

MODULE 1

CALCULUS OF VECTOR FUNCTIONS

Vector valued functions of single variable, derivative of vector functions and geometrical interpretation, motion along a curve - velocity, acceleration and speed. Concept of scalar and vector fields, Gradient and its properties, directional derivative, divergence and curl, Line integrals of vector fields, work as line integral, conservative vector fields, independence of path and potential functions (results with out proof)

(Text 1 : Relevant Topics from sections 12.1, 12.2, 12.6, 13.6, 15.1
15.2, 15.3)

Vector valued functions of a real variable.

A vector valued function in two dimensional space defined for $a \leq t \leq b$ can be expressed as $\vec{r} = x(t)\hat{i} + y(t)\hat{j}$, $a \leq t \leq b$ and in three dimensional space it is expressed as $\vec{r} = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$, $a \leq t \leq b$ where $x(t)$, $y(t)$, $z(t)$ are real functions of a real variable t . These functions are also called components of the vector valued function $\vec{r}(t)$.

Let \vec{r}_0 be the position vector of a point P. Then the equation of a line passing through the point P and parallel to the vector \vec{v} is given by $\vec{r} = \vec{r}_0 + t\vec{v}$ where t is a real number. If \vec{r}_0 and \vec{r}_1 are position vectors of two points P and Q, then the vector equation of the line through P and Q is given by

$\vec{q} = \vec{q}_0 + t(\vec{q}_1 - \vec{q}_0)$ where t is a real parameter.

Q1) Find the vector equation of the circular helix whose parametric equation is $x = a \cos t$ $y = a \sin t$ $z = ct$

Ans: Vector equation $\vec{q}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$
 $= a \cos t \mathbf{i} + a \sin t \mathbf{j} + ct \mathbf{k}$

Q2) Express the curve of intersection of the surfaces $y = x^2$ and $z = x^3$ in parametric form and vector form.

Ans: we choose $x = t$ as a parameter

The parametric equation of the curve of intersection of the two surfaces is $x = t$, $y = t^2$, $z = t^3$.

The vector equation of the above curve is

$$\begin{aligned}\vec{q}(t) &= x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \\ &= t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k} \text{ where } t \text{ is a parameter.}\end{aligned}$$

Q3) Find the cartesian equation of a curve whose vector equation is

$$\vec{q}(t) = 2t\mathbf{i} + \frac{2}{1+t^2}\mathbf{j}$$

Ans: Comparing with $\vec{q}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$

$$x(t) = 2t \quad y(t) = \frac{2}{1+t^2}$$

eliminating t between x only we get the cartesian equation

$$\mathbf{j} = \frac{2}{1 + \frac{x^2}{4}}$$

$$\mathbf{y} = \frac{8}{4+x^2} //$$

Limit and continuity

Let $\vec{r}(t)$ be a vector valued function defined in a neighbourhood of a point $t=a$, except that $\vec{r}(t)$ need not be defined at a . A vector \vec{L} is said to be the limit of the vector valued function $\vec{r}(t)$ if

$$\lim_{t \rightarrow a} |\vec{r}(t) - \vec{L}| = 0 \quad \text{and in this case we}$$

$$\text{denote } \lim_{t \rightarrow a} \vec{r}(t) = \vec{L}.$$

If $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ then

$$\begin{aligned}\lim_{t \rightarrow a} \vec{r}(t) &= \left\langle \lim_{t \rightarrow a} x(t), \lim_{t \rightarrow a} y(t), \lim_{t \rightarrow a} z(t) \right\rangle \\ &= \lim_{t \rightarrow a} x(t)\mathbf{i} + \lim_{t \rightarrow a} y(t)\mathbf{j} + \lim_{t \rightarrow a} z(t)\mathbf{k}\end{aligned}$$

(Q) Find $\lim_{t \rightarrow 0} \vec{r}(t)$ where $\vec{r}(t) = e^t\mathbf{i} + \sin 2t\mathbf{j} + 3 \cos \pi t \mathbf{k}$

