1

GRAPH THEORY AND COMBINATORICS

Module I

Introductory concepts - What is graph – Application of graphs – finite and infinite graphs – Incidence and Degree – Isolated vertex, pendent vertex and Null graph. Paths and circuits – Isomorphism, sub graphs, walks, paths and circuits, connected graphs, disconnect graphs.

General Tips

1. $d(v) \leq n-1$ (n = vertices)

2. $\sum_{i=1}^{n} d(v_i) = 2e$

3. $\left.\begin{matrix}\sum \text{odd degree} \\ \text{Vertices}\end{matrix}\right\} = \text{even}$

4. $\left.\begin{matrix}\text{n Vertices} \\ \text{are having}\end{matrix}\right\} \Rightarrow \frac{n(n-1)}{2}$ edges

5. $\left.\begin{matrix}\text{n vertices} \\ \text{are having}\end{matrix}\right\}$ $2^{\frac{n(n-1)}{2}}$ possible graphs

6. Walk - no repeated edges, vertices may be repeated.
7. path - A walk no repeated vertices & no repeated edges
8. Closed walk - Starting & ending Vertices should be same
9. Circuit - No repeated edges & No repeated Vertices & Starting ending vertices should be same.

10. $\left.\begin{matrix}\text{length of} \\ \text{path}\end{matrix}\right\}$ - Count number of lines (edges) in a path.

Module 1 Theorems:

Theorem 1:
The sum of degrees of a graph is even, being twice the number of edges

Theorem 2:
The number of vertices of odd degree in a graph is always even.

Theorem 3:
A graph G is disconnected if and only if its vertex set V can be partitioned into two nonempty disjoint subsets V_1 and V_2 such that there exists no edge in G whose one end vertex is in subset V_1 and the other in subset V_2

Theorem 4:
If a graph has exactly two vertices of odd degree there must be a path joining these two vertices.

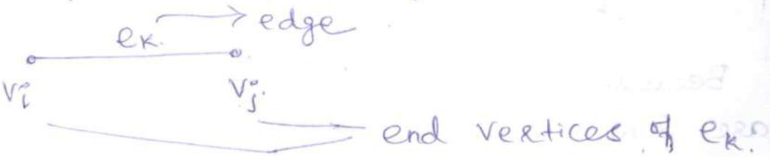
Theorem 5:
A simple graph with n vertices and k components can have at most $\frac{(n-k)(n-k+1)}{2}$ edges.

Theorem 6

The maximum no. of edges in a simple graph with n vertices $n(n-1)$

Graph: (or) linear graph:

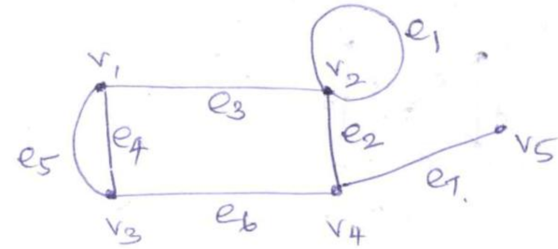
$G = (V, E)$ Consists of Set of objects $V = \{v_1, v_2, \dots\}$ Called vertices and another Set $E = \{e_1, e_2, \dots\}$ are called edges, such that each edge e_k is an unordered pair (v_i, v_j) of vertices.



graph Representation:

* Diagram with V & E
* Vertices (V) are represented as points
* Edges (E) are represented as a line

Example:



Self loop: ⇒ e_1 is self loop

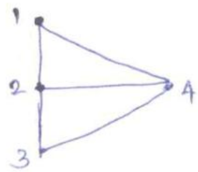
An edge having the Same Vertex as both its end vertices is called a self loop.

Parallel edges: ⇒ e_4 & e_5 parallel edges

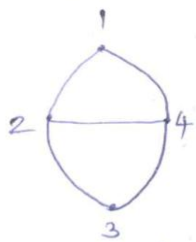
More than one edge associated with Same pair of vertices are called as parallel edges

Graph Types
- Simple graph ⇒ Self loops & parallel edges are not allowed
- general graph ⇒ self loops & parallel edges are allowed.

Same graph drawn differently:



equal to

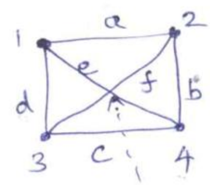


Because incidence between edges and vertices are Same in both diagram.

Note:

Some times two edges may seem to intersect at a point that doesnot represent a vertex.

Ex:



$\Rightarrow$ Here, e and f have no Common vertex

It's not a vertex.

Other Names:

Graph => linear Complex, 1-complex, One dimensional complex.

Vertex => a node, a junction, a point, 0-cell (zero) 0-Simplex (zero)

Edge => Branch, a line, an element, 1-cell arc, 1-simplex

General Terms => Graph, Vertex, Edge

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Graph Theory – Fundamentals

Graph is a diagram of points and lines connected to the points. It has at least one line joining a set of two vertices with no vertex connecting itself. The concept of graphs in graph theory stands up on some basic terms such as point, line, vertex, edge, degree of vertices, properties of graphs, etc.

Point

A point is a particular position in a one-dimensional, two-dimensional, or three-dimensional space. For better understanding, a point can be denoted by an alphabet. It can be represented with a dot.

Example

Here, the dot is a point named 'a'.

Line

A **Line** is a connection between two points. It can be represented with a solid line.

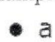
Example

Here, 'a' and 'b' are the points. The link between these two points is called a line.

Vertex

A vertex is a point where multiple lines meet. It is also called a **node**. Similar to points, a vertex is also denoted by an alphabet.

Example



Here, the vertex is named with an alphabet 'a'.

Edge

An edge is the mathematical term for a line that connects two vertices. Many edges can be formed from a single vertex. Without a vertex, an edge cannot be formed. There must be a starting vertex and an ending vertex for an edge.

Example

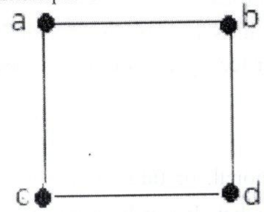
Here, 'a' and 'b' are the two vertices and the link between them is called an edge.

(b)

Graph

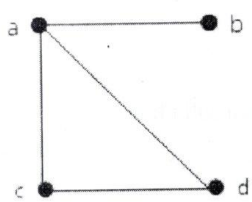
A graph 'G' is defined as G = (V, E) Where V is a set of all vertices and E is a set of all edges in the graph.

Example 1



In the above example, ab, ac, cd, and bd are the edges of the graph. Similarly, a, b, c, and d are the vertices of the graph.

Example 2

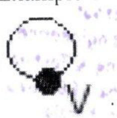


In this graph, there are four vertices a, b, c, and d, and four edges ab, ac, ad, and cd.

Loop or self loop

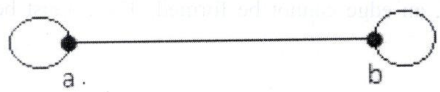
In a graph, if an edge is drawn from vertex to itself, it is called a loop.

Example 1



In the above graph, V is a vertex for which it has an edge (V, V) forming a loop.

Example 2



Degree of Vertex Number of edges incident on a vertex V, with self loop counted twice.

It is the number of vertices adjacent to a vertex V.

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For simple graph, $deg(v) \leq n-1$
where, $n \Rightarrow$ no. of vertices in the graph.

Notation – deg(V).

In a simple graph with n number of vertices, the degree of any vertices is –

$deg(v) \leq n - 1 \; \forall \; v \in G$

A vertex can form an edge with all other vertices except by itself. So the degree of a vertex will be up to the **number of vertices in the graph minus 1**. This 1 is for the self-vertex as it cannot form a loop by itself. If there is a loop at any of the vertices, then it is not a Simple Graph.

