MODULE - I

Partial Differential Equations

Introduction

A differential equation which involves partial derivatives is called a partial differential equation. (PDE)

Eg 1. $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 0$

2. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = z$

3 $\frac{\partial^2 u}{\partial x \partial y} = \left(\frac{\partial u}{\partial x}\right)^3$

The order of a PDE is the order of the highest partial derivative occuring in the equation.

order of eg 1 is 1

order of eg 2 and 3 is 2

The degree of a PDE is the degree of the highest order partial derivative occuring in the equation.

Degree of egs 1, 2 and 3 is 1

Note

If z is a function of two independent variables x and y, then we denote

$\frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y} = q$, $\frac{\partial^2 z}{\partial x^2} = r$, $\frac{\partial^2 z}{\partial x \partial y} = s$, $\frac{\partial^2 z}{\partial y^2} = t$

Formation of PDE

PDE can be formed either by the elimination of arbitrary constants or by the elimination of arbitrary functions. If the number of arbitrary constants to be eliminated is equal to the number of independent variables, the PDE that arise are of the first order. If the number of arbitrary constants to be eliminated is more than the number of independent variables, the PDE's obtained are of second or higher order.

If the PDE is obtained by elimination of arbitrary functions, then the order of the PDE is in general equal to the number of arbitrary functions eliminated

Problems

Form PDE from the following equations by eliminating the arbitrary constants

1. $z = ax + by + ab$

Ans: No. of independent variables = 2 = no. of arbitrary constants

Order of required PDE = 1

Here $z = ax + by + ab$ —— ①

Differentiating equation ① partially w.r.t. x, we get

$$\frac{\partial z}{\partial x} = a \Rightarrow p = a$$

Similarly, $\frac{\partial z}{\partial y} = b, \Rightarrow q = b$

Substituting in equation ①, we get

$$\underline{z = px + qy + pq}$$

2. $z = ax + by + a^2 + b^2$

Ans: No. of independent variables = 2 = no. of arbitrary constants

$\therefore$ order of required PDE = 1

Here $z = ax + by + a^2 + b^2$ —— ①

Differentiating ① partially w.r.t. x, we get

$\frac{\partial z}{\partial x} = a$

Similarly $\frac{\partial z}{\partial y} = b$

Substituting in ①, we get

$z = px + qy + p^2 + q^2$

3. $z = (x^2+a)(y^2+b)$

Ans: No. of independent variables = 2 = no. of arbitrary constants

$\therefore$ order of required PDE = 1

Here $z = (x^2+a)(y^2+b)$ —— ①

Differentiating ① partially w.r.t. x, we get

$\frac{\partial z}{\partial x} = (y^2+b)\,2x \Rightarrow p = (y^2+b)\,2x$

$\Rightarrow y^2+b = \frac{p}{2x}$

Similarly, $\frac{\partial z}{\partial y} = (x^2+a)\,2y \Rightarrow q = (x^2+a)\,2y$

$\Rightarrow x^2+a = \frac{q}{2y}$

$\therefore$ ① $\Rightarrow z = \frac{q}{2y} \cdot \frac{p}{2x}$

$\Rightarrow 4xyz = pq$

4. $z = (x-a)^2 + (y-b)^2$

Ans: No. of independent variables = 2 = no. of arbitrary constants

$\therefore$ order of required PDE = 1

Here $z = (x-a)^2 + (y-b)^2$ —— ①

$\therefore \frac{\partial z}{\partial x} = 2(x-a) \Rightarrow p = 2(x-a) \Rightarrow x-a = \frac{p}{2}$

$\frac{\partial z}{\partial y} = 2(y-b) \rightarrow q = 2(y-b) \Rightarrow y-b = \frac{q}{2}$

$\therefore$ ① $\Rightarrow z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2$

$\Rightarrow z = \frac{p^2}{4} + \frac{q^2}{4} \Rightarrow z = \frac{p^2+q^2}{4}$