Ans $\lim_{t \rightarrow 0} \vec{r}(t) = \lim_{t \rightarrow 0} e^t\mathbf{i} + \lim_{t \rightarrow 0} \sin 2t\mathbf{j} + \lim_{t \rightarrow 0} 3 \cos \pi t \mathbf{k}$
 $= \mathbf{i} + 3\mathbf{k}$

A vector valued function $\vec{r}(t)$ defined at all points in some neighbourhood of a point $t=a$ is said to be continuous at $t=a$ if $\lim_{t \rightarrow a} \vec{r}(t) = \vec{r}(a)$. The vector valued function $\vec{r}(t)$ is said to be continuous in an interval I if the vector $\vec{r}(t)$ is continuous at each point $t \in I$.

A vector valued function $\vec{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ at $t=a$ is continuous if and only if the component functions $x(t)$, $y(t)$ and $z(t)$ are continuous at $t=a$.

Derivative of vector valued functions.

A vector function $\vec{q}'(t)$ is said to be the derivative of the vector function $\vec{q}(t)$ if

$$\vec{q}'(t) = \lim_{h \rightarrow 0} \frac{\vec{q}(t+h) - \vec{q}(t)}{h} \quad \text{Provided the limit}$$

exists and is non-zero.

The vector function $\vec{q}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ is differentiable at t if and only if the component functions $x(t), y(t), z(t)$ are differentiable at t . The derivative of $\vec{q}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ is given by $\frac{d}{dt}(\vec{q}(t)) = \frac{d}{dt}x(t)\mathbf{i} + \frac{d}{dt}y(t)\mathbf{j} + \frac{d}{dt}z(t)\mathbf{k}$

- (Q5) Show that the space curve given by $\vec{q}(t) = 2t^2\mathbf{i} + (3t^2+1)\mathbf{j} + 5\sin t\mathbf{k}$ is differentiable curve.

Ans: Comparing with $\vec{q}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$

$$x(t) = 2t^2, \quad y(t) = 3t^2+1, \quad z(t) = 5\sin t$$

We know that the vector valued function is differentiable iff its component functions are differentiable. Note that the functions $x(t) = 2t^2, y(t) = 3t^2+1, z(t) = 5\sin t$ are differentiable functions of real variable t . Hence the given vector function is differentiable.

- (Q6) Find the derivative of $\vec{q}(t) = (1+t^3)\mathbf{i} + te^{-t}\mathbf{j} + \sin 2t\mathbf{k}$

$$\begin{aligned} \frac{d}{dt}\vec{q}(t) &= 3t^2\mathbf{i} + \left[t\bar{e}^{-t} + \int \bar{e}^{-t} dt \right] \mathbf{j} + 2\cos 2t \mathbf{k} \\ &= 3t^2\mathbf{i} + [-t\bar{e}^{-t} - \bar{e}^{-t}] \mathbf{j} + 2\cos 2t \mathbf{k} \end{aligned}$$

Q7) Find the derivative of $\vec{r}(t) = t^2 \mathbf{i} + e^t \mathbf{j} - (2\omega s \pi t) \mathbf{k}$

$$\frac{d\vec{r}}{dt} = 2t \mathbf{i} + e^t \mathbf{j} + (2\pi s \omega t) \mathbf{k}$$

Rules of Differentiation

1) $\frac{d}{dt}(c) = 0$

2) $\frac{d}{dt}(k \vec{r}(t)) = k \frac{d}{dt} \vec{r}(t)$

3) $\frac{d}{dt} [\vec{r}_1(t) + \vec{r}_2(t)] = \frac{d}{dt} \vec{r}_1(t) + \frac{d}{dt} \vec{r}_2(t)$

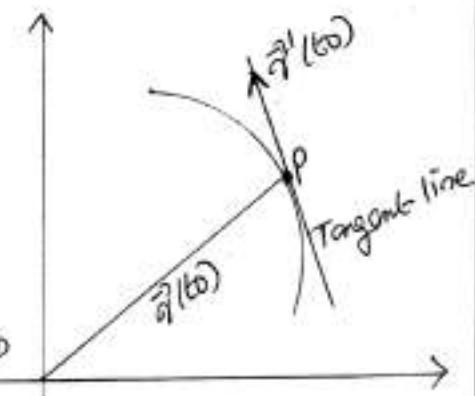
4) $\frac{d}{dt} [\vec{r}_1(t) - \vec{r}_2(t)] = \frac{d}{dt} \vec{r}_1(t) - \frac{d}{dt} \vec{r}_2(t)$