Degree of vertex can be considered under two cases of graphs –

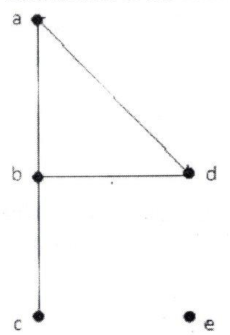
- Undirected Graph
- Directed Graph

Degree of Vertex in an Undirected Graph

An undirected graph has no directed edges. Consider the following examples.

Example 1

Take a look at the following graph –



X

Pendent Vertex:
degree(v) is 1

Isolated vertex:
degree(v) is 0.

In the above Undirected Graph,

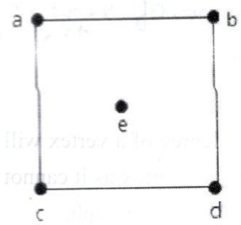
- deg(a) = 2, as there are 2 edges meeting at vertex 'a'.
- deg(b) = 3, as there are 3 edges meeting at vertex 'b'.
- deg(c) = 1, as there is 1 edge formed at vertex 'c'

  So 'c' is a **pendent vertex**.
- deg(d) = 2, as there are 2 edges meeting at vertex 'd'.
- deg(e) = 0, as there are 0 edges formed at vertex 'e'.

  So 'e' is an **isolated vertex**.

Example 2

Take a look at the following graph –



In the above graph,

deg(a) = 2, deg(b) = 2, deg(c) = 2, deg(d) = 2, and deg(e) = 0.

The vertex 'e' is an isolated vertex. The graph does not have any pendent vertex.

Degree of Vertex in a Directed Graph

In a directed graph, each vertex has an **indegree** and an **outdegree**.

Indegree of a Graph

- Indegree of vertex V is the number of edges which are coming into the vertex V.
- **Notation** − $deg^{+}(V)$.

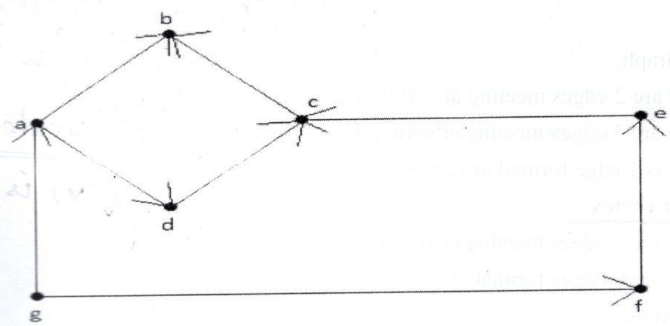
Outdegree of a Graph

- Outdegree of vertex V is the number of edges which are going out from the vertex V.
- **Notation** − $deg^{-}(V)$.

Consider the following examples.

Example 1

Take a look at the following directed graph. Vertex 'a' has two edges, 'ad' and 'ab', which are going outwards. Hence its outdegree is 2. Similarly, there is an edge 'ga', coming towards vertex 'a'. Hence the indegree of 'a' is 1.



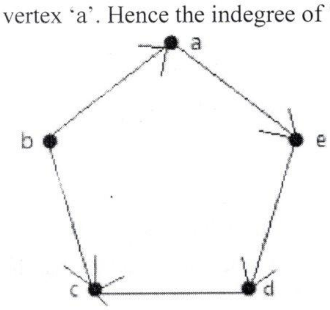
The indegree and outdegree of other vertices are shown in the following table –

(9)

Vertex	Indegree	Outdegree
a	1	2
b	2	0
c	2	1
d	1	1
e	1	1
f	1	1
g	0	2

Example 2

Take a look at the following directed graph. Vertex 'a' has an edge 'ae' going outwards from vertex 'a'. Hence its outdegree is 1. Similarly, the graph has an edge 'ba' coming towards vertex 'a'. Hence the indegree of 'a' is 1.



The indegree and outdegree of other vertices are shown in the following table −

Vertex	Indegree	Outdegree
a	1	1
b	0	2
c	2	0
d	1	1
e	1	1

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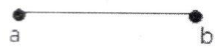
Pendent Vertex

By using degree of a vertex, we have a two special types of vertices. A vertex with degree one is called a pendent vertex.

Example



Here, in this example, vertex 'a' and vertex 'b' have a connected edge 'ab'. So with respect to the vertex 'a', there is only one edge towards vertex 'b' and similarly with respect to the vertex 'b', there is only one edge towards vertex 'a'. Finally, vertex 'a' and vertex 'b' has degree as one which are also called as the pendent vertex.

Isolated Vertex

A vertex with degree zero is called an isolated vertex.

Example

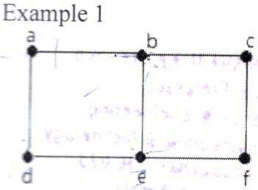
Here, the vertex 'a' and vertex 'b' has a no connectivity between each other and also to any other vertices. So the degree of both the vertices 'a' and 'b' are zero. These are also called as isolated vertices.

Adjacency

Here are the norms of adjacency –

- In a graph, two vertices are said to be **adjacent,** if there is an edge between the two vertices. Here, the adjacency of vertices is maintained by the single edge that is connecting those two vertices.
- In a graph, two edges are said to be adjacent, if there is a common vertex between the two edges. Here, the adjacency of edges is maintained by the single vertex that is connecting two edges.

Example 1

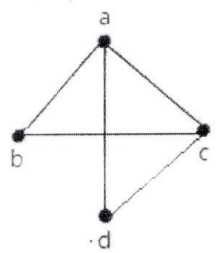


In the above graph –

- 'a' and 'b' are the adjacent vertices, as there is a common edge 'ab' between them.
- 'a' and 'd' are the adjacent vertices, as there is a common edge 'ad' between them.

- ab' and 'be' are the adjacent edges, as there is a common vertex 'b' between them.
- be' and 'de' are the adjacent edges, as there is a common vertex 'e' between them.

Example 2

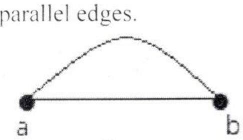


In the above graph –

- a' and 'd' are the adjacent vertices, as there is a common edge 'ad' between them.
- 'c' and 'b' are the adjacent vertices, as there is a common edge 'cb' between them.
- 'ad' and 'cd' are the adjacent edges, as there is a common vertex 'd' between them.
- ac' and 'cd' are the adjacent edges, as there is a common vertex 'c' between them.

Parallel Edges

In a graph, if a pair of vertices is connected by more than one edge, then those edges are called parallel edges.

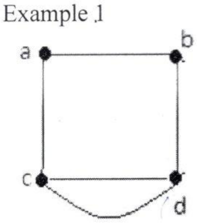


In the above graph, 'a' and 'b' are the two vertices which are connected by two edges 'ab' and 'ab' between them. So it is called as a parallel edge.

Multi Graph

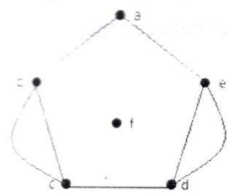
A graph having parallel edges is known as a Multigraph.

Example 1



In the above graph, there are five edges 'ab', 'ac', 'cd', 'cd', and 'bd'. Since 'c' and 'd' have two parallel edges between them, it a Multigraph.

Example 2

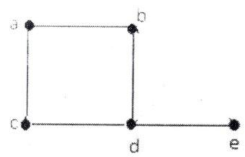


In the above graph, the vertices ‘b’ and ‘c’ have two edges. The vertices ‘e’ and ‘d’ also have two edges between them. Hence it is a Multigraph.

Degree Sequence of a Graph

If the degrees of all vertices in a graph are arranged in descending or ascending order, then the sequence obtained is known as the degree sequence of the graph.