$\Rightarrow 4z = p^2 + q^2$

5 $(x-a)^2 + (y-b)^2 - z^2 = 1.$

Ans: No. of independent variables = 2 = no. of arbitrary constants

$\therefore$ order of required PDE = 1

Here $(x-a)^2 + (y-b)^2 - z^2 = 1$ —— ①

Differentiating ① w.r.t. x partially, we get

$2(x-a) - 2z \cdot \frac{\partial z}{\partial x} = 0$

$\Rightarrow 2(x-a) - 2zp = 0$

$\Rightarrow 2(x-a) = 2zp \Rightarrow x-a = zp$

Similarly $2(y-b) - 2z\frac{\partial z}{\partial y} = 0$

$\Rightarrow 2(y-b) - 2zq = 0$

$\Rightarrow 2(y-b) = 2zq$

$\Rightarrow y-b = zq$

$\therefore$ ① $\Rightarrow (zp)^2 + (zq)^2 - z^2 = 1$

$\Rightarrow z^2p^2 + z^2q^2 - z^2 = 1$

$\Rightarrow z^2(p^2+q^2-1) = 1$

Form the PDE by eliminating arbitrary functions from the following relations.

U.Q 1. $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$

Ans: No. of arbitrary functions $= 1$

$\therefore$ order of required PDE $= 1$

$z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$ —— ①

From ①, $\frac{\partial z}{\partial x} = 2f'\left(\frac{1}{x} + \log y\right) \cdot \frac{-1}{x^2}$

$\Rightarrow p = \frac{-2}{x^2} f'\left(\frac{1}{x} + \log y\right)$

$\Rightarrow -\frac{x^2 p}{2} = f'\left(\frac{1}{x} + \log y\right)$ —— ②

Also from ①, $\frac{\partial z}{\partial y} = 2y + 2f'\left(\frac{1}{x} + \log y\right) \cdot \frac{1}{y}$

$\Rightarrow q = 2y + 2 \cdot \frac{-x^2 p}{2} \cdot \frac{1}{y}$ (using ②)

$\Rightarrow q = \frac{2y^2 - x^2 p}{y}$

$\Rightarrow qy = 2y^2 - x^2 p \Rightarrow \underline{px^2 + qy = 2y^2}$

2. $z = f\left(\frac{y}{x}\right)$

Ans: $z = f\left(\frac{y}{x}\right)$ — ①

From ①, $\frac{\partial z}{\partial x} = f'\left(\frac{y}{x}\right) \cdot \frac{-y}{x^2}$

$\Rightarrow p = f'\left(\frac{y}{x}\right) \cdot \frac{-y}{x^2} \Rightarrow \frac{x^2 p}{-y} = f'\left(\frac{y}{x}\right)$ — ②

Also from ①, $\frac{\partial z}{\partial y} = f'\left(\frac{y}{x}\right) \cdot \frac{1}{x}$

$\Rightarrow q = f'\left(\frac{y}{x}\right) \cdot \frac{1}{x}$

$\Rightarrow q = \frac{x^2 p}{-y} \cdot \frac{1}{x}$ (using ②)

$\Rightarrow -yq = xp \quad \Rightarrow \underline{\underline{px + qy = 0}}$

3. $xyz = \phi(x+y+z)$

Ans: $xyz = \phi(x+y+z)$ —— (1)

Differentiating equation (1) partially w.r.t. x, we get

$$y\left(x\frac{\partial z}{\partial x} + z\right) = \phi'(x+y+z)\left(1+\frac{\partial z}{\partial x}\right)$$

$$\Rightarrow y(xp+z) = \phi'(x+y+z)(1+p)$$

$$\Rightarrow \frac{y(xp+z)}{1+p} = \phi'(x+y+z) \quad \text{—— (2)}$$

Also differentiating (1) partially w.r.t. y, we get

$$x\left(y\frac{\partial z}{\partial y} + z\right) = \phi'(x+y+z)\left(1+\frac{\partial z}{\partial y}\right)$$

$$\Rightarrow x(yq+z) = \phi'(x+y+z)(1+q)$$

$$\Rightarrow xyq + xz = \frac{y(xp+z)}{1+p}\cdot(1+q) \quad (\text{using } (2))$$

$$\Rightarrow \underline{\underline{(xyq+xz)(1+p) = (yxp+yz)(1+q)}}$$

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Linear PDE of the first order - Lagrange's linear equation

A PDE involving first order partial derivatives p and q only is called a PDE of the first order.

The PDE of the form $Pp + Qq = R$, where P, Q and R are functions of x, y, z is the standard form of a linear PDE of the first order and is called Lagrange's linear equation.

Solution of Lagrange's equation.

1. Form the auxilliary equation (A.E) $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$
2. Solve the A.E's by the method of grouping or by the method of multipliers or both to get two independent solutions $u=a$ and $v=b$, where a and b are arbitrary constants
3. The general solution is $\phi(u,v)=0$

Problems

1. Solve $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$

Ans: $P=\sqrt{x}$, $Q=\sqrt{y}$, and $R=\sqrt{z}$ when compairing with $Pp+Qq=R$.