5) $\frac{d}{dt} [f(t) \vec{r}(t)] = f(t) \frac{d}{dt} \vec{r}(t) + \frac{d}{dt} [f(t)] \vec{r}(t)$

Tangent lines to graph of vector-valued functions.

Let P be a point on the graph of a vector valued function $\vec{r}(t)$, and let $\vec{r}(t_0)$ be the radius vector from the origin to P .

$\vec{r}'(t_0)$ a tangent vector to the graph of $\vec{r}(t)$ at $\vec{r}(t_0)$ and we call the line through P that is parallel to the tangent vector the tangent line to the graph of $\vec{r}(t)$ at $\vec{r}(t_0)$.



Let $\vec{r}_0 = \vec{r}(t_0)$ and $\vec{v}_0 = \vec{r}'(t_0)$. It follows that the tangent line to the graph of $\vec{r}(t)$ at \vec{r}_0 is given by the vector equation

$$\vec{r} = \vec{r}_0 + t \vec{v}_0$$

Q8) Find the equation of tangent to the curve $\vec{r} = t^2 \mathbf{i} - 2 \cos t \mathbf{k}$ at the point $t = \frac{1}{2}$ on the curve.

Ans: $\vec{r}(t) = t^2 \mathbf{i} - 2\cos \pi t \mathbf{k}$

$$\vec{r}_0 = \vec{r}\left(\frac{1}{2}\right) = \frac{1}{4} \mathbf{i} + 0 \mathbf{k}$$

$$\vec{r}'(t) = 2t \mathbf{i} + 2\pi \sin \pi t \mathbf{k}$$

$$\vec{r}'_0 = \vec{r}'\left(\frac{1}{2}\right) = 1 \mathbf{i} + 2\pi \mathbf{k}$$

equation of the tangent line $\vec{r}(t) = \vec{r}_0 + t \vec{r}'_0$

$$\vec{r}(t) = \frac{1}{4} \mathbf{i} + t(1 \mathbf{i} + 2\pi \mathbf{k})$$

parametric form

$$x = \frac{1}{4} + t \quad y = 2\pi t$$

$=$

Q9) Find the parametric equation of the tangent to the curve

$$\vec{r}(t) = t^2 \mathbf{i} + 2t^3 \mathbf{j} + 3t \mathbf{k} \text{ at } t=1$$

$$\vec{r}_0 = \vec{r}(1) = 1 \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$$

$$\vec{r}'(t) = 2t \mathbf{i} + 6t^2 \mathbf{j} + 3 \mathbf{k}$$

$$\vec{r}'_0 = \vec{r}'(1) = 2 \mathbf{i} + 6\mathbf{j} + 3\mathbf{k}$$

equation of the tangent line is $\vec{r}(t) = \vec{r}_0 + t \vec{r}'_0$

$$(e) \vec{r}(t) = (1 \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + t(2 \mathbf{i} + 6\mathbf{j} + 3\mathbf{k})$$

$$\vec{r}(t) = (1+2t) \mathbf{i} + (2+6t) \mathbf{j} + (3+3t) \mathbf{k}$$

parametric equation is $x(t) = 1+2t \quad y = 2+6t \quad z = 3+3t$

Cartesian form is

$$\frac{x-1}{2} = \frac{y-2}{6} = \frac{z-3}{3}$$

Q10) Find the parametric equations of the tangent line to the circular helix: $x = \cos t \quad y = \sin t \quad z = t$ where $t=0$ and use that result to find parametric equations for the tangent line at the point where $t=\pi$.