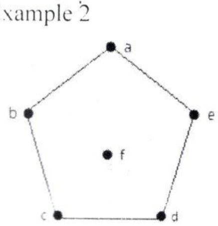
Example 1



Vertex	A	b	c	d	e
Connecting to	b,c	a,d	a,d	c,b,e	d
Degree	2	2	2	3	1

In the above graph, for the vertices {d, a, b, c, e}, the degree sequence is {3, 2, 2, 2, 1}.

Example 2



In the above graph, for the vertices {a, b, c, d, e, f}, the degree sequence is {2, 2, 2, 2, 2, 0}.

Vertex	A	b	c	d	e	f
Connecting to	b,e	a,c	b,d	c,e	a,d	-
Degree	2	2	2	2	2	0

Some properties

Distance between Two Vertices

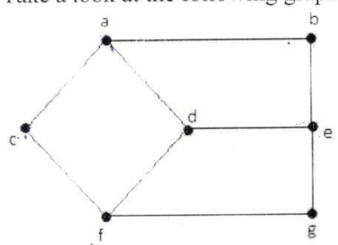
It is number of edges in a shortest path between Vertex U and Vertex V. If there are multiple paths connecting two vertices, then the shortest path is considered as the distance between the two vertices.

Notation – d(U,V)

There can be any number of paths present from one vertex to other. Among those, you need to choose only the shortest one.

Example

Take a look at the following graph –



Here, the distance from vertex 'd' to vertex 'e' or simply 'de' is 1 as there is one edge between them. There are many paths from vertex 'd' to vertex 'e' –

- da, ab, be
- df, fg, ge
- de (It is considered for distance between the vertices)
- df, fc, ca, ab, be
- da, ac, cf, fg, ge

Eccentricity of a Vertex

The maximum distance between a vertex to all other vertices is considered as the eccentricity of vertex.

Notation – e(V)

The distance from a particular vertex to all other vertices in the graph is taken and among those distances, the eccentricity is the highest of distances.

Example

In the above graph, the eccentricity of 'a' is 3.

The distance from 'a' to 'b' is 1 ('ab'),

from 'a' to 'c' is 1 ('ac');

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from 'a' to 'd' is 1 ('ad'),

from 'a' to 'e' is 2 ('ab'-'be') or ('ad'-'de'),

from 'a' to 'f' is 2 ('ac'-'cf') or ('ad'-'df'),

from 'a' to 'g' is 3 ('ac'-'cf'-'fg') or ('ad'-'df'-'fg').

So the eccentricity is 3, which is a maximum from vertex 'a' from the distance between 'ag' which is maximum.

In other words, a = 3

e(b) = 3

e(c) = 3

e(d) = 2

e(e) = 3

e(f) = 3

e(g) = 3

Radius of a Connected Graph

The minimum eccentricity from all the vertices is considered as the radius of the Graph G. The minimum among all the maximum distances between a vertex to all other vertices is considered as the radius of the Graph G.

Notation – r(G)

From all the eccentricities of the vertices in a graph, the radius of the connected graph is the minimum of all those eccentricities.

Example – In the above graph r(G) = 2, which is the minimum eccentricity for 'd'.

Diameter of a Graph

The maximum eccentricity from all the vertices is considered as the diameter of the Graph G. The maximum among all the distances between a vertex to all other vertices is considered as the diameter of the Graph G.

Notation – d(G)

From all the eccentricities of the vertices in a graph, the diameter of the connected graph is the maximum of all those eccentricities.

Example – In the above graph, d(G) = 3; which is the maximum eccentricity.

Central Point

If the eccentricity of a graph is equal to its radius, then it is known as the central point of the graph. If

e(V) = r(V),

then 'V' is the central point of the Graph 'G'.

Example – In the example graph, 'd' is the central point of the graph.

e(d) = r(d) = 2

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Centre

The set of all central points of 'G' is called the centre of the Graph.

Example − In the example graph, {'d'} is the centre of the Graph.

Circumference

The **number of edges in the longest cycle of 'G'** is called as the circumference of 'G'.

Example − In the example graph, the circumference is 6, which we derived from the longest cycle a-c-f-g-e-b-a or a-c-f-d-e-b-a.

Girth

The number of edges in the shortest cycle of 'G' is called its Girth.

Notation − g(G).

Example − In the example graph, the Girth of the graph is 4, which we derived from the shortest cycle a-c-f-d-a or d-f-g-e-d or a-b-e-d-a.

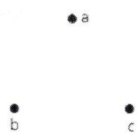
Sum of Degrees of Vertices Theorem

Types of Graphs

Null Graph

A **graph having no edges** is called a Null Graph.

Example



In the above graph, there are three vertices named 'a', 'b', and 'c', but there are no edges among them. Hence it is a Null Graph.

Trivial Graph

A **graph with only one vertex** is called a Trivial Graph.

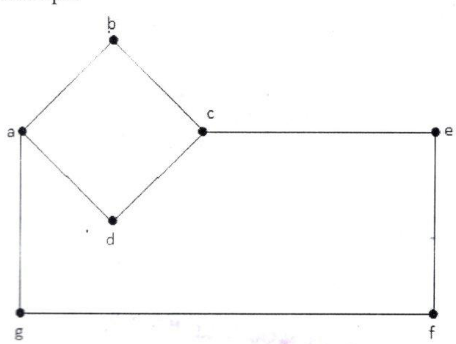
Example

In the above shown graph, there is only one vertex 'a' with no other edges. Hence it is a Trivial graph.

Non-Directed Graph

A non-directed graph contains edges but the edges are not directed ones.

Example



In this graph, 'a', 'b', 'c', 'd', 'e', 'f', 'g' are the vertices, and 'ab', 'bc', 'cd', 'da', 'ag', 'gf', 'ef' are the edges of the graph. Since it is a non-directed graph, the edges 'ab' and 'ba' are same. Similarly other edges also considered in the same way.

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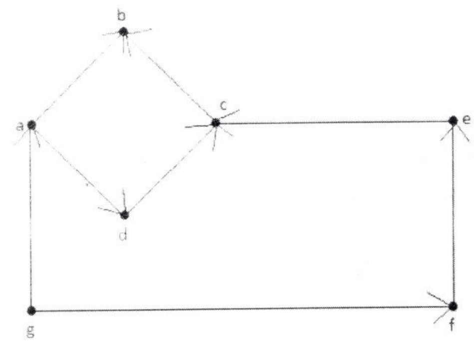
Directed Graph

In a directed graph, each edge has a direction.

Example



In the above graph, we have seven vertices 'a', 'b', 'c', 'd', 'e', 'f', and 'g', and eight edges 'ab', 'cb', 'dc', 'ad', 'ec', 'fe', 'gf', and 'ga'. As it is a directed graph, each edge bears an arrow mark that shows its direction. Note that in a directed graph, 'ab' is different from 'ba'.

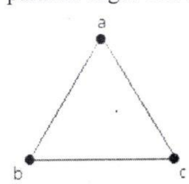
Simple Graph

A graph **with no loops** and **no parallel edges** is called a simple graph.

- The maximum number of edges possible in a single graph with 'n' vertices is $^nC_2$ where $^nC_2 = n(n-1)/2$.
- The number of simple graphs possible with 'n' vertices = $2^{nc_2} = 2^{n(n-1)/2}$.

Example

In the following graph, there are 3 vertices with 3 edges which is maximum excluding the parallel edges and loops. This can be proved by using the above formulae.