A.E is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\Rightarrow \frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}} \quad \text{——} \ (1)$$

From (1) $\frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}} \Rightarrow \frac{dx}{x^{1/2}} = \frac{dy}{y^{1/2}}$

$$\Rightarrow x^{-1/2}dx = y^{-1/2}dy$$

Integrating both sides $\Rightarrow \int x^{-1/2}dx = \int y^{-1/2}dy$

$$\Rightarrow \frac{x^{-1/2+1}}{-1/2+1} = \frac{y^{-1/2+1}}{-\frac{1}{2}+1} + a$$, where a is constant of integration

$$\Rightarrow \frac{x^{1/2}}{\frac{1}{2}} = \frac{y^{1/2}}{\frac{1}{2}} + a \Rightarrow 2x^{1/2} = 2y^{1/2} + a$$

$$\Rightarrow 2\sqrt{x} = 2\sqrt{y} + a$$

$$\Rightarrow 2(\sqrt{x} - \sqrt{y}) = a$$

$$\Rightarrow \sqrt{x} - \sqrt{y} = a$$

$$\Rightarrow u = \sqrt{x} - \sqrt{y}$$

Also ① $\Rightarrow \frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}} \Rightarrow \frac{dy}{y^{1/2}} = \frac{dz}{z^{1/2}}$

$$\Rightarrow y^{-1/2} dy = z^{-1/2} dz$$

Integrating both sides $\Rightarrow \int y^{-1/2} dy = \int z^{-1/2} dz$

$$\Rightarrow 2y^{1/2} = 2z^{1/2} + b$$

$$\Rightarrow 2(\sqrt{y} - \sqrt{z}) = b$$

$$\Rightarrow \sqrt{y} - \sqrt{z} = b \Rightarrow V = \sqrt{y} - \sqrt{z}$$

$\therefore$ The solution is $\phi(u, v) = 0$

$$\Rightarrow \phi(\sqrt{x} - \sqrt{y}, \sqrt{y} - \sqrt{z}) = 0$$

2. solve $p \tan x + q \tan y = \tan z$

Ans. Here $P = \tan x$, $Q = \tan y$ and $R = \tan z$.

$\therefore$ AE is $\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$ ——— ①

From ①, $\frac{dx}{\tan x} = \frac{dy}{\tan y} \Rightarrow \cot x \, dx = \cot y \, dy$

Integrating both sides, $\int \cot x\,dx = \int \cot y\,dy$

$$\Rightarrow \log \sin x = \log \sin y + \log a$$

$$\Rightarrow \log \sin x - \log \sin y = \log a$$

$$\Rightarrow \log\left(\frac{\sin x}{\sin y}\right) = \log a$$

$$\Rightarrow \frac{\sin x}{\sin y} = a \quad \Rightarrow \quad u = \frac{\sin x}{\sin y}$$

Also from ①, $\frac{dy}{\tan y} = \frac{dz}{\tan z}$

Similarly integrating we get $\frac{\sin y}{\sin z} = b$

$$\Rightarrow V = \frac{\sin y}{\sin z}$$

$\therefore$ The solution is $\phi(u,v) = 0$

$$\Rightarrow \phi\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$$

3. Solve $y^2p - xyq = x(z-2y)$

Ans: $P = y^2$, $Q = -xy$, $R = x(z-2y)$

$\therefore$ AE is $\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$ —— ①

From ①, $\frac{dx}{y^2} = \frac{dy}{-xy} \Rightarrow \frac{dx}{y} = \frac{dy}{-x}$

$$\Rightarrow -x\,dx = y\,dy$$

Integrating $\Rightarrow \int -x\,dx = \int y\,dy$

$$\Rightarrow -\frac{x^2}{2} = \frac{y^2}{2} + \frac{a}{2}$$

$$\Rightarrow -x^2 = y^2 + a$$

$$\Rightarrow x^2 + y^2 = a \Rightarrow u = x^2 + y^2$$

Also from ①, $\frac{dy}{-xy} = \frac{dz}{x(z-2y)}$

$$\Rightarrow \frac{dy}{-y} = \frac{dz}{z-2y}$$

$$\Rightarrow (z-2y)dy = -y\,dz$$

$\Rightarrow zdy - 2ydy = -ydz$

$\Rightarrow zdy + ydz = 2ydy$

$\Rightarrow d(zy) = 2y\,dy$

Integrating $\Rightarrow \int d(zy) = \int 2y\,dy$

$\Rightarrow zy = \frac{2y^2}{2} + b$

$\Rightarrow zy - y^2 = b \Rightarrow V = zy - y^2$

$\therefore$ solution is $\phi(u,v) = 0 \Rightarrow \phi(x^2+y^2,\ zy - y^2) = 0$

