MODULE - II

CONTINUOUS PROBABILITY DISTRIBUTIONS

Continuous random variable

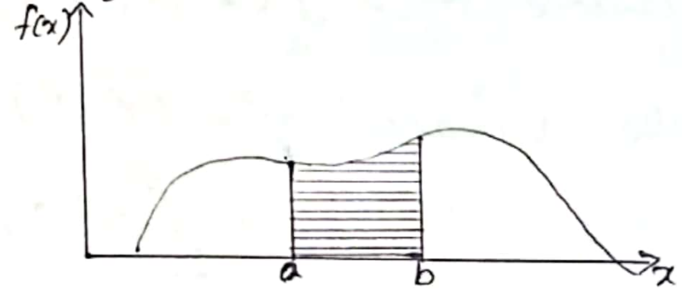
If x is a random variable which can take all values in an interval, then x is called a continuous random variable.

Probability density function

Let x be a continuous random variable. Then a probability distribution or probability density function (Pdf) of x is a function $f(x)$ such that for any two numbers a and b with $a \leq b$,

$$P(a \leq x \leq b) = \int_a^b f(x)dx$$

ie; the probability that x takes on a value in the interval $[a, b]$ is the area above this interval and under the graph of the density function. The graph of $f(x)$ is often referred to as the density curve.



For $f(x)$ to be a pdf, it must satisfy

(1) $f(x) \geq 0$ for all x

(2) $\int_{-\infty}^{\infty} f(x)dx$ = area under the entire graph of $f(x) = 1$

Note

If x is a continuous random variable, then

$P(a \leq x \leq b) = P(a \leq x < b) = P(a < x \leq b) = P(a < x < b)$.

Cumulative distribution function (cdf) or distribution function.

If X is a continuous random variable and $f(x)$ as its pdf, then $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)dx$ is called the cdf or distribution function of X.

Note

1. The pdf of a continuous random variable X is the derivative of the distribution function.

   ie; $f(x) = \frac{d(F(x))}{dx} = F'(x)$

2. $P(X \geq x) = \int_{x}^{\infty} f(x)dx$

3. $P(a < x < b) = F(b) - F(a)$.

Mean and Variance

Mean, $\mu = \int_{-\infty}^{\infty} x f(x)dx = E(X)$

Variance $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$

Also, Variance $\sigma^2 = E(X^2) - [E(X)]^2$

$= \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2$

Problems

1. Check whether $f(x) = \begin{cases} 2e^{-2x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$ is a pdf.

Ans: Clearly $f(x) > 0$

$\int_{-\infty}^{\infty} f(x)dx = \int_{0}^{\infty} 2e^{-2x}dx = \left(2\frac{e^{-2x}}{-2}\right)_0^{\infty} = 0 - -1 = 1$

$\therefore f(x)$ is a pdf.

2. Find k, so that $f(x) = \begin{cases} 0 & , x \le 0 \\ kxe^{-4x^2}, & x > 0 \end{cases}$ is a pdf.

Ans: Since $f(x)$ is a pdf, $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\Rightarrow \int_0^{\infty} kxe^{-4x^2} dx = 1$$

$$\Rightarrow k\int_0^{\infty} xe^{-4x^2} dx = 1$$

Put $u = 4x^2$

$\Rightarrow \frac{du}{dx} = 8x$

$\Rightarrow du = 8x dx$

$\Rightarrow x dx = \frac{du}{8}$

$$\Rightarrow k\int_0^{\infty} e^{-u}\frac{du}{8} = 1$$

$$\Rightarrow \frac{k}{8}\left[\frac{e^{-u}}{-1}\right]_0^{\infty} = 1$$

$$\Rightarrow \frac{-k}{8}\left[0-1\right] = 1$$

$$\Rightarrow \frac{k}{8} = 1 \Rightarrow \underline{\underline{k = 8}}$$

3. Given the probability density $f(x) = \frac{k}{1+x^2}$ for $-\infty < x < \infty$

Find k.

Ans: Since $f(x)$ is a pdf, $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{k}{1+x^2} dx = 1$$

$$\Rightarrow k.\left(\tan^{-1}x\right)_{-\infty}^{\infty} = 1$$

$$\Rightarrow k\left(\tan^{-1}\infty - \tan^{-1}(-\infty)\right) = 1$$

$$\Rightarrow k\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = 1$$

$$\Rightarrow k.\ \frac{2\pi}{2} = 1$$

$$\Rightarrow k\pi = 1$$

$$\Rightarrow \underline{\underline{k = \frac{1}{\pi}}}$$

Uniform Distribution (rectangular distribution)

A rv X is said to have uniform distribution or rectangular distribution over a finite interval (α, β) if its pdf is given by $f(x) = \begin{cases} \frac{1}{\beta-\alpha}, & \alpha < x < \beta \\ 0, & \text{Otherwise} \end{cases}$