The maximum number of edges with n=3 vertices −

$^nC_2 = n(n-1)/2$
$= 3(3-1)/2$
$= 6/2$
$= 3$ edges

The maximum number of simple graphs with n=3 vertices −

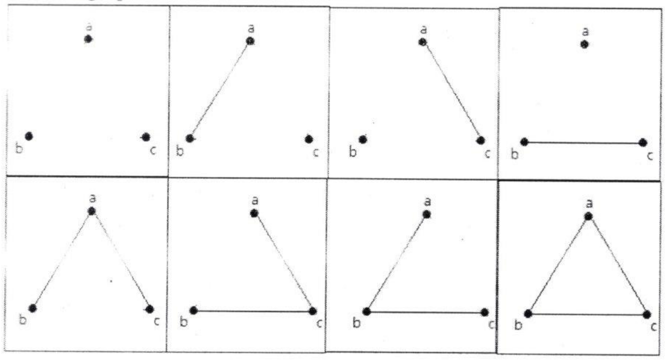
$2^{nC_2} = 2^{n(n-1)/2}$

$2^{3(3-1)/2}$

$= 2^3$

$= 8$

These 8 graphs are as shown below –

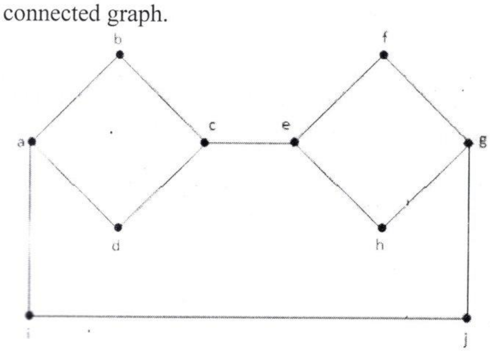


Connected Graph

A graph G is said to be connected **if there exists a path between every pair of vertices**. There should be at least one edge for every vertex in the graph. So that we can say that it is connected to some other vertex at the other side of the edge.

Example

In the following graph, each vertex has its own edge connected to other edge. Hence it is a connected graph.

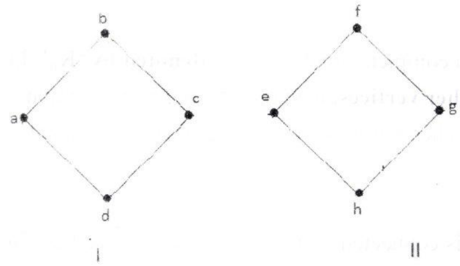


Disconnected Graph

A graph G is disconnected, if it does not contain at least two connected vertices.

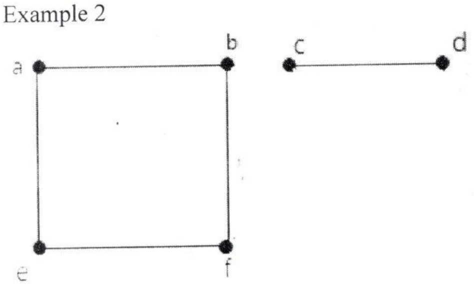
Example 1

The following graph is an example of a Disconnected Graph, where there are two components, one with 'a', 'b', 'c', 'd' vertices and another with 'e', 'f', 'g', 'h' vertices.



The two components are independent and not connected to each other. Hence it is called disconnected graph.

Example 2



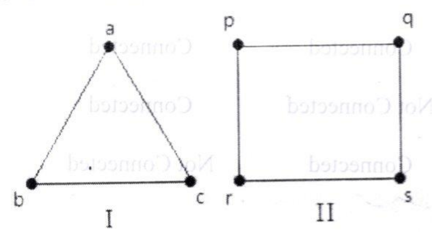
In this example, there are two independent components, a-b-f-e and c-d, which are not connected to each other. Hence this is a disconnected graph.

Regular Graph

A graph G is said to be regular, **if all its vertices have the same degree**. In a graph, if the degree of each vertex is 'k', then the graph is called a 'k-regular graph'.

Example

In the following graphs, all the vertices have the same degree. So these graphs are called regular graphs.



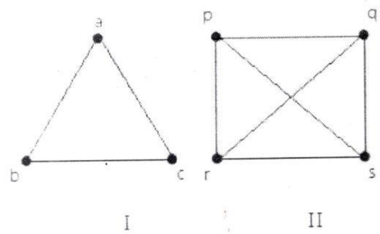
In both the graphs, all the vertices have degree 2. They are called 2-Regular Graphs.

Complete Graph

A simple graph with 'n' mutual vertices is called a complete graph and it is **denoted by 'K_n'**. In the graph, **a vertex should have edges with all other vertices,** then it called a complete graph. In other words, if a vertex is connected to all other vertices in a graph, then it is called a complete graph.

Example

In the following graphs, each vertex in the graph is connected with all the remaining vertices in the graph except by itself.



In graph I,

	a	b	c
a	Not Connected	Connected	Connected
b	Connected	Not Connected	Connected
c	Connected	Connected	Not Connected

In graph II,

	p	q	r	s
p	Not Connected	Connected	Connected	Connected
q	Connected	Not Connected	Connected	Connected
r	Connected	Connected	Not Connected	Connected
s	Connected	Connected	Connected	Not Connected

Cycle Graph

A simple graph with 'n' vertices (n >= 3) and 'n' edges is called a cycle graph if all its edges form a cycle of length 'n'.

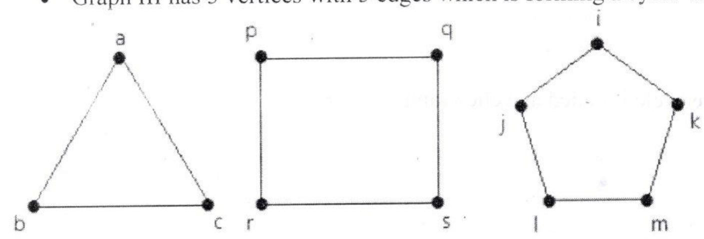
If the **degree of each vertex in the graph is two,** then it is called a Cycle Graph.

Notation – C_n

Example

Take a look at the following graphs –

- Graph I has 3 vertices with 3 edges which is forming a cycle 'ab-bc-ca'.
- Graph II has 4 vertices with 4 edges which is forming a cycle 'pq-qs-sr-rp'.
- Graph III has 5 vertices with 5 edges which is forming a cycle 'ik-km-ml-lj-ji'.



I II III

Hence all the given graphs are cycle graphs.

<u>Wheel Graph</u>

A wheel graph is obtained from a cycle graph C_{n-1} by adding a new vertex. That new vertex is called a **Hub** which is connected to all the vertices of C_n.

doubt

Notation – W_n

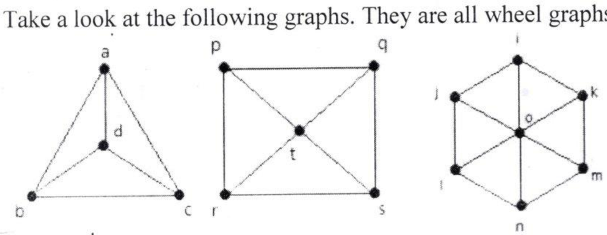
No. of edges in W_n = No. of edges from hub to all other vertices +
No. of edges from all other nodes in cycle graph without a hub.

$= (n-1) + (n-1)$

$= 2(n-1)$

Example

Take a look at the following graphs. They are all wheel graphs.



I(W_4) II(W_5) III(W_7)

In graph I, it is obtained from C_3 by adding an vertex at the middle named as 'd'. It is denoted as W_4.

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Number of edges in $W_4 = 2(n-1) = 2(3) = 6$

In graph II, it is obtained from C_4 by adding a vertex at the middle named as 't'. It is denoted as W_5.

Number of edges in $W_5 = 2(n-1) = 2(4) = 8$

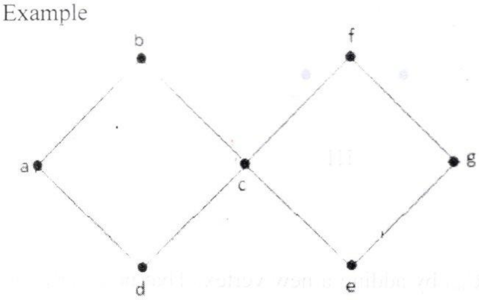
In graph III, it is obtained from C_6 by adding a vertex at the middle named as 'o'. It is denoted as W_7.