4. Solve $\frac{y^2z}{x} p + xzq = y^2$

Ans: Here $P = \frac{y^2z}{x}$, $Q = xz$, $R = y^2$

AE is $\frac{dx}{\frac{y^2z}{x}} = \frac{dy}{xz} = \frac{dz}{y^2}$

$\Rightarrow \frac{xdx}{y^2z} = \frac{dy}{xz} = \frac{dz}{y^2}$ —— ①

From ①, $\frac{xdx}{y^2z} = \frac{dy}{xz} \Rightarrow x^2dx = y^2dy$

Integrating $\Rightarrow \int x^2dx = \int y^2dy$

$\Rightarrow \frac{x^3}{3} = \frac{y^3}{3} + \frac{a}{3}$

$\Rightarrow x^3 - y^3 = a \Rightarrow u = x^3 - y^3$

Also from ①, $\frac{xdx}{y^2z} = \frac{dz}{y^2} \Rightarrow xdx = zdz$

Integrating $\Rightarrow \int xdx = \int zdz$

$\Rightarrow \frac{x^2}{2} = \frac{z^2}{2} + \frac{b}{2}$

$\Rightarrow x^2 - z^2 = b \Rightarrow V = x^2 - z^2$

The solution is $\phi(u,v) = 0 \Rightarrow \phi(x^3 - y^3,\ x^2 - z^2) = 0$

Charpits Method

This is a general method for finding the complete solution of non-linear PDE of the first order.

Let the given equation be $f(x,y,z,p,q)=0$ —— ①

Charpits auxiliary equations are

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}}$$

$$\Rightarrow \frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} \quad \text{—— ②}$$

Using ① and ② find the values of p and q. Substituting p and q in $dz = p\,dx + q\,dy$ and then integrating we get the complete solution of ①.

Problems

U.Q

1. Solve $(p^2+q^2)y = qz$ —— ①

Ans: Given equation is $(p^2+q^2)y = qz \Rightarrow (p^2+q^2)y - qz = 0$

$\therefore f(x,y,z,p,q) = (p^2+q^2)y - qz.$

$\frac{\partial f}{\partial p} = 2py$, $\frac{\partial f}{\partial q} = 2qy - z$, $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = p^2+q^2$ and

$\frac{\partial f}{\partial z} = -q$

$\therefore$ charpits auxiliary equations are

$$\frac{dx}{-2py} = \frac{dy}{-(2qy-z)} = \frac{dz}{-p\cdot 2py - q(2qy-z)} = \frac{dp}{0+p\cdot -q} = \frac{dq}{p^2+q^2+q\cdot -q} \quad \text{—— ②}$$

$$\Rightarrow \frac{dx}{-2py} = \frac{dy}{z-2qy} = \frac{dz}{-2p^2y - 2q^2y + qz} = \frac{dp}{-pq} = \frac{dq}{p^2}$$

From ②, $\frac{dp}{-pq} = \frac{dq}{p^2} \Rightarrow p\,dp = -q\,dq$

$\Rightarrow \int p\,dp = -\int q\,dq$

$$\Rightarrow \frac{p^2}{2} = \frac{-q^2}{2} + \frac{a}{2}$$

$$\Rightarrow p^2+q^2 = a \quad \text{--- (3)}$$

$\therefore$ (1) $\Rightarrow ay - qz = 0 \Rightarrow ay = qz \Rightarrow q = \frac{ay}{z}$

$\therefore$ (3) $\Rightarrow p^2 + \left(\frac{ay}{z}\right)^2 = a \Rightarrow p^2 + \frac{a^2y^2}{z^2} = a$

$$\Rightarrow p^2 = a - \frac{a^2y^2}{z^2}$$

$$= \frac{az^2 - a^2y^2}{z^2}$$

$$\Rightarrow p = \sqrt{\frac{az^2-a^2y^2}{z^2}} = \frac{\sqrt{az^2-a^2y^2}}{z}$$