Mean of uniform distribution

$$\text{Mean}, \mu = \int_{-\infty}^{x} x f(x) dx$$

$$= \int_{\alpha}^{\beta} x \cdot \frac{1}{\beta-\alpha} dx$$

$$= \frac{1}{\beta-\alpha} \cdot \left(\frac{x^2}{2}\right)_{\alpha}^{\beta}$$

$$= \frac{1}{\beta-\alpha} \left(\frac{\beta^2-\alpha^2}{2}\right)$$

$$= \frac{1}{\beta-\alpha} \cdot \frac{(\beta-\alpha)(\beta+\alpha)}{2} = \frac{\beta+\alpha}{2}$$

$$\therefore \text{Mean } \mu = E(X) = \frac{\alpha+\beta}{2}$$

Variance of uniform distribution

$$\text{Variance}, \sigma^2 = E(X^2) - (E(X))^2$$

$$= \int_{-\infty}^{\infty} x^2 f(x) dx - \left(\frac{\alpha+\beta}{2}\right)^2$$

$$= \int_{\alpha}^{\beta} x^2 \frac{1}{\beta-\alpha} dx - \frac{(\alpha+\beta)^2}{4}$$

$$= \frac{1}{\beta-\alpha} \cdot \left(\frac{x^3}{3}\right)_{\alpha}^{\beta} - \frac{(\alpha^2+2\alpha\beta+\beta^2)}{4}$$

$$= \frac{1}{\beta-\alpha}\,\frac{(\beta^3-\alpha^3)}{3} - \left(\frac{\alpha^2+2\alpha\beta+\beta^2}{4}\right)$$

$$= \frac{1}{\beta-\alpha}\cdot\frac{(\beta-\alpha)(\beta^2+\alpha\beta+\alpha^2)}{3} - \left(\frac{\alpha^2+2\alpha\beta+\beta^2}{4}\right)$$

$$= \frac{\beta^2+\alpha\beta+\alpha^2}{3} - \left(\frac{\alpha^2+2\alpha\beta+\beta^2}{4}\right)$$

$$= \frac{4(\beta^2+\alpha\beta+\alpha^2)-3(\alpha^2+2\alpha\beta+\beta^2)}{12}$$

$$= \frac{4\beta^2+4\alpha\beta+4\alpha^2-3\alpha^2-6\alpha\beta-3\beta^2}{12}$$

$$= \frac{\beta^2-2\alpha\beta+\alpha^2}{12} = \frac{(\beta-\alpha)^2}{12} = \underline{\underline{\frac{(\alpha-\beta)^2}{12}}}$$

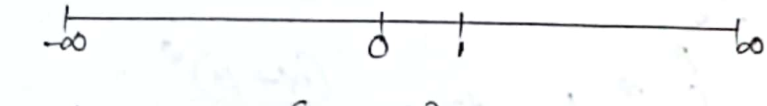
Problems

1. Find the distribution function of a rv having uniform distribution on $(0,1)$.

Ans: The pdf of uniform distribution on $(0,1)$ is

$$f(x) = \begin{cases} \frac{1}{1-0} & , 0<x<1 \\ 0 & , \text{otherwise} \end{cases}$$

$$\Rightarrow \quad f(x) = \begin{cases} 1, & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$$



(i) when $x \in (-\infty, 0)$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = 0$$

(ii) when $x \in (0,1)$

$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$$

$$= 0 + \int_0^x 1\,dx = (x)_0^x = x$$

(iii) when $x \in (1, \infty)$

$$F(x) = \int_{-\infty}^{0} f(x)dx + \int_0^1 f(x)dx + \int_1^x f(x)dx$$

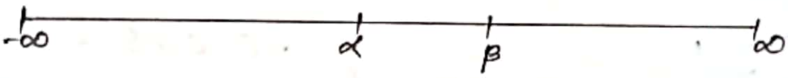
$$= 0 + \int_0^1 1\,dx + 0 = (x)_0^1 = 1$$

$$\therefore \quad F(x) = \begin{cases} 0, & x \leq 0 \\ x, & 0 \leq x \leq 1 \\ 1, & x \geq 1 \end{cases}$$

2. Find the distribution function of a random variable having uniform distribution on (α, β)

Ans: The pdf of uniform distribution on (α, β) is

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha}, & \alpha < x < \beta \\ 0, & \text{otherwise.} \end{cases}$$



(i) when $x \in (-\infty, \alpha)$

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)dx = 0$$

(ii) when $x \in (\alpha, \beta)$

$$F(x) = P(X \leq x) = \int_{-\infty}^{\alpha} f(x)dx + \int_{\alpha}^{x} f(x)dx$$

$$= 0 + \int_{\alpha}^{x} \frac{1}{\beta - \alpha} dx$$

$$= \frac{1}{\beta - \alpha} (x)_{\alpha}^{x} = \frac{x - \alpha}{\beta - \alpha}$$

