = 6$	
5	6	$\underset{6}{\Delta} f(5) = \frac{19-6}{6-5} = 13$		
6	19			

$\Delta^4 f(x)$	$\Delta^5 f(x)$
$\underset{1,2,4,5}{\Delta^4} f(0) = \frac{1-1}{5} = 0$	$\underset{1,2,4,5,6}{\Delta^5} f(0) = 0$
$\underset{2,4,5,6}{\Delta^4} f(1) = \frac{1-1}{6-1} = 0$	

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Newton's divided difference formula is

$$f(x) = f(0) + (x-0)\underset{1}{\Delta} f(0) + (x-0)(x-1)\underset{1,2}{\Delta^2} f(0)$$

$$+ (x-0)(x-1)(x-2)\underset{1,2,4}{\Delta^3} f(0) + \cdots$$

$$= 1 + x \cdot 13 + x(x-1) \cdot -6 + x(x-1)(x-2) \cdot 1$$

$$= 1 + 13x - 6x(x-1) + (x^2 - x)(x-2)$$

$$= 1 + 13x - 6x^2 + 6x + x^3 - 2x^2 - x^2 + 2x$$

$$= \underline{\underline{x^3 - 9x^2 + 21x + 1}}$$

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Numerical integration

Numerical integration is a process of finding the numerical value of a definite integral $I = \int_a^b f(x)dx$ where $f(x)$ is not known explicitly. The function $f(x)$ is given analytically by a formula or empirically by a table of values.

1. **Trapezoidal rule**

Here we subdivide the interval of integration, $a \le x \le b$ into n subintervals of equal length $h = \frac{b-a}{n}$, $a = x_0 < x_1 < x_2 < \dots < x_n = b$, then

$$I = \int_a^b f(x)dx$$

$$= \frac{h}{2}\left[y_0 + y_n + 2(y_1 + y_2 + \dots + y_{n-1})\right],$$ where

$y_0, y_1, y_2, \dots y_n$ are the values of $f(x)$ corresponding to $x_0, x_1, x_2, \dots x_n$ respectively.

2. **Simpson's rule** (simpson's one third rule)

Here we divide the interval of integration $a \le x \le b$ into an even number of equal subintervals (say $n = 2m$) of length $h = \frac{b-a}{n}$, then

$$I = \int_a^b f(x)dx$$

$$= \frac{h}{3}\left[y_0 + y_n + 4(y_1 + y_3 + \dots + y_{n-1}) + 2(y_2 + y_4 + \dots + y_{n-2})\right]$$

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Problems

1. Find the value of $\int_0^1 x^4 dx$, taking (a) $h=0.5$ and (b) $h=0.25$ using trapezoidal rule.

Ans: $f(x)=x^4$, $a=0$, $b=1$

(a) $h=0.5$

x	0	0.5	1
$y=f(x)$	0	0.0625	1

$x_0=0$, $x_1=0.5$, $x_2=1$

$y_0=0$, $y_1=0.0625$, $y_2=1$.

Here $n=2$

$$\therefore I = \int_0^1 x^4 dx = \frac{h}{2}\left[y_0+y_2+2(y_1)\right]$$

$$= \frac{0.5}{2}\left[0+1+2(0.0625)\right]$$

$$= \underline{\underline{0.28125}}$$

(b) $h=0.25$

x	0	0.25	0.5	0.75	1
$y=f(x)$	0	0.003906	0.0625	0.3164	1
	y_0	y_1	y_2	y_3	y_4

$$I=\int_0^1 x^4 dx = \frac{h}{2}\left[y_0+y_4+2(y_1+y_2+y_3)\right]$$

$$= \frac{0.25}{2}\left[0+1+2(0.003906+0.0625+0.3164)\right]$$

$$= \underline{\underline{0.2207}}$$

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2. Find $\int_0^1 \frac{dx}{1+x^2}$ taking 5 subintervals by trapezoidal rule correct to 5 significant digits.

Ans: $f(x) = \frac{1}{1+x^2}$, $a=0$, $b=1$, $n=5$

$\therefore h = \frac{b-a}{n} = \frac{1-0}{5} = 0.2$

x :	0	0.2	0.4	0.6	0.8	1
$f(x)$:	1	0.961538	0.862069	0.735294	0.609756	0.5
	y_0	y_1	y_2	y_3	y_4	y_5

$$I = \int_0^1 \frac{dx}{1+x^2}$$

$$= \frac{h}{2}\left[y_0 + y_5 + 2(y_1 + y_2 + y_3 + y_4)\right]$$

$$= \frac{0.2}{2}\left[1 + 0.5 + 2(0.961538 + 0.862069 + 0.735294 + 0.609756)\right]$$

$$= \underline{\underline{0.78373}}$$

3. Evaluate the integral $\int_0^\pi \sin x\, dx$ using trapezoidal rule and Simpson's rule by dividing the range into ten equal parts.

Ans: $f(x) = \sin x$, $a=0$, $b=\pi$, $n=10$

$\therefore h = \frac{b-a}{n} = \frac{\pi - 0}{10} = \frac{\pi}{10}$

x :	0	$\pi/10$	$2\pi/10$	$3\pi/10$	$4\pi/10$
$f(x)$:	0	0.309017	0.587785	0.809017	0.951057
	y_0	y_1	y_2	y_3	y_4

$5\pi/10$	$6\pi/10$	$7\pi/10$	$8\pi/10$	$9\pi/10$	π
1	0.951057	0.809017	0.587785	0.309017	0
y_5	y_6	y_7	y_8	y_9	y_{10}

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$$\therefore \int_0^{\pi} \sin x\,dx = \frac{h}{2}\left[y_0 + y_{10} + 2(y_1 + y_2 + \cdots + y_9)\right]$$

$$= \frac{\pi}{20}\left[0 + 0 + 2(0.309017 + 0.587785 + 0.809017 + 0.951057 + 1 + 0.951057 + 0.809017 + 0.587785 + 0.309017)\right]$$

$$= \underline{\underline{1.98257}}$$

By simpson's rule

$$\int_0^{\pi} \sin x\,dx = \frac{h}{3}\left[y_0 + y_{10} + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8)\right]$$

$$= \frac{\pi}{30}\left[0 + 0 + 4(0.309017 + 0.809017 + 1 + 0.809017 + 0.309017) + 2(0.587785 + 0.951057 + 0.951057 + 0.587785)\right]$$

$$= \underline{\underline{2.00121}}$$

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