Substituting p and q in $dz = pdx + q\,dy$, we get

$$dz = \frac{\sqrt{az^2-a^2y^2}}{z}dx + \frac{ay}{z}dy = \frac{\sqrt{az^2-a^2y^2}\,dx + ay\,dy}{z}$$

$$\Rightarrow z\,dz = \sqrt{az^2-a^2y^2}\,dx + ay\,dy$$

$$\Rightarrow z\,dz - ay\,dy = \sqrt{az^2-a^2y^2}\,dx$$

$$\Rightarrow \frac{z\,dz - ay\,dy}{\sqrt{az^2-a^2y^2}} = dx$$

Put $t = az^2 - a^2y^2$

$\Rightarrow dt = 2az\,dz - 2a^2y\,dy$

$= 2a(z\,dz - ay\,dy)$

$\Rightarrow \frac{dt}{2a} = z\,dz - ay\,dy$

$$\Rightarrow \frac{dt}{2a\sqrt{t}} = dx$$

$$\Rightarrow \frac{1}{2a} t^{-1/2} dt = dx$$

$$\Rightarrow \int \frac{1}{2a} t^{-1/2} dt = \int dx$$

$$\Rightarrow \frac{1}{2a} \cdot \frac{t^{1/2}}{\frac{1}{2}} = x + b$$

$$\Rightarrow \frac{t^{1/2}}{a} = x + b \Rightarrow \frac{\sqrt{t}}{a} = x + b$$

$$\Rightarrow \sqrt{az^2-a^2y^2} = a(x+b) = ax + b$$

$$\Rightarrow \sqrt{az^2-a^2y^2} = ax + b$$

Squaring both sides $\Rightarrow az^2 - a^2y^2 = (ax+b)^2$

$$\Rightarrow az^2 = (ax+b)^2 + a^2y^2$$

$$\Rightarrow z^2 = \frac{(ax+b)^2 + a^2y^2}{a}$$

$$\Rightarrow z^2 = (ax+b)^2 + a^2y^2$$

U.Q

2. Solve $2xz - px^2 - 2qxy + pq = 0$

Ans: Given equation is $2xz - px^2 - 2qxy + pq = 0$ — ①

$\therefore f(x, y, z, p, q) = 2xz - px^2 - 2qxy + pq$

$$\frac{\partial f}{\partial p} = -x^2 + q \quad , \quad \frac{\partial f}{\partial q} = -2xy + p, \quad \frac{\partial f}{\partial x} = 2z - 2px - 2qy$$

$$\frac{\partial f}{\partial y} = -2qx, \quad \frac{\partial f}{\partial z} = 2x$$

$\therefore$ charpits auxiliary equations are

$$\frac{dx}{-(-x^2+q)} = \frac{dy}{-(-2xy+p)} = \frac{dz}{-p(-x^2+q) - q(-2xy+p)} =$$

$$\frac{dp}{2z - 2px - 2qy + p\cdot 2x} = \frac{dq}{-2qx + q\cdot 2x}$$

$$\Rightarrow \frac{dx}{x^2-q} = \frac{dy}{2xy-p} = \frac{dz}{x^2p - pq + 2xyq - pq}$$

$$= \frac{dp}{2z - 2qy} = \frac{dq}{0} \quad \text{—— ②}$$

From ②, $\frac{dq}{0}$ = each ratio $\Rightarrow dq = 0 \Rightarrow \int dq = \int 0$