(iii) when $x \in (\beta, \infty)$

$$F(x) = P(X \leq x) = \int_{-\infty}^{\alpha} f(x)dx + \int_{\alpha}^{\beta} f(x)dx + \int_{\beta}^{x} f(x)dx$$

$$= 0 + \int_{\alpha}^{\beta} \frac{1}{\beta-\alpha} dx + 0$$

$$= \frac{1}{\beta-\alpha} (x)_{\alpha}^{\beta} = \frac{\beta-\alpha}{\beta-\alpha} = 1$$

$$\therefore F(x) = \begin{cases} 0 , & x \le \alpha \\ \frac{x-\alpha}{\beta-\alpha} , & \alpha \le x \le \beta \\ 1 , & x \geq \beta \end{cases}$$

Exponential distribution

A continuous r.v is said to have an exponential distribution with parameter β if X has the probability density function

$$f(x) = \begin{cases} \frac{1}{\beta} e^{-x/\beta}, & x>0,\ \beta>0 \\ 0, & \text{elsewhere.} \end{cases}$$

Note

If X follows poisson distribution with parameter λ, then the time between successive arrivals follows exponential distribution with parameter $\beta = \frac{1}{\lambda}$.

Mean of exponential distribution

Mean $\mu = E(X)$

$$= \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_{0}^{\infty} x \frac{1}{\beta} e^{-x/\beta} dx$$

$$= \frac{1}{\beta}\left[x \frac{e^{-x/\beta}}{-1/\beta} - 1 \cdot \frac{e^{-x/\beta}}{1/\beta^2} \right]_0^{\infty}$$

$$= \frac{1}{\beta}\left[0 - 0 - \left(0 - \frac{1}{1/\beta^2}\right)\right]$$

$$= \frac{1}{\beta} \cdot \beta^2 = \underline{\underline{\beta}}$$

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Variance of exponential distribution

Variance $\sigma^2 = E(X^2) - (E(X))^2$

$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx - \beta^2$$

$$= \int_{0}^{\infty} x^2 \cdot \frac{1}{\beta} e^{-x/\beta}\,dx - \beta^2$$

$$= \frac{1}{\beta}\left[x^2 \frac{e^{-x/\beta}}{-1/\beta} - 2x \cdot \frac{e^{-x/\beta}}{1/\beta^2} + 2\frac{e^{-x/\beta}}{-1/\beta^3}\right]_0^{\infty} - \beta^2$$

$$= \frac{1}{\beta}\left[0 - \left(0 - 0 + 2 \cdot \frac{1}{-1/\beta^3}\right)\right] - \beta^2$$

$$= \frac{1}{\beta} \cdot 2\beta^3 - \beta^2 = 2\beta^2 - \beta^2 = \underline{\underline{\beta^2}}$$

Problems

1. The amount of time that a surveillance camera will run without having to be reset is a rv having the exponential distribution with $\beta = 50$ days. Find the probabilities that such a camera will (a) have to be reset in less than 20 days (b) not have to be reset in atleast 60 days.

Ans: Here $\beta = 50$

$$\therefore f(x) = \begin{cases} \frac{1}{50} e^{-x/50} & , x > 0 \\ 0 & , \text{elsewhere} \end{cases}$$

(a) $P(X < 20) = \int_{-\infty}^{20} f(x)\,dx = \int_0^{20} \frac{1}{50} e^{-x/50}\,dx$

$$= \frac{1}{50}\left(\frac{e^{-x/50}}{-1/50}\right)_0^{20}$$

$$= -\left(e^{-20/50} - e^0\right) = -e^{-2/5} - 1 = \underline{\underline{0.3297}}$$

(b) $P(X \geq 60) = \int_{60}^{\infty} f(x)\,dx$

$$= \int_{60}^{\infty} \frac{1}{50} e^{-x/50}\,dx$$

$$= \frac{1}{50}\left(\frac{e^{-x/50}}{-1/50}\right)_{60}^{\infty}$$

$$= -\left(e^{-\infty} - e^{-60/50}\right)$$

$$= -\left(0 - e^{-6/5}\right) = e^{-6/5} = \underline{\underline{0.3012}}$$

2. A consulting engineer receives on average 0.7 requests per week. If the no. of requests follows a poisson distribution, find the probabilities that the time between successive requests for consulting will be (a) less than 0.5 week (b) more than 3 weeks.