Ans Vector equation of the helix is -

$$\vec{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + tk \mathbf{k}$$

$$\vec{r}_0 = \cos b_0 \mathbf{i} + \sin b_0 \mathbf{j} + b_0 k \mathbf{k}$$

$$\vec{r}'(t) = -\sin t \mathbf{i} + \cos t \mathbf{j} + k \mathbf{k}$$

$$\vec{v}_0 = \vec{r}'(b_0) = -\sin b_0 \mathbf{i} + \cos b_0 \mathbf{j} + k \mathbf{k}$$

$$\vec{r}(t) = \vec{r}_0 + t \vec{v}_0$$

$$= (\cos b_0 \mathbf{i} + \sin b_0 \mathbf{j} + b_0 k) + t[-\sin b_0 \mathbf{i} + \cos b_0 \mathbf{j} + k]$$

$$= (\cos b_0 - t \sin b_0) \mathbf{i} + (\sin b_0 + t \cos b_0) \mathbf{j} + (b_0 + t) \mathbf{k}$$

parametric equations of the tangent line at $t = b_0$ are

$$x = \cos b_0 - t \sin b_0 \quad y = \sin b_0 + t \cos b_0 \quad z = b_0 + t$$

parametric equation of the tangent line at $t = \pi$ is

$$x = \cos \pi - t \sin \pi, \quad y = \sin \pi - t \cos \pi \quad z = \pi + t$$

$$\text{i.e.) } x = -t \quad y = 1 \quad z = \pi + t$$

Q(ii) Let $\vec{q}_1(t) = (\cos t) \mathbf{i} + (\sin t) \mathbf{j} + t^2 \mathbf{k}$

$$\vec{q}_2(t) = (t^2 - t) \mathbf{i} + (2t - 2) \mathbf{j} + (1nt) \mathbf{k}$$

The graph of $\vec{q}_1(t)$ and $\vec{q}_2(t)$ intersect at the origin. Find the degree measure of the acute angle between the tangent lines to the graphs of $\vec{q}_1(t)$ and $\vec{q}_2(t)$ at the origin.

Ans.

Integration of vector valued functions

To integrate a vector valued function integrate its components.

i.e. If $\vec{q}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ then

$$\int \vec{q}(t) dt = \int x(t) dt \mathbf{i} + \int y(t) dt \mathbf{j} + \int z(t) dt \mathbf{k} + \mathbf{c}$$

Q15) Compute $\int_0^3 (i^0 3t^2 - j 4t^3 + k 8t) dt$

$$= [t^3 i^0 - t^4 j + 4t^2 k]_0^3$$

$$= [8i^0 + -16j + 16k]$$

Q16) Evaluate $\int_0^3 |it + j2t^2| dt$

$$= \int_0^3 \sqrt{t^2 + 4t^4} dt$$

$$= \int_0^{17} t \sqrt{1+4t^2} dt \quad u = 1+4t^2$$

$$= \int_{\mathbf{1}}^{17} \sqrt{u} \frac{du}{8} \quad \frac{du}{dt} = 8t \quad \frac{du}{8} = t dt$$

$$= \frac{1}{8} \times \frac{2}{3} [u^{3/2}]_{\mathbf{1}}^{17} \quad t=0 \Rightarrow u=1$$

$$= \frac{1}{8} \times \frac{2}{3} [17^{3/2} - 1] \quad t=2 \Rightarrow u=17$$

$$= \frac{(17)^{3/2} - 1}{12} //$$

Q17) If $\frac{d\vec{q}}{dt} = i^0 2t - j 4t^3$ and $\vec{q}(1) = 2i^0 + 4j$ find $\vec{q}(t)$

Ans: $\vec{q}(t) = \int \left(\frac{d\vec{q}}{dt} \right) dt = \int (2t\mathbf{i}^0 - 4t^3\mathbf{j}) dt$

$$\vec{q}(t) = t^2 i^0 - t^4 j + \mathbf{c}$$

$$\vec{q}(1) = i^0 - j + \mathbf{c}$$

given $2i^0 + 4j = i^0 - j + \mathbf{c} \quad \therefore \mathbf{c} = 2i^0 + 4j - i^0 + j = i^0 + 5j$

$$\begin{aligned}\therefore \vec{r}(t) &= t^2 \mathbf{i}^0 - t^4 \mathbf{j}^0 + t^0 + 5\mathbf{j}^0 \\ &= (t^2 + 1)\mathbf{i}^0 + (5 - t^4)\mathbf{j}^0 //\end{aligned}$$