$$\Rightarrow q = a$$

Putting $q = a$ in ①, $\Rightarrow 2xz - px^2 - 2axy + pa = 0$

$$\Rightarrow -px^2 + pa = 2axy - 2xz$$

$$\Rightarrow p(-x^2 + a) = 2x(ay - z)$$

$$\Rightarrow p = \frac{2x(ay-z)}{a - x^2}$$

Substituting p and q in $dz = pdx + qdy$, we get

$$dz = \frac{2x(ay-z)}{a-x^2}dx + ady$$

$$\Rightarrow dz - ady = \frac{2x(ay-z)}{a-x^2}dx$$

$$\Rightarrow \frac{dz - ady}{ay-z} = \frac{2x}{a-x^2}dx$$

Put $u = ay - z$

$\Rightarrow du = ady - dz$

$= -(dz - ady)$

$\Rightarrow dz - ady = -du$

Put $v = a - x^2$

$\Rightarrow dv = 0 - 2xdx$

$\Rightarrow 2xdx = -dv$

$$\Rightarrow \frac{-du}{u} = \frac{-dv}{v}$$

Integrating $\Rightarrow -\int \frac{du}{u} = -\int \frac{dv}{v}$

$$\Rightarrow -\log u = -\log v + \log b$$

$$\Rightarrow -\log u + \log v = \log b$$

$$\Rightarrow \log u - \log v = \log b$$

$$\Rightarrow \log\left(\frac{u}{v}\right) = \log b$$

$$\Rightarrow \log\left(\frac{ay-z}{a-x^2}\right) = \log b$$

$$\Rightarrow \frac{ay-z}{a-x^2} = b \Rightarrow ay - z = b(a-x^2)$$

$$\Rightarrow \underline{\underline{z = ay - b(a-x^2)}}$$

3. Solve $2z + p^2 + qy + 2y^2 = 0$

Ans: Given equation is $2z + p^2 + qy + 2y^2 = 0$ —— ①

$\therefore f(x, y, z, p, q) = 2z + p^2 + qy + 2y^2$

$$\frac{\partial f}{\partial p} = 2p, \quad \frac{\partial f}{\partial q} = y, \quad \frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = q + 4y, \quad \frac{\partial f}{\partial z} = 2$$

Charpits auxiliary equations are

$$\frac{dx}{-2p} = \frac{dy}{-y} = \frac{dz}{-p\,2p - q\,y} = \frac{dp}{0 + p\cdot 2} = \frac{dq}{q + 4y + 2q}$$

$$\Rightarrow \frac{dx}{-2p} = \frac{dy}{-y} = \frac{dz}{-2p^2-qy} = \frac{dp}{2p} = \frac{dq}{3q+4y} \quad \text{--- (2)}$$

From (2), $\frac{dx}{-2p} = \frac{dp}{2p} \Rightarrow -dx = dp$

$\Rightarrow \int -dx = \int dp \Rightarrow -x = p + a$

$\Rightarrow p = -x + a$

Putting $p = a - x$ in (1), we get

$2z + (a-x)^2 + qy + 2y^2 = 0$

$\Rightarrow qy = -2z - (a-x)^2 - 2y^2$

$\Rightarrow q = -\frac{(2z + (a-x)^2 + 2y^2)}{y}$

Substituting p and q in $dz = pdx + qdy$, we get

$dz = (-x+a)dx + -\frac{(2z + (a-x)^2 + 2y^2)}{y} dy$

$= (a-x)dx - \frac{1}{y}(2z + (a-x)^2 + 2y^2)dy$

Multiplying both sides by $2y^2$, we get

$2y^2dz = 2y^2(a-x)dx - \frac{2y^2}{y}(2z + (a-x)^2 + 2y^2)dy$

$\Rightarrow 2y^2dz = 2y^2(a-x)dx - 4yzdy - 2y(a-x)^2dy - 4y^3dy$

$\Rightarrow 2y^2dz + 4yzdy = 2y^2(a-x)dx - 2y(a-x)^2dy - 4y^3dy$

$\Rightarrow d(2y^2z) = d(-y^2(a-x)^2) - 4y^3dy$

Integrating $\rightarrow \int d(2y^2z) = \int d(-y^2(a-x)^2) - \int 4y^3dy$

$\Rightarrow 2y^2z = -y^2(a-x)^2 - 4\frac{y^4}{4} + b$

$\Rightarrow 2y^2z = -(y^2(a-x)^2 + y^4) + b$

$\Rightarrow 2y^2z + y^2(a-x)^2 + y^4 = b$

$\Longrightarrow y^2(2z + (a-x)^2 + y^2) = b$

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Equations Solvable by Direct integration

These equations which contain only one partial derivative can be solved by direct integration. However, in place of the constants of integration, we must use arbitrary functions of the variable kept constant.