Ans: The time between successive requests follows an exponential distribution with $\beta = \frac{1}{\lambda} = \frac{1}{0.7}$

$$\therefore f(x) = \begin{cases} \frac{1}{1/0.7} e^{-x/(1/0.7)}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

$$= \begin{cases} 0.7\, e^{-0.7x}, & x > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

(a) $P(x<0.5) = \int_{-\infty}^{0.5} f(x)dx$

$= \int_{0}^{0.5} 0.7e^{-0.7x} dx$

$= 0.7\left(\frac{e^{-0.7x}}{-0.7}\right)_{0}^{0.5}$

$= -\left(e^{-0.7\times0.5} - e^{0}\right) = -e^{-0.35} + 1$

$= \underline{\underline{0.2953}}$

(b) $P(x>3) = \int_{3}^{\infty} f(x)dx$

$= \int_{3}^{\infty} 0.7e^{-0.7x} dx$

$= 0.7\left(\frac{e^{-0.7x}}{-0.7}\right)_{3}^{\infty}$

$= 0.7^{-}\left(0 - e^{-2.1}\right) = e^{-2.1} = \underline{\underline{0.1224}}$

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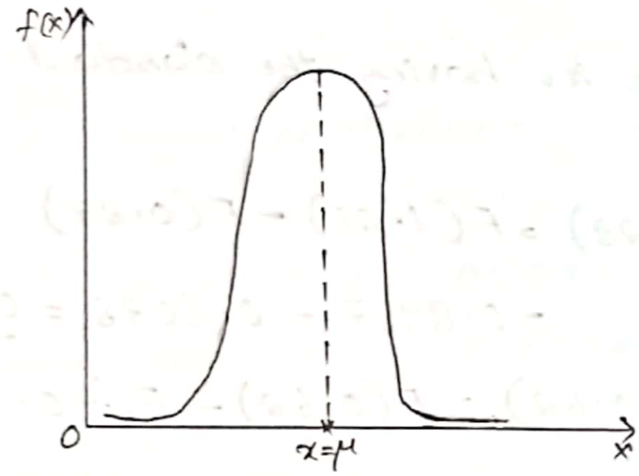
Normal Distribution

A continuous rv X is said to have a normal distribution with parameters μ and σ if its pdf is

$$f(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}, \quad \sigma>0, \; -\infty < x < \infty$$

The parameters of the distribution μ and σ are the mean and standard deviation of the distribution. The normal curve is unimodal and bell shaped with highest point over the mean μ. It is symmetrical about a vertical line through μ.



The normal distribution with parameter values $\mu=0$ and $\sigma=1$ is called the standard normal distribution and its entries are the values of

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2}\,dt = P(Z \le z)$$

Note

1. $P(a \le z \le b) = F(b) - F(a)$
2. $F(-z) = 1 - F(z)$
3. $z = \dfrac{x-\mu}{\sigma}$
4. $P(a \le x \le b) = P\left(\dfrac{a-\mu}{\sigma} \le \dfrac{x-\mu}{\sigma} \le \dfrac{b-\mu}{\sigma}\right)$

$$= P\left(\frac{a-\mu}{\sigma} \le z \le \frac{b-\mu}{\sigma}\right)$$

$$= F\left(\frac{b-\mu}{\sigma}\right) - F\left(\frac{a-\mu}{\sigma}\right)$$

Problems

1. Find the probabilities that a rv having the standard normal distribution will take on a value

(a) between 0.87 and 1.28

(b) between −0.34 and 0.62

(c) greater than 0.85

(d) greater than −0.65

Ans: Let z be a rv having the standard normal distribution

(a) $P(0.87 < z < 1.28) = F(1.28) - F(0.87)$

$$= 0.8997 - 0.8078 = \underline{\underline{0.0919}}$$

(b) $P(-0.34 < z < 0.62) = F(0.62) - F(-0.34)$

$$= 0.7324 - 0.3669 = \underline{\underline{0.3655}}$$

(c) $P(z > 0.85) = 1 - P(z \le 0.85)$

$$= 1 - F(0.85)$$

$$= 1 - 0.8023 = \underline{\underline{0.1977}}$$

(d) $P(z > -0.65) = 1 - P(z \le -0.65)$

$$= 1 - F(-0.65)$$

$$= 1 - 0.2578 = \underline{\underline{0.7422}}$$

Q.2. The time to microwave a bag of pop corn using the automatic setting can be treated as a rv having a normal distribution with standard deviation 10 seconds. If the probability is 0.8212 that the bag will take less than 282.5 seconds to pop, find the probability that it will take longer than 258.3 seconds to pop.