Number of edges in $W_4 = 2(n-1) = 2(6) = 12$

Cyclic Graph

A graph **with at least one** cycle is called a cyclic graph.

Example

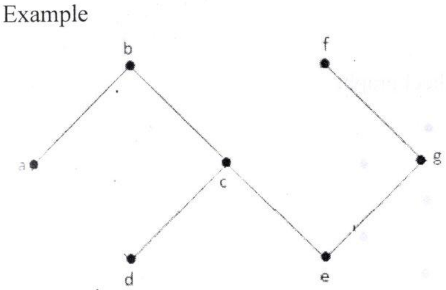


In the above example graph, we have two cycles a-b-c-d-a and c-f-g-e-c. Hence it is called a cyclic graph.

Acyclic Graph

A graph **with no cycles** is called an acyclic graph.

Example



In the above example graph, we do not have any cycles. Hence it is a non-cyclic graph.

Bipartite Graph

A simple graph $G = (V, E)$ with vertex partition $V = \{V_1, V_2\}$ is called a bipartite graph **if every edge of E joins a vertex in V_1 to a vertex in V_2.**

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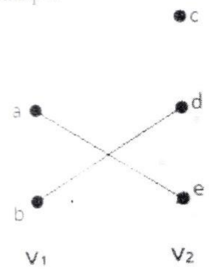
In general, a Bipertite graph has two sets of vertices, let us say, V_1 and V_2, and if an edge is drawn, it should connect any vertex in set V_1 to any vertex in set V_2.

Example



In this graph, you can observe two sets of vertices – V_1 and V_2. Here, two edges named 'ae' and 'bd' are connecting the vertices of two sets V_1 and V_2.

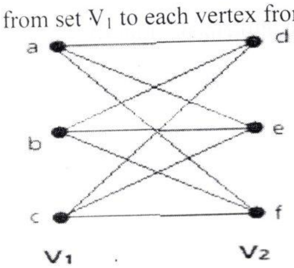
Complete Bipartite Graph

A bipartite graph 'G', G = (V, E) with partition V = {V_1, V_2} is said to be a complete bipartite graph if every vertex in V_1 is connected to every vertex of V_2.

In general, a complete bipartite graph connects each vertex from set V_1 to each vertex from set V_2.

Example

The following graph is a complete bipartite graph because it has edges connecting each vertex from set V_1 to each vertex from set V_2.



If $|V_1| = m$ and $|V_2| = n$, then the complete bipartite graph is denoted by $K_{m,n}$.

- $K_{m,n}$ has (m+n) vertices and (mn) edges.
- $K_{m,n}$ is a regular graph if m=n.

In general, **a complete bipartite graph is not a complete graph**.

$K_{m,n}$ is a complete graph if m=n=1.

'G' is a bipartite graph if 'G' has no cycles of odd length. A special case of bipartite graph is a **star graph**.

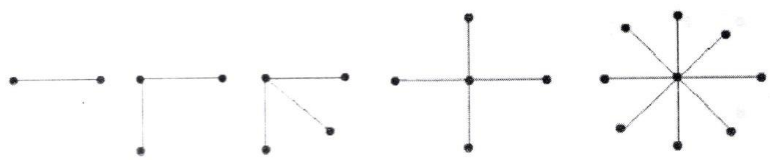
mxn edges.

m+n vertics

Star Graph

A complete bipartite graph of the form $K_{1,n-1}$ is a star graph with n-vertices. A star graph is a complete bipartite graph if a single vertex belongs to one set and all the remaining vertices belong to the other set.

Example



In the above graphs, out of 'n' vertices, all the 'n–1' vertices are connected to a single vertex. Hence it is in the form of $K_{1,n-1}$ which are star graphs.

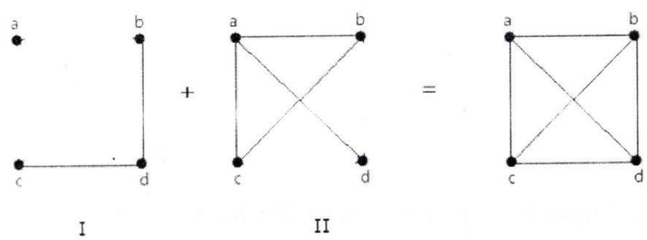
Complement of a Graph

Let '$G-$' be a simple graph with some vertices as that of 'G' and an edge {U, V} is present in '$G-$', if the edge is not present in G. It means, two vertices are adjacent in '$G-$' if the two vertices are not adjacent in G.

If the edges that exist in graph I are absent in another graph II, and if both graph I and graph II are combined together to form a complete graph, then graph I and graph II are called complements of each other.

Example .

In the following example, graph-I has two edges 'cd' and 'bd'. Its complement graph-II has four edges.



Note that the edges in graph-I are not present in graph-II and vice versa. Hence, the combination of both the graphs gives a complete graph of 'n' vertices.

Note – A combination of two complementary graphs gives a complete graph.

If 'G' is any simple graph, then

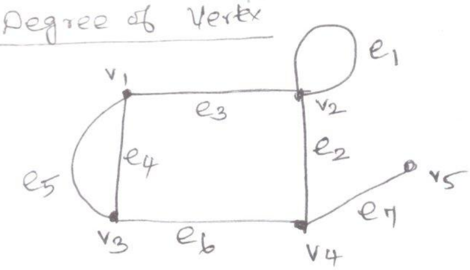
$|E(G)| + |E('G-')| = |E(K_n)|$, where n = number of vertices in the graph.

Types of Graph

1. Directed graph ⇒ direction, is included (→)
2. Undirected graph ⇒ No direction
3. Simple graph ⇒ No selfloop & parallel edges
4. Null graph (N_n) ⇒ no edges
5. Trivial graph ⇒ only one vertex graph.
6. Regular graph (r-regular; r → degree) ⇒ All vertices are having same degree
7. cubic graph ⇒ All vertices degree is equal to 3
8. Connected graph ⇒ It has exactly one connected component
9. Disconnected graph ⇒ two (or) more components (graph) in the G
10. Complete graph (K_n) ~~(or) clique~~ ⇒ A vertex is connected to all other vertices in a graph (Mesh)
    graph with n vertices & n edges
11. Cycle graph (C_n) ⇒ degree of each vertex is 2
12. Wheel graph (W_n) ⇒ A new vertex is added in cycle graph, which is connected to all vertices.
13. Bipartite graph ($K_{m,n}$) ⇒ Two vertices set V_1 & V_2 are created then an edge should join from V_1 to V_2.
14. Cyclic graph ⇒ A graph with atleast one cycle
15. Acyclic graph ⇒ A graph without any cycle
16. Complement of graph ⇒ If two vertices are adjacent in G then the two vertices are not adjacent in G'. & vice versa.
17. Finite graph ⇒ Finite number of edges & vertices
18. Infinite graph ⇒ Infinite number of edges & vertices
19. weighted graph ⇒ some constant value is assigned to each edge.

In Bipartite graph, $V_1 = m$ vertices $V_2 = n$ vertices then $K_{m,n}$ has

Find the Degree of Vertex



[∴ This is a General graph. It Contains both self loop & parallel edges]

$deg(v_1) = 3$

$deg(v_2) = 4$ [∴ for self loops (ie) twice]

$deg(v_3) = 3$

$deg(v_4) = 3$

$deg(v_5) = 1$

v_5 is a pendent vertex

e_4 & e_5 are parallel edges

e_1 is a self loop.

$$\therefore \boxed{\sum_{i=1}^{n} d(v_i) = 2e}$$

The sum of the degrees of all the vertices is twice the number of edges.

ie. Total number of edges in the Graph is 7

$$deg(v_1) + deg(v_2) + d(v_3) + d(v_4) + d(v_5) = 2(7)$$

$$3 + 4 + 3 + 3 + 1 = 14$$

$$14 = 14$$

L.H.S = R.H.S proved.