Motion along a curve - velocity and acceleration

Let $\vec{r}(t)$ be the vector valued function describing position vector of a moving particle then the instantaneous velocity and acceleration are given by

$$\vec{v}(t) = \frac{d}{dt} \vec{r}(t)$$

$$\vec{a}(t) = \frac{d}{dt} \vec{v}(t) = \frac{d^2}{dt^2} \vec{r}(t)$$

Speed of the particle is given by $|\vec{v}(t)| = \left| \frac{d}{dt} \vec{r}(t) \right|$

We can compute velocity from acceleration and displacement from velocity by integration

$$\vec{r}(t) = \int \vec{v}(t) dt + c_1 \quad \text{and} \quad \vec{v}(t) = \int \vec{a}(t) dt + c_2$$

The displacement of the particle in the time interval $t_1 \leq t \leq t_2$ is given by $\int_{t_1}^{t_2} \vec{v}(t) dt = \vec{r}(t_2) - \vec{r}(t_1)$

The distance travelled in the interval $t_1 \leq t \leq t_2$ is given by $\int_{t_1}^{t_2} |\vec{v}(t)| dt$.

- Q18) A particle moves along a path $x=t$, $y=t^2$, $z=t^3$. Find its instantaneous velocity and acceleration at time t .

Ans:

$$\vec{r}(t) = t^1 \mathbf{i}^0 + t^2 \mathbf{j}^0 + t^3 \mathbf{k}^0$$

$$\vec{v}(t) = \frac{d}{dt} \vec{r}(t) = \mathbf{i}^0 + 2t\mathbf{j}^0 + 3t^2\mathbf{k}^0$$

$$\vec{a}(t) = \frac{d}{dt} \vec{v}(t) = 2\mathbf{j}^0 + 6t\mathbf{k}^0$$

Q19) Find the velocity at the time $t=\pi$ of a particle moving along the curve $\vec{r}(t) = e^t \sin t \mathbf{i} + e^t \cos t \mathbf{j} + t \mathbf{k}$

Ans: $\vec{v}(t) = \frac{d}{dt} \vec{r}(t)$
 $= (e^t \cos t + \sin t e^t) \mathbf{i} + (-e^t \sin t + \cos t e^t) \mathbf{j} + \mathbf{k}$
 $\vec{v}(t) \text{ at } t=\pi, \vec{v}(t) = -e^\pi \mathbf{i} + -e^\pi \mathbf{j} + \mathbf{k}$

Q20) A particle moves in three dimensional space with velocity $\vec{v}(t) = \mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$. where t is the time variable. Find the coordinates of the particle when $t=1$ given that the particle is at $(-1, 2, 4)$ when $t=0$.

Ans: position vector of the particle is given by

$$\begin{aligned}\vec{r}(t) &= \int \vec{v}(t) dt \\ &= \int (\mathbf{i} + t\mathbf{j} + t^2\mathbf{k}) dt \\ &= t\mathbf{i} + \frac{t^2}{2}\mathbf{j} + \frac{t^3}{3}\mathbf{k} + C\end{aligned}$$

$$\vec{r}(0) = C$$

$$\text{given that } \vec{r}(0) = -\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$$

$$\therefore C = -\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$$

$$\therefore \vec{r}(t) = \left(t\mathbf{i} + \frac{t^2}{2}\mathbf{j} + \frac{t^3}{3}\mathbf{k} \right) + (-\mathbf{i} + 2\mathbf{j} + 4\mathbf{k})$$

$$\vec{r}(t) = (t-1)\mathbf{i} + \left(\frac{t^2}{2} + 2\right)\mathbf{j} + \left(\frac{t^3}{3} + 4\right)\mathbf{k}$$

$$\vec{r}(1) = 0\mathbf{i} + \frac{5}{2}\mathbf{j} + \frac{13}{3}\mathbf{k}$$

\therefore The particle is at the point $(0, \frac{5}{2}, \frac{13}{3})$ when $t=1$