Ans: $\sigma = 10$

$P(X < 282.5) = 0.8212$

$\Rightarrow P\left(\frac{X-\mu}{10} < \frac{282.5-\mu}{10}\right) = 0.8212$

$\Rightarrow F\left(\frac{282.5-\mu}{10}\right) = 0.8212$

From the table $P(0.92) = 0.8212$

$\therefore \frac{282.5-\mu}{10} = 0.92$

$\Rightarrow 282.5 - \mu = 0.92 \times 10$

$\Rightarrow \mu = 282.5 - 9.2 = \underline{\underline{273.3}}$

Now $P(X > 258.3) = P\left(\frac{X-273.3}{10} > \frac{258.3-273.3}{10}\right)$

$= P(z > -1.5)$

$= 1 - P(z \le -1.5) = 1 - F(-1.5)$

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$= 1 - 0.0668$

$= \underline{\underline{0.9332}}$

Joint probability density function

Let X and Y be continuous random variables. A joint probability density function $f(x,y)$ for these two variables is a function satisfying

$f(x,y) \geq 0$ and $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

Also $P(a \leq X \leq b,\ c \leq Y \leq d) = \int_a^b \int_c^d f(x,y)\,dy\,dx$.

Marginal probability density function

The marginal probability density functions of X and Y, denoted by $f_X(x)$ and $f_Y(y)$ respectively are given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \quad \text{for} \quad -\infty < x < \infty$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx \quad \text{for} \quad -\infty < y < \infty$$

Independent random variables.

Two random variables X and Y are said to be independent if for every pair of x and y values

$f(x,y) = f_X(x) \cdot f_Y(y)$, where X and Y are continuous

If it is not satisfied, then X and Y are said to be dependent.

1. A bank operates both a drive-up facility and a walk-up window. On a randomly selected day, let X = the proportion of time that the drive-up facility is in use and Y = the proportion of time that the walk-up window is in use. Let

$$f(x,y) = \begin{cases} \frac{6}{5}(x+y^2), & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Check whether $f(x,y)$ is a joint pdf or not.

(b) Find $P(0 \le x \le \frac{1}{4},\ 0 \le y \le \frac{1}{4})$

(c) Find the marginal probability density functions of X and Y. and also $P(\frac{1}{4} \le Y \le \frac{3}{4})$

Ans: (a) If $f(x,y)$ is a joint pdf, then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1.$$

Here $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^1\int_0^1 \frac{6}{5}(x+y^2)\,dx\,dy$$

$$= \int_0^1 \frac{6}{5}\left(\frac{x^2}{2} + y^2 x\right)_0^1 dy$$

$$= \int_0^1 \frac{6}{5}\left(\frac{1}{2} + y^2 - 0\right) dy$$

$$= \frac{6}{5}\left(\frac{1}{2}y + \frac{y^3}{3}\right)_0^1$$

$$= \frac{6}{5}\left(\frac{1}{2} + \frac{1}{3} - 0\right)$$

$$= \frac{6}{5}\left(\frac{5}{6}\right) = \underline{\underline{1}}$$

$\therefore$ $$f(x,y) = \begin{cases} \frac{6}{5}(x+y^2), & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$ is a joint pdf.

(b) $P\left(0 \le x \le \frac{1}{4}, 0 \le y \le \frac{1}{4}\right) = \int_0^{1/4}\int_0^{1/4} \frac{6}{5}(x+y^2)\,dx\,dy$

$$= \int_0^{1/4} \frac{6}{5}\left(\frac{x^2}{2} + y^2 x\right)_0^{1/4} dy$$

$$= \int_0^{1/4} \frac{6}{5}\left(\frac{1}{32} + \frac{y^2}{4} - 0\right) dy$$

$$= \int_0^{1/4} \frac{6}{5}\left(\frac{1}{32} + \frac{y^2}{4}\right) dy = \frac{7}{640} = \underline{\underline{0.0109}}$$

(c) $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

$$= \int_0^1 \frac{6}{5}(x+y^2)\,dy$$

$$= \frac{6}{5}\left(xy + \frac{y^3}{3}\right)_0^1$$

$$= \frac{6}{5}\left(x + \frac{1}{3}\right) = \frac{6}{5}x + \frac{6}{5}\cdot\frac{1}{3}$$

$$= \underline{\underline{\frac{6}{5}x + \frac{2}{5}}}$$

$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

$$= \int_0^1 \frac{6}{5}(x+y^2)\,dx$$

$$= \frac{6}{5}\left(\frac{x^2}{2} + y^2 x\right)_0^1$$

$$= \frac{6}{5}\left(\frac{1}{2} + y^2\right) = \frac{6}{5}\cdot\frac{1}{2} + \frac{6}{5}\cdot y^2$$

$$= \underline{\underline{\frac{3}{5} + \frac{6}{5}y^2}}$$

$P\left(\frac{1}{4} \le Y \le \frac{3}{4}\right) = \int_{1/4}^{3/4} f_Y(y)\,dy$

$$= \int_{1/4}^{3/4} \left(\frac{3}{5} + \frac{6}{5}y^2\right) dy$$

$$= \frac{37}{80} = \underline{\underline{0.4625}}$$

2\. If $f(x,y) = kye^{-x}$, $x>0, 0<y<2$ is the joint pdf of the random variable x and Y, find the value of k.