Degree Sequence:

Vertex	v_1	v_2	v_3	v_4	v_5
Connecting to	v_2, v_3 v_3	v_1, v_4 (v_2, v_2) selfloop	v_1, v_1 v_4	v_2, v_5 v_3	v_4
Degree	3	4	3	3	1

Arrange the vertex in ascending sequence based on degree. $v_5\ v_1\ v_3\ v_4\ v_2$ of degree sequence $\{1, 3, 3, 3, 4\}$

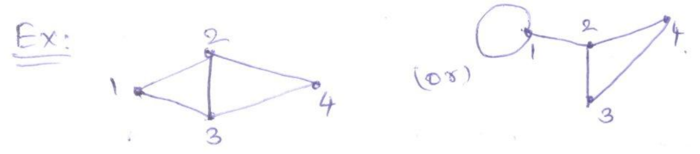
descending sequence

Finite and Infinite Graphs:

Finite:

A graph with a finite number of vertices as well as a finite number of edges is called a finite graph.

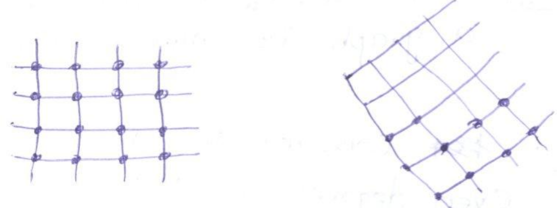
Ex:



Infinite:

A graph with an infinite number of vertices as well as an Infinite number of edges is called an infinite graph.

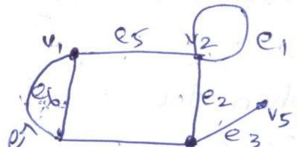
Ex:



Incidence and Degree (or) valency:

Incidence: when a vertex v_i is an end vertex of some edge e_j, v_i and e_j are said to be incident with (on (or) to) each other.

Degree: The number of edges incident on a vertex v_i, with self-loops counted twice is called the degree $d(v_i)$ of vertex v_i

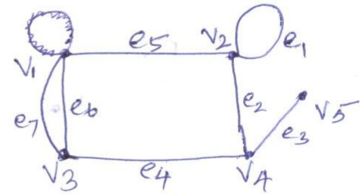


* e_2, e_3, e_4 are incident with vertex v_4
* e_2 and e_3 are adjacent

Theorem (1): The sum of the degrees of a graph is even, being twice the number of edges

Consider a graph G with e edges and n vertices $v_1, v_2, \ldots v_n$. Since each edge contributes two degrees, the sum of the degree of all vertices in G is twice the number of edges in G.

$$\sum_{i=1}^{n} d(v_i) = 2e$$



In this example. (previous)

$$d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5)$$

$$= 3 + 4 + 3 + 3 + 1 = \underline{14} \Rightarrow$$ Twice the number of edges

Theorem (2): The number of vertices of odd degree in a graph is always even.

Proof: ~~Let~~ Consider the vertices with odd and even degrees separately

$$\therefore \sum_{i=1}^{n} d(v_i) = \sum_{even} d(v_j) + \sum_{odd} d(v_k)$$

$$\underbrace{2e}_{R.H.S} = \underbrace{\sum_{even} d(v_j) + \sum_{odd} d(v_k)}_{R.H.S}$$ [∵ Substitute Theorem (1)]

Since R.H.S is even and $\sum_{even} d(v_j)$ is also even, Therefore, $\sum_{odd} d(v_k)$ is even.

This is only possible when the number of ·

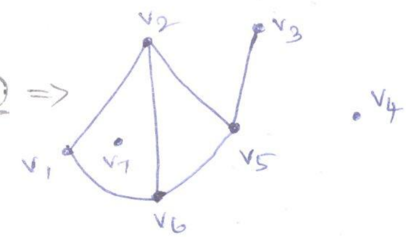
Isolated Vertex:

A vertex having no incident edge is called an Isolated vertex.

(or)

Isolated vertices are vertices with zero degree.

Figure (1) =>



Vertices v_4 and v_7 are Isolated vertex.

bcz, $d(v_4)$ & $d(v_7) = 0$.

Pendant vertex (or) End Vertex:

A vertex of degree one is called pendant vertex (or) end vertex.

vertex v_3 is a end vertex

In Figure (1) $d(v_3) = 1$

Null graph:

* A graph without any edges is called a null graph.

* Every vertex in a null graph is an Isolated ~~graph~~. vertex. (bcz) $deg(v) = 0$.

Ex:

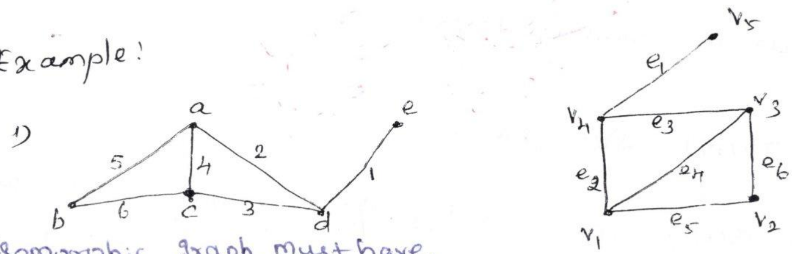
③ ISOMORPHISM.

Any two graph number of vertices & edges are same
→ edge connectivity should be same

Two graphs G and G' are said to be isomorphism if there is a one-one correspondence between their vertices and between their edges. It is denoted by $G \cong G'$.

Example:

1)



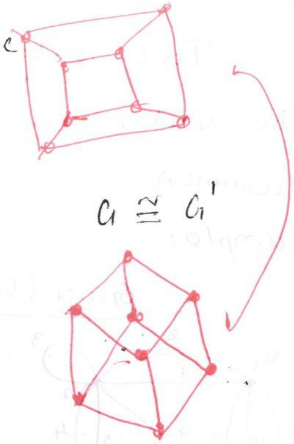
Isomorphic graph must have,

1. Same number of vertices
2. Same number of edges
3. An equal number of vertices with a given degree.

The vertices a, b, c, d & e correspond to v_1, v_2, v_3, v_4 & v_5 respectively.

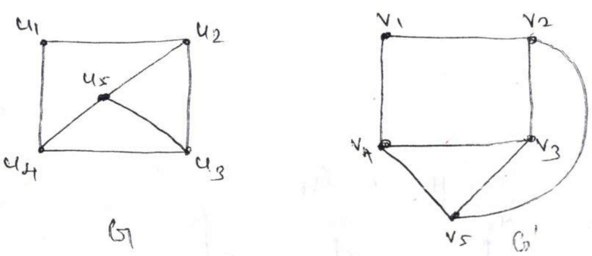
The edges 1, 2, 3, 4, 5 & 6 correspond to e_1, e_2, e_3, e_4, e_5 & e_6 respectively

∴ G & G' are isomorphic



$G \cong G'$

2)



G G'

3)

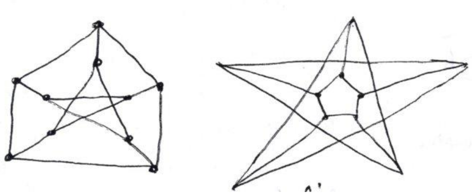
G G'

$G \cong G'$

G & G' are 3 regular graph.

G is complete bipartite graph.