Q21) A particle moves along a path with vector equation

$\vec{r}(t) = \mathbf{i} \cos(\pi t) + \mathbf{j} \sin(\pi t) + kt$. Find the distance travelled and the displacement of the particle during the time interval $1 \leq t \leq 5$.

$$\text{Ans: } \vec{\gamma}(t) = i^0 \cos(\pi t) + j^0 \sin(\pi t) + kt$$

The displacement of the particle is $= \vec{\gamma}(5) - \vec{\gamma}(1)$

$$= [i^0 \cos(5\pi) + j^0 \sin(5\pi) + 5k] - [i^0 \cos\pi + j^0 \sin\pi + k]$$

$$= [-i^0 + 0j + 5k] - [-i^0 + 0j + k]$$

$$= \underline{\underline{4k}}$$

Distance travelled is given by $\int_{t_1}^{t_2} |\vec{v}(t)| dt$

$$\int_{t_1}^{t_2} |-i^0 \pi \sin(\pi t) + j\pi \cos(\pi t) + k| dt$$

$$\int_1^5 \sqrt{\pi^2 \sin^2(\pi t) + \pi^2 \cos^2(\pi t) + 1} dt$$

$$\int_1^5 \sqrt{\pi^2 + 1} dt$$

$$\sqrt{\pi^2 + 1} [t]_1^5 = \underline{\underline{4\sqrt{\pi^2 + 1}}}$$

Scalar field and vector field.

Let R be a region of space at each point of which a scalar $\phi = \phi(x, y, z)$ is given, then ϕ is called scalar function and R is called scalar field.

Let R be a region of space at each point of which a vector $\vec{v} = \vec{v}(x, y, z)$ is given then \vec{v} is called a vector point function and R is called a vector field.

Gradient of a scalar field.

The Gradient of the scalar field $\phi(x, y, z)$ is denoted by $\nabla\phi$ or $\text{grad } \phi$ and is defined as

$$\nabla \phi = \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \phi$$

$$\text{grad } \phi = \nabla \phi = i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z}$$

The hamilton operator or del operator is $\nabla = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}$

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Directional Derivative

The directional derivative of ϕ in the direction of unit vector \vec{u} at the point (x_0, y_0, z_0) is denoted by $D_{\vec{u}} \phi(x_0, y_0, z_0)$ and is defined as $D_{\vec{u}} \phi(x_0, y_0, z_0) = \nabla \phi \cdot \vec{u}$

Q2) Find the gradient of the following.

(i) $z = 4x - 10y$

$$\begin{aligned}\nabla z &= \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) (4x - 10y) \\ &= \underline{\underline{i}4 - 10j}\end{aligned}$$

(ii) $z = e^{-3y} \cos 5x$

$$\begin{aligned}\nabla z &= \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} \right) (e^{-3y} \cos 5x) \\ &= i \frac{\partial}{\partial x} e^{-3y} \cos 5x + j \frac{\partial}{\partial y} e^{-3y} \cos 5x \\ &= i e^{-3y} (-3 \sin 5x) 5 + j e^{-3y} (-3) \cos 5x \\ &= \underline{\underline{-5e^{-3y} \sin 5x i - 3e^{-3y} \cos 5x j}}\end{aligned}$$

(iii) $f(x, y) = (x^2 + xy)^3$ at the point $(-2, -1)$

$$\nabla f = \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} \right) (x^2 + xy)^3$$

$$= i 3(x^2 + xy)^2 (2x + y) + j 3(x^2 + xy)^2 \cdot x$$

$$\nabla f(-2, -1) = -540i - 216j$$

$$(N) f(x,y,z) = xy^2z^3 \text{ at } (1,1,1)$$

$$\begin{aligned}\nabla f &= \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) (xy^2z^3) \\ &= i \frac{\partial}{\partial x} (xy^2z^3) + j \frac{\partial}{\partial y} (xy^2z^3) + k \frac{\partial}{\partial z} (xy^2z^3) \\ &= y^2z^3 i + 2xyz^3 j + 3xy^2z^2 k\end{aligned}$$

$$\nabla f_{(1,1,1)} = i + 2j + 3k$$

Q23) Find the directional derivative of $f(x,y) = xe^y$ at $(1,1)$ in the direction of the vector $i - j$

Ans: D.D of f in the direction of the unit vector \vec{u} is $(\nabla f) \cdot \vec{u}$

$$\begin{aligned}\nabla f &= \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) xe^y \\ &= i \frac{\partial}{\partial x} xe^y + j \frac{\partial}{\partial y} xe^y + k \frac{\partial}{\partial z} xe^y \\ &= i e^y + j x e^y\end{aligned}$$

$$\nabla f_{(1,1)} = i e + j e$$

$$\text{Directional derivative} = \nabla f \cdot \vec{u}$$

$$= (i e + j e) \cdot \left(\frac{i - j}{\sqrt{2}} \right)$$

$$= \frac{e}{\sqrt{2}} - \frac{e}{\sqrt{2}} = 0$$

Q24) Find the directional derivative of $f(x,y) = xe^y - ye^x$ at point $P(0,0)$ in the direction of $5P - 2j$

$$\nabla f = i \frac{\partial}{\partial x} (xe^y - ye^x) + j \frac{\partial}{\partial y} (xe^y - ye^x)$$

$$= i[e^x - ye^x] + j[ye^x - e^x]$$

$$\nabla f(0,0) = i - j$$

$$\vec{v} = 5i - 2j \quad \hat{v} = \frac{5i - 2j}{\sqrt{29}}$$

$$\begin{aligned} D \cdot D &= \nabla f \cdot \hat{v} \\ &= (i - j) \cdot \frac{(5i - 2j)}{\sqrt{29}} \end{aligned}$$

$$= \frac{5+2}{\sqrt{29}} = \frac{1}{\sqrt{29}}$$

Q) Find the directional derivative of $f(x,y) = e^{-x} \sec y$ at $P(0, \pi/4)$ in the direction of the origin

$$\begin{aligned} \nabla f &= i \frac{\partial}{\partial x} e^{-x} \sec y + j \frac{\partial}{\partial y} e^{-x} \sec y \\ &= -i e^{-x} \sec y + j e^{-x} \sec y \tan y \end{aligned}$$

$$\nabla f(0, \frac{\pi}{4}) = -i \frac{\sqrt{2}}{\sqrt{2}} - i \sqrt{2} + j \sqrt{2}$$

$$\vec{PO} = -\frac{\pi}{4} j$$

$$\hat{PO} = \frac{-\frac{\pi}{4} j}{\frac{\pi}{4}} = -j$$

$$\begin{aligned} D \cdot D &= \nabla f \cdot \hat{PO} = (-i \sqrt{2} + j \sqrt{2}) \cdot (-j) \\ &= -\sqrt{2} \end{aligned}$$

Q) Find the directional derivative of $f(x,y) = e^{2xy}$ at $P(5,0)$ in the direction of $\vec{v} = -\frac{3}{5}i + \frac{4}{5}j$

$$\begin{aligned} \nabla f &= i \frac{\partial}{\partial x} e^{2xy} + j \frac{\partial}{\partial y} e^{2xy} \\ &= i e^{2xy} [2y] + j [e^{2xy} \cdot 2x] \end{aligned}$$

$$\nabla f(5,0) = 10i + 10j \quad \hat{v} = \frac{-\frac{3}{5}i + \frac{4}{5}j}{\sqrt{1}}$$

$$\begin{aligned} \mathbf{D} \cdot \mathbf{D} &= \nabla f \cdot \hat{\mathbf{u}} \\ &= (10\mathbf{j}) \cdot \left(-\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j} \right) \\ &= 10 \times \frac{4}{5} = \underline{\underline{8}} \end{aligned}$$

Q) Let $\vec{r} = xi + yj + zk