Ans: Given $f(x,y)$ is a joint pdf

$$\therefore \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$\Rightarrow \int_0^{\infty}\left[\int_0^2 kye^{-x}\,dy\right]dx = 1$$

$$\Rightarrow \int_0^{\infty} ke^{-x}\cdot\left(\frac{y^2}{2}\right)_0^2 dx = 1$$

$$\Rightarrow \int_0^{\infty} ke^{-x}\left(\frac{2^2}{2} - 0\right)dx = 1$$

$$\Rightarrow \int_0^{\infty} k\ 2e^{-x}\,dx = 1$$

$$\Rightarrow 2k\left(\frac{e^{-x}}{-1}\right)_0^{\infty} = 1$$

$$\Rightarrow 2k\left(0 - \frac{1}{-1}\right) = 1 \Rightarrow 2k = 1 \Rightarrow \underline{\underline{k = \frac{1}{2}}}$$

3\. If two random variables x and Y have the pdf $f(x,y) = ke^{-(2x+y)}$, $x,y>0$. Evaluate k?

Ans: Given $f(x,y)$ is a pdf

$$\therefore \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$\Rightarrow \int_0^{\infty}\int_0^{\infty} ke^{-(2x+y)}\,dx\,dy = 1$$

$$\Rightarrow \int_0^{\infty}\int_0^{\infty} k\,e^{-2x}\cdot e^{-y}\,dx\,dy = 1$$

$$\Rightarrow \int_0^\infty ke^{-y}\left(\frac{e^{-2x}}{-2}\right)_0^\infty dy = 1$$

$$\Rightarrow \int_0^\infty ke^{-y}\left(\frac{0-1}{-2}\right) dy = 1$$

$$\Rightarrow \int_0^\infty ke^{-y} \cdot \frac{1}{2} dy = 1$$

$$\Rightarrow \frac{k}{2}\left(\frac{e^{-y}}{-1}\right)_0^\infty = 1 \Rightarrow \frac{k}{2}\left(\frac{0-1}{-1}\right) = 1$$

$$\Rightarrow \frac{k}{2} = 1 \Rightarrow \underline{\underline{k = 2}}$$

Conditional probability density function

Let X and Y be two continuous r.v's with joint pdf $f(x,y)$ and marginal pdf's $f_X(x)$ and $f_Y(y)$. Then the conditional probability density function of Y given that $X = x$ is

$$f(Y/X) = \frac{f(x,y)}{f_X(x)}, \quad -\infty < y < \infty$$

and the conditional probability density function of X given that $Y = y$ is

$$f(X/Y) = \frac{f(x,y)}{f_Y(y)}$$

Note

1. $P(X < x) = \int_{-\infty}^{x} f_X(x)\,dx$

2. $P(X > x) = \int_{x}^{\infty} f_X(x)\,dx$

3. $P(X<x, Y<y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(x,y)\,dy\,dx$

Problems

V.Imp 1. If the joint distribution of random variables X and Y be $f(x,y) = \begin{cases} e^{-(x+y)}, & x \geq 0,\ y \geq 0 \\ 0, & \text{otherwise} \end{cases}$, find

(a) $P(X<1)$

(b) $P(X+Y<1)$

Ans: (a) $P(X<1) = \int_{-\infty}^{1} f_X(x)\,dx$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_{0}^{\infty} e^{-(x+y)}\,dy$$

$$= \int_{0}^{\infty} e^{-x}\cdot e^{-y}\,dy$$

$$= e^{-x}\left(\frac{e^{-y}}{-1}\right)_0^{\infty} = e^{-x}\left(0 - \frac{1}{-1}\right)$$

$$= e^{-x}$$

$$\therefore P(X<1) = \int_{-\infty}^{1} f_X(x)\,dx$$

$$= \int_{0}^{1} e^{-x}\,dx = \left(\frac{e^{-x}}{-1}\right)_0^{1}$$

$$= \frac{e^{-1}-1}{-1} = 1-e^{-1} = 1-\frac{1}{e}$$

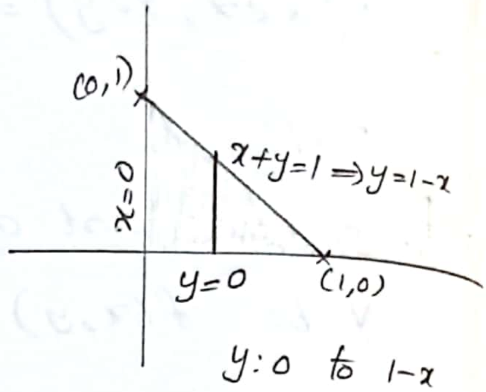
(b) $P(X+Y<1)$

$$f(x,y) = \begin{cases} e^{-(x+y)}, & x \geq 0,\ y \geq 0 \\ 0 ; & \text{otherwise} \end{cases}$$