Problems

1. Solve $\frac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + \sin(2x-y) = 0$.

Ans: Given equation is $\frac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + \sin(2x-y) = 0$ — ①

Integrating ① w.r.t. x, keeping y fixed we get

$$\frac{\partial^2 z}{\partial x \partial y} + \frac{18x^2y^2}{2} + \frac{-\cos(2x-y)}{2} = 0 + f(y) \quad \text{— ②}$$

Integrating ② w.r.t. x, keeping y fixed we get

$$\frac{\partial z}{\partial y} + \frac{18x^3y^2}{2\times 3} + \frac{-\sin(2x-y)}{2\times 2} = 0 + xf(y) + g(y)$$

$$\Rightarrow \frac{\partial z}{\partial y} + 3x^3y^2 - \frac{\sin(2x-y)}{4} = xf(y) + g(y) \quad \text{— ③}$$

Integrating ③ w.r.t. y, keeping x fixed we get

$$z + \frac{3x^3y^3}{3} - \frac{-\cos(2x-y)}{4\times -1} = x\int f(y)\,dy + \int g(y)\,dy + w(x)$$

$$\Rightarrow z + x^3y^3 - \frac{\cos(2x-y)}{4} = x\int f(y)\,dy + \int g(y)\,dy + w(x) \quad \text{— ④}$$

Let $\int f(y)\,dy = u(y)$ and $\int g(y)\,dy = v(y)$, then

$$④ \Rightarrow z + x^3y^3 - \frac{\cos(2x-y)}{4} = x\,u(y) + v(y) + w(x)$$

$$\Rightarrow z = \frac{\cos(2x-y)}{4} - x^3y^3 + x\,u(y) + v(y) + w(x)$$

where $u, v,$ and w are arbitrary functions

2. Solve $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$ for which $\frac{\partial z}{\partial y} = -2\sin y$ when $x=0$ and $z=0$ when y is an odd multiple of $\pi/2$.

Ans: Given equation is $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$ —— (1)

Integrating (1) w.r.t. x keeping y fixed we get

$$\frac{\partial z}{\partial y} = -\cos x \sin y + f(y) \quad \text{—— (2)}$$

Given when $x=0$, $\frac{\partial z}{\partial y} = -2\sin y$

Also when $x=0$, (2) $\Rightarrow \frac{\partial z}{\partial y} = -\sin y + f(y)$

$\therefore -2\sin y = -\sin y + f(y)$

$\Rightarrow f(y) = -2\sin y + \sin y = -\sin y$

$\therefore$ (2) $\Rightarrow \frac{\partial z}{\partial y} = -\cos x \sin y - \sin y$ —— (3)

Integrating (3) w.r.t. y keeping x fixed we get

$$z = \cos x \cos y + \cos y + g(x) \quad \text{—— (4)}$$

Given when y is an odd multiple of $\frac{\pi}{2}$, $z=0$

Also when y is an odd multiple of $\frac{\pi}{2}$, (4) $\Rightarrow z = 0 + 0 + g(x)$

$\Rightarrow z = g(x)$

$\therefore g(x) = 0$

Hence from (4), $z = \cos x \cos y + \cos y$

$$\Rightarrow z = (1 + \cos x)\cos y$$

Method of separation of variables

The method of separation of variables is applicable to a large number of linear homogeneous equations where all the terms of PDE contain dependent variable. This method reduces a PDE in 2 independent variables to 2 ordinary differential equations.

Suppose that the given PDE contains 2 independent variables x, y and one dependent variable u. Then we first assume that the equation possesses product solution of the form $u(x,y) = X(x)\,Y(y)$ —— ①
where X is a function of x only and Y is a function of y only. Substituting equation ① in given PDE we get 2 ordinary differential equations one in each of the unknowns X and Y which can be then solved for X and Y. Finally the general solution is obtained by substituting X and Y in ①.

Problems

1. Find the solution of $u_x + u_y = 0$ by separation of variables

Ans: Given equation is $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$ —— ①

Let the solution of ① be $u(x,y) = X(x)\,Y(y)$ —— ②

Differentiating ② partially w.r.t. x and y we get

$\frac{\partial u}{\partial x} = \frac{dX}{dx} \cdot Y = X'Y$ and

$\frac{\partial u}{\partial y} = X \cdot \frac{dY}{dy} = XY'$

Substituting these values in ①, we get

$$X'Y + XY' = 0$$

$$\Rightarrow X'Y = -XY'$$

$$\Rightarrow \frac{X'}{X} = \frac{-Y'}{Y} \quad \text{---} ③$$

L.H.S is a function of x only and the R.H.S is a function of y only. ∴ ③ can only be true if each side is equal to the same constant k, which is called separation constant

$$\therefore \frac{X'}{X} = \frac{-Y'}{Y} = k$$

This gives two ordinary differential equations

$$\frac{X'}{X} = k \quad \text{---} ④ \quad \text{and} \quad \frac{-Y'}{Y} = k \quad \text{---} ⑤$$