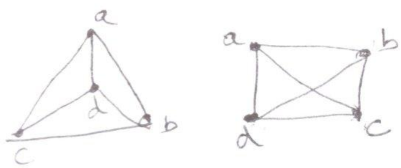
4)



$G \cong G'$.

① Isomorphic graphs have same degree list

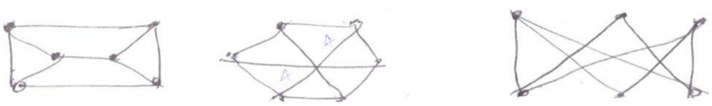
② If the degree



Both are Isomorphic.

All are Isomorphic graphs.

Both are Isomorphic graphs.



A All are not isomorphic graphs. C

D

Steps:

1. Label the graph.
2. Check the 1-1 correspondence.
3. After, Check all other rules

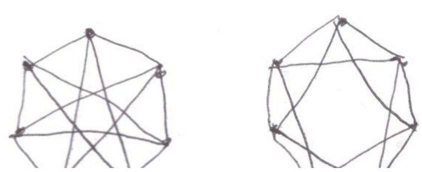
} Simple graph.

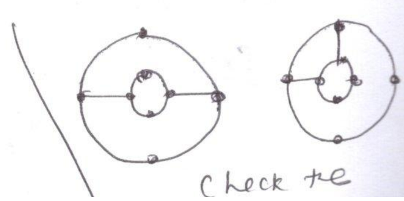
Steps:

1. Label the graph
2. Check all general rules of Isomorphism.
3. Check 1-1 Correspondence

} Complex graph.

Ex:





Check the

Relabel the Vertices.

Isomorphism: Rules:

① Two graphs should have same number of Vertices

② Two graphs should have same number of edges

③ If one graph has parallel edges ~~and~~ the other graphs should have parallel edge

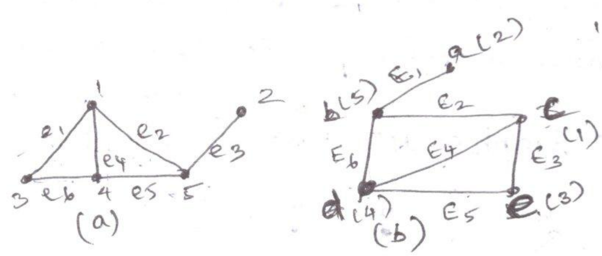
④ If one graph has a loop then othee graphs should have a loop

⑤ If one graph has a vertex of degree k then othee graphs have the same

⑥ If one graph is connected then othee graphs should be connected

⑦ If one graph has a cycle and othee graphs should have a cycle.

Ex:



~~1, 2, 3, 4, 5 correspond to b, a, e, c, d~~

$E_1 = e_3$ $E_4 = e_4$
$E_6 = e_5$ $E_5 = e_6$
$E_3 = e_1$ $E_6 = e_5$

① No selfloop & No parallel edge

② Both graphs have same number of edges & Vertices

③ degree of graph a : $\begin{pmatrix} \overset{1}{3}, \overset{2}{1}, \overset{3}{2}, \overset{4}{3}, \overset{5}{3} \end{pmatrix} \Rightarrow \overset{2}{1}, \overset{3}{2}, \overset{1}{3}, \overset{4}{3}, \overset{5}{3} \rightarrow$ Ascending

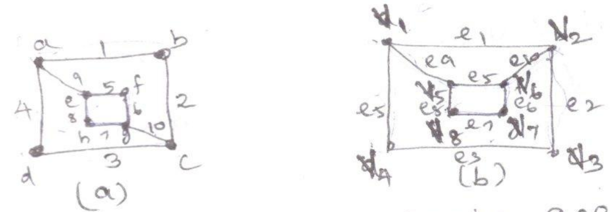
degree of graph b : $\begin{pmatrix} \underset{a}{1}, \underset{b}{3}, \underset{c}{3}, \underset{d}{3}, \underset{e}{2} \end{pmatrix} \Rightarrow \underset{a}{1}, \underset{e}{2}, \underset{b}{3}, \underset{c}{3}, \underset{d}{3} \rightarrow$ Ascending

④ Two graphs are connected graphs

④ Both graphs are having cycles.

| Graph (a) & Graph (b) are isomorphic |

Ex : 2



Explain why the two graphs are not isomorphic.

Sol :

no of Vertices of (a) & (b) = 8

no of edges of (a) & (b) = 10

① Both graphs are having same no; of Vertices & edges

② No self loop & parallel edges

③ degree of graph (a): (3, 2, 3, 2, 3, 2, 3, 2) = (2, 2, 2, 2, 3, 3, 3, 3)

degree of graph (b): (3, 3, 2, 2, 3, 3, 2, 2) = (2, 2, 2, 2, 3, 3, 3, 3)

④ Both graphs are connected

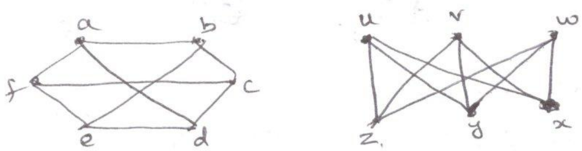
The Vertices a, b, c, d, e, f, g & h correspond to v_1, v_2, v_3, v_4, v_5, v_6, v_7 & v_8.

The edges 1, 2, 3, 4, 5, 6, 7, 8, 9, correspond to e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9. But 10 is not correspond to e_{10}

One-one correspondence is not here

The two graphs are Not isomorphic

Ex : 3



① (3, 3, 3, 3, 3, 3)   (3, 3, 3, 3, 3, 3)

② equal Vertices & edges

③ No self loop & parallel edges

SUBGRAPHS:

A graph $G_1(V_1, X_1)$ is said to be a subgraph of a graph $G = (V, X)$ if $V_1 \subseteq V$ & $X_1 \subseteq X$.

(or)

If all vertices & all edges of G_1 are in G and each edge of G_1 has the same end vertices in G_1 as in G.

EDGE DISJOINT SUBGRAPH

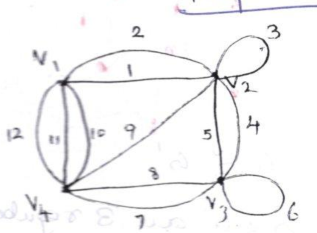
Two subgraphs G_1 & G_2 of a graph G are said to be edge disjoint if G_1 & G_2 do not have any edges in common.

VERTEX DISJOINT SUBGRAPH

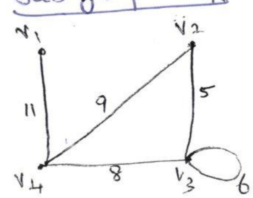
Two subgraphs G_1 & G_2 of a graph G are said to be vertex disjoint if G_1 & G_2 do not have any vertexes in common.

Example:

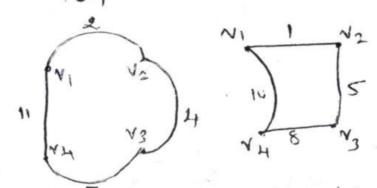
Graph (G)



Sub Graph (G_1)

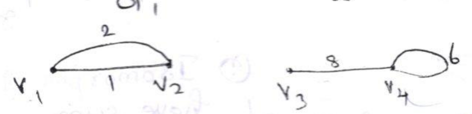


G_1 G_2



Edge disjoint subgraph.

G_1 G_2



Vertex disjoint Subgraph.

Subgraph observations:

1. Every graph is its own subgraph.
2. A subgraph of a subgraph of G is a subgraph of G.
3. A single vertex in a graph G is a subgraph of G.
4. A single edge in G, together

4 **WALK (or) Edge train (or) Train:**

A walk of a graph G is an alternating sequence of points (vertices) and edges, $v_0, x_1, v_1, x_2, \ldots v_{n-1}, x_n, v_n$. beginning and ending with vertices. such that edge x_i is incident with v_{i-1} and v_i.