We have $x=0,\ y=0$, $x+y=1$

x	0	1
y	1	0

$$P(X+Y<1) = \int_0^1 \int_0^{1-x} f(x,y)\,dy\,dx$$



$y: 0$ to $1-x$

$x: 0$ to 1

$$= \int_0^1 \int_0^{1-x} e^{-(x+y)}\,dy\,dx$$

$$= \int_0^1 \left[\int_0^{1-x} e^{-x} \cdot e^{-y}\,dy\right] dx$$

$$= \int_0^1 e^{-x} \cdot \left(\frac{e^{-y}}{-1}\right)_0^{1-x} dx$$

$$= \int_0^1 e^{-x} \left(\frac{e^{-(1-x)} - 1}{-1}\right) dx$$

$$= \int_0^1 \left(\frac{e^{-x} \cdot e^{-1} \cdot e^{x} - e^{-x}}{-1}\right) dx$$

$$= \int_0^1 -\left(e^{-1} - e^{-x}\right) dx$$

$$= \int_0^1 \left(e^{-x} - e^{-1}\right) dx$$

$$= \left(\frac{e^{-x}}{-1} - e^{-1} \cdot x\right)_0^1$$

$$= \frac{e^{-1}}{-1} - e^{-1} - \left(\frac{1}{-1} - 0\right)$$

$$= -e^{-1} - e^{-1} + 1 \quad = \underline{\underline{1 - 2e^{-1}}} = 0.264$$

Expectation of a 2-D random variable.

Let (X, Y) be 2-D continuous random variable and let $g: R^2 \to R$ be a continuous function, then the expectation of $g(X,Y)$ is

$$\boxed{E(g(X,Y)) = \int_x \int_y g(x,y) \cdot f(x,y)\, dy\, dx}\,,$$

where $f(x,y)$ is the joint pdf of (X,Y).

Properties of expectation

1. $E(X+Y) = E(X) + E(Y)$
2. If X and Y are independent, $E(XY) = E(X) \cdot E(Y)$.
3. $E(X) = \int_x x \cdot f_X(x)\, dx$

$E(Y) = \int_y y\, f_Y(y)\, dy$

Problems

1. Let X and Y be two random variables with joint pdf $f(x,y) = \begin{cases} 4xy\,; & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0\,; & \text{otherwise} \end{cases}$. Find $E(XY)$?

Ans: $E(XY) = \int_x \int_y xy \cdot f(x,y)\, dy\, dx$

$$= \int_0^1 \int_0^1 xy \cdot 4xy\, dy\, dx$$

$$= \int_0^1 \int_0^1 4x^2y^2\, dy\, dx$$

$$= \int_0^1 4x^2 \left(\frac{y^3}{3}\right)_0^1 dx$$

$$= \int_0^1 4x^2 \left(\frac{1}{3}\right) dx = \int_0^1 \frac{4}{3} x^2 dx = \frac{4}{9}$$

2. The joint pdf of a continuous 2-D random variable (X, Y) is $f(x,y) = \begin{cases} \frac{x(1+3y^2)}{4}; & 0<x<2, 0<y<1 \\ 0 & ; \text{elsewhere} \end{cases}$

Show that X and Y are independent using expectation?

Ans: If X and Y are independent $E(XY) = E(X) \cdot E(Y)$

$$E(XY) = \int_x \int_y xy\, f(x,y)\, dy\, dx$$

$$= \int_0^2 \int_0^1 xy\, \frac{x(1+3y^2)}{4} dy\, dx$$

$$= \int_0^2 \int_0^1 \frac{x^2}{4} (y+3y^3)\, dy\, dx$$

$$= \int_0^2 \frac{x^2}{4} \left(\frac{y^2}{2} + \frac{3y^4}{4}\right)_0^1 dx$$

$$= \int_0^2 \frac{x^2}{4} \left(\frac{1}{2} + \frac{3}{4}\right) dx$$

$$= \int_0^2 \frac{x^2}{4} \left(\frac{10}{8}\right) dx = \frac{10}{32} \int_0^2 x^2 dx$$

$$= \frac{10}{32} \cdot \left(\frac{x^3}{3}\right)_0^2 = \frac{10}{32} \cdot \frac{8}{3}$$

$$= \frac{10}{12} = \frac{5}{6} \quad — ①$$

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$$E(X) = \int_x x\, f_X(x)\,dx$$