$$④ \Rightarrow \frac{X'}{X} = k \Rightarrow \frac{1}{X} \cdot \frac{dX}{dx} = k$$

$$\Rightarrow \frac{dX}{X} = k\,dx$$

Integrating $\Rightarrow \int \frac{dX}{X} = \int k\,dx$

$$\Rightarrow \log X = kx + c_1$$

$$\Rightarrow X = e^{kx + c_1} \quad \text{---} ⑥$$

$$⑤ \Rightarrow \frac{-Y'}{Y} = k \Rightarrow \frac{-1}{Y} \cdot \frac{dY}{dy} = k$$

$$\Rightarrow \frac{-dY}{Y} = k\,dy$$

Integrating $\Rightarrow -\int \frac{dY}{Y} = \int k\,dy$

$$\Rightarrow -\log Y = ky + c_2$$

$$\Rightarrow \log Y = -(ky + c_2)$$

$$\Rightarrow Y = e^{-(ky+c_2)} \quad \text{--- (7)}$$

Substituting (6) and (7) in equation (2), we get the solution as $u(x,y) = e^{kx+c_1} e^{-(ky+c_2)}$

$$= e^{kx+c_1-ky-c_2}$$

$$= e^{kx-ky+c_1-c_2}$$

$$= e^{k(x-y)} \cdot e^{c_1-c_2} = \underline{c\, e^{k(x-y)}}$$

U.Q

2. Solve $3u_x + 2u_y = 0$ with $u(x,0) = 4e^{-x}$ by the method of separation of variables.

Ans: Given equation is $3\frac{\partial u}{\partial x} + 2 \cdot \frac{\partial u}{\partial y} = 0$ --- (1)

Let the solution of (1) be $u(x,y) = X(x)\,Y(y)$ --- (2)

From (2), $\frac{\partial u}{\partial x} = X'Y$ and $\frac{\partial u}{\partial y} = XY'$

$\therefore$ (1) $\Rightarrow 3X'Y + 2XY' = 0$

$$\Rightarrow 3X'Y = -2XY'$$

$$\Rightarrow \frac{3X'}{X} = \frac{-2Y'}{Y} = k$$

$\Rightarrow \frac{3X'}{X} = k$ --- (3) and $\frac{-2Y'}{Y} = k$ --- (4)

(3) $\Rightarrow 3X' = kX$

$\Rightarrow 3\frac{dX}{dx} = kX$

$\Rightarrow \frac{3dX}{X} = k\,dx$

$\Rightarrow \int \frac{3dX}{X} = \int k\,dx$

$\Rightarrow 3\log X = kx + c_1$

$\Rightarrow \log X = \frac{kx+c_1}{3}$

$\Rightarrow X = e^{\frac{kx+c_1}{3}}$

(4) $\Rightarrow \frac{-2Y'}{Y} = k$

$\Rightarrow -\frac{2}{Y}\frac{dY}{dy} = k$

$\Rightarrow -\frac{2dY}{Y} = k\,dy$

$\Rightarrow \int -\frac{2dY}{Y} = \int k\,dy$

$\Rightarrow -2\log Y = ky + c_2$

$\Rightarrow \log Y = \frac{ky+c_2}{-2}$

$\Rightarrow Y = e^{\frac{ky+c_2}{-2}}$

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$$\therefore (2) \Rightarrow u(x,y) = e^{\frac{kx+c_1}{3}} \cdot e^{\frac{ky+c_2}{-2}}$$

$$= e^{\frac{kx}{3}} \cdot e^{\frac{c_1}{3}} \cdot e^{\frac{ky}{-2}} \cdot e^{\frac{c_2}{-2}}$$

$$= e^{\frac{kx}{3} + \frac{ky}{-2}} \cdot c$$

$$= e^{k\left(\frac{x}{3} - \frac{y}{2}\right)} c \quad \text{——} \quad (5)$$

Given $u(x,0) = 4e^{-x}$

when $y=0$, $(5) \Rightarrow u(x,0) = e^{k\left(\frac{x}{3}\right)} \cdot c$

$$\therefore 4e^{-x} = c e^{\frac{kx}{3}}$$

$$\Rightarrow c = 4 \quad \text{and} \quad \frac{k}{3} = -1$$

$$\Rightarrow k = -3$$

$$\therefore (5) \Rightarrow u(x,y) = 4e^{-3\left(\frac{x}{3} - \frac{y}{2}\right)}$$

$$= 4e^{-x + \frac{3y}{2}}$$