No edges appears more than once in a walk

TERMINAL VERTICES

Vertices with which a walk begins and ends are called its terminal vertices.

TRAIL

A walk is called a trail if all its edges are distinct.

PATH (or) Elementary Path (or) Simple path:

A walk is called a path if all its vertices are distinct. The no. of edges in a path is called the length a path.

CLOSED WALK:

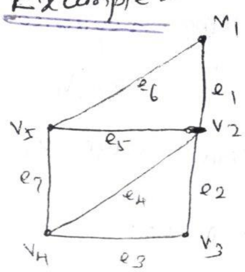
A walk, to begin and end at the same vertex. such a walk is called a closed walk.

Open walk: A walk is not closed which is called an open walk. (The terminal vertices are distinct)

CIRCUIT

A closed walk in which no vertex appears more than once is called a circuit. A circuit is also called a cycle, elementary cycle, circular path & polygon.

Example:-



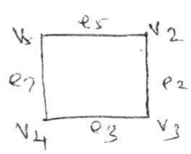
$v_1\ e_1\ v_2\ e_5\ v_5\ e_7\ v_4\ e_4\ v_2\ e_2\ v_3$ is a walk

v_1 & v_3 are terminal vertices, (or) initial vertices.

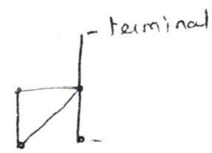
$v_1\ e_1\ v_2\ e_5\ v_5\ e_7\ v_4\ e_3\ v_3\ e_2\ v_2$ is a trail

$v_1\ e_6\ v_5\ e_5\ v_2\ e_2\ v_3$ is a path. Path of length is 3

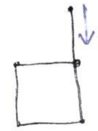
$v_2\ e_2\ v_3\ e_3\ v_4\ e_7\ v_5\ e_5\ v_2$ is a circuit or cycle.



cycle.



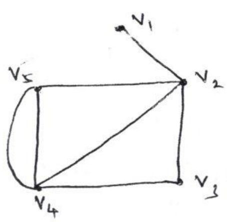
walk



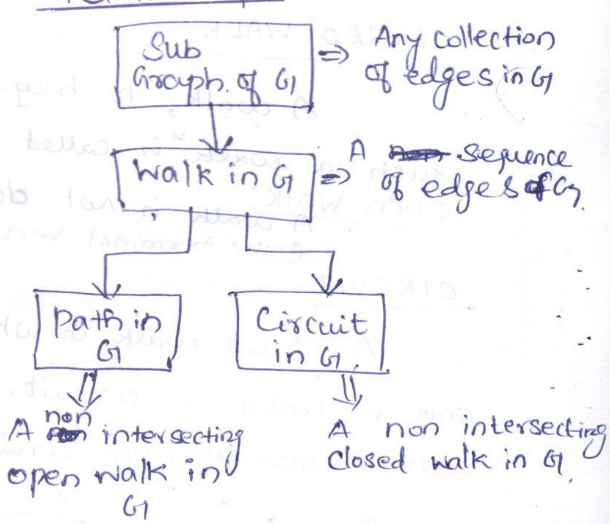
trail

path P_3.

i)



Relationships:



(5) Note:

1. The terminal vertices of a path are of degree 1.
2. The intermediate vertices of a path are of degree 2.

CONNECTED GRAPH.

A graph G is said to be connected if there is atleast one path between every pair of vertices in G. otherwise it is disconnected.

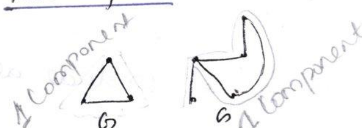
Note :-

1. A null graph of more than one vertex is disconnected
2. A disconnected graph consists of two or more connected graphs. Each of these connected subgraph is called components.

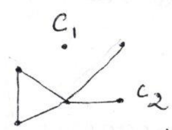
Disconnected graph:

A graph G is said to be disconnected if there is no path between every pair of vertices in G.

Example:-



connected Graph



Disconnected Graph with 2 component.

Theorem 3:

A graph G is disconnected if and only if its vertex set V can be partitioned into two nonempty disjoint subsets V_1 and V_2 such that there exists no edge in G whose one end vertex is in subset V_1 and the other in subset V_2

Proof:-

Consider a graph G which have vertex set as atleast two partitions, say V_1 & V_2.

Consider two arbitrary vertices a & b of G, such that $a \in V_1$ & $b \in V_2$.

No path can exist between vertices a & b.

Otherwise, there would be at least one edge whose one end vertex would be in V_1 & the other in V_2. but in G. partition exists in vertex set. which is $\rightarrow\leftarrow$ (opposite) to our assumption.

Hence G is not connected.

Conversely, let G be a disconnected graph.

Consider a vertex a in G.

let V_1 be the set of all vertices that are joined by paths to a.

Since G is disconnected, V_1 does not include all vertices of G.

The remaining vertices form a vertex set V_2.

$\therefore$ No vertex in V_1 is joined to any in V_2 by an edge.

Hence the partitioned exists.

(6) Theorem 4

(Hand Shaking Lemma) (connected (or) disconnected).

If a graph has exactly two vertices of odd degree, there must be a path joining those two vertices. (No two pendent vertex should be there)

Proof:

Consider a graph G with all even vertices except vertices v_1 & v_2, which are odd.

WKT, The no. of vertices of odd degree in a graph is always even.

$\therefore$ no graph can have an odd no. of odd degree vertices.

In graph G, v_1 & v_2 must belong to the same component.

Hence there exists a path between them.

Theorem 5:

(7) A simple graph with n vertices and k component can have at most $(n-k)(n-k+1)/2$ edges.

Proof:

Let the no. of vertices in each of the k-components of a graph G be $n_1, n_2, n_3, \ldots n_k$.

$\therefore$ we $n_1 + n_2 + n_3 + \cdots + n_k = n$, $n_i \geq 1$.

$$\Rightarrow \sum_{i=1}^{k} n_i - \ldots = n - k \;\forall\; n_i \geq 1. \longrightarrow (1)$$

WKT, an algebraic in equality,

$$\sum^{k} n_i^2 \leq n^2 - (k-1)(2n-k) \longrightarrow (2)$$

In a simple graph G, the maximum no. of edges in the i^{th} component is $\frac{n_i(n_i-1)}{2}$.

A0

$\therefore$ The maximum no. of edges in G is.

$$\sum_{i=1}^{k} \frac{n_i(n_i-1)}{2} = \frac{1}{2}\sum_{i=1}^{k} n_i^2 - n_i$$

$$\leq \frac{1}{2}\left[n^2-(k-1)(2n-k) - n\right]$$

$$\leq \frac{1}{2}\left[n^2 - 2nk + k^2 + 2n - k - n\right]$$

$$= \frac{1}{2}\left[n^2 - 2nk + n - k + k^2\right]$$

$$= \frac{1}{2}\left[n^2 - nk - nk + n - k + k^2\right]$$

$$= \frac{1}{2}\left[n(n-k) - k(n-k) + (n-k)\right]$$

$$= \frac{1}{2}\left[(n-k)(n-k+1)\right]$$

Rational Rose

Hence proved.

No need

Note!:

Algebraic - Inequality.

$\sum_{i=1}^{k} n_i - 1 = n-k$

squaring on both sides $\quad (n_i - 1 \geq 0.)$

$\left(\sum_{i=1}^{k} n_i - 1\right)^2 = (n-k)^2$

$n^2 - 2nk + k^2$

$n_i^2 \quad 2n_i + k$ ... k^2

$\leq 2n + k + n^2 - 2nk + k^2$ (k times) (by (1))

$\leq n^2 - 2n(k-1) + k(k-1)$

$\leq n^2 - (k-1)(2n-k).$

$n.$