$$f_X(x) = \int_y f(x,y)\,dy = \int_0^1 \frac{x(1+3y^2)}{4}\,dy$$

$$= \frac{x}{4}\left(y + \frac{3y^3}{3}\right)_0^1 = \frac{x}{4}(1+1)$$

$$= \frac{2x}{4} = \frac{x}{2}, \quad 0<x<2$$

$$\therefore E(X) = \int_x x \cdot \frac{x}{2}\,dx$$

$$= \int_0^2 \frac{x^2}{2}\,dx = \frac{4}{3}$$

$$E(Y) = \int_y y \cdot f_Y(y)\,dy$$

$$f_Y(y) = \int_x f(x,y)\,dx = \int_0^2 \frac{x(1+3y^2)}{4}\,dx$$

$$= \left(\frac{1+3y^2}{4}\right)\left(\frac{x^2}{2}\right)_0^2$$

$$= \left(\frac{1+3y^2}{4}\right)\frac{4}{2} = \frac{1+3y^2}{2}, \quad 0<y<1$$

$$\therefore E(Y) = \int_0^1 y \cdot \left(\frac{1+3y^2}{2}\right)dy$$

$$= \frac{1}{2}\int_0^1 (y+3y^3)\,dy = \frac{5}{8}$$

$$E(X)\cdot E(Y) = \frac{4}{3}\cdot\frac{5}{8} = \frac{5}{6} \quad \text{——— (2)}$$

From (1) and (2) $E(XY) = E(X)\cdot E(Y)$

$\therefore$ X and Y are independent.

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Memoryless property of exponential distribution

If X is an exponential random variable, then $P(X \geq t+t_0 / X \geq t_0) = P(X > t)$.

Problems

1. The life length X of an electric component follows exponential distribution with parameter $\beta = 2$. Find the probability that the component survives atleast 10 months, given that already it had survived more than 9 months?

Ans: Here $\beta = 2$

$$\therefore f(x) = \begin{cases} \frac{1}{\beta} e^{-x/\beta}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{2} e^{-x/2}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Memoryless property can be used here.

$$P(X \geq 10 / X \geq 9) = P(X \geq 9+1 / X \geq 9)$$

$$= P(X > 1)$$

$$= \int_1^\infty f(x)\,dx$$

$$= \int_1^\infty \frac{1}{2} e^{-x/2}\,dx$$

$$= \frac{1}{2} \left(\frac{e^{-x/2}}{-1/2} \right)_1^\infty$$

$$= \frac{\frac{1}{2}\left(e^{-\infty} - e^{-1/2}\right)}{-1/2} = -(0 - e^{-1/2})$$

$$= e^{-1/2} = \underline{\underline{0.6065}}$$

Central limit theorem.

If $x_1, x_2, \ldots x_n$ are independently and identically distributed random variables (i.i.d random variables) with $E(x_i) = \mu$ and $var(x_i) = \sigma^2$ for $i = 1, 2, \ldots n$. Then $S_n = x_1 + x_2 + \cdots + x_n$ follows normal distribution with mean $n\mu$ and variance $n\sigma^2$.

Note

If $x_1, x_2, \ldots, x_n$ are i.i.d random variables with $E(x_i) = \mu$ and $var(x_i) = \sigma^2$ for $i = 1, 2, \ldots n$, then the random variable, sample mean $\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}$ follows normal distribution with mean μ and variance $\frac{\sigma^2}{n}$.

Problems

1. A random sample of size 100 is taken from a population whose mean is 60 and variance 400. Using Central limit theorem, with what probability can we assert that the mean of the sample will not differs from $\mu = 60$ by more than 4?

Ans: Let $x_1, x_2, \ldots x_{100}$ be the sample values from the given population

Given $\mu = E(x_i) = 60$ and $\sigma^2 = var(x_i) = 400$

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_{100}}{100}$$

By central limit theorem $\bar{x} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

$$\Rightarrow \bar{x} \sim N\left(60, \frac{400}{100}\right)$$

$$\Rightarrow \bar{x} \sim N(60, 4)$$

$$P\left[|\bar{x} - \mu| < 4\right] = P(-4 \leq \bar{x} - \mu \leq 4)$$

$$= P\left(\frac{-4}{2} \leq \frac{\bar{x} - \mu}{2} \leq \frac{4}{2}\right)$$ (since here $\sigma^2 = 4$, $\sigma = \sqrt{4} = 2$)

$$= P(-2 \leq z \leq 2)$$

$$= F(2) - F(-2)$$

$$= 0.9772 - 0.0228$$

$$= \underline{\underline{0.9544}}$$

H.W

2. A random sample of size 100 is taken from a population whose mean is 80 and variance is 400. Using CLT, with what probability can we assert that the mean of the sample will not differ from $\mu = 80$ by more than 6?

Ans: $\underline{\underline{0.9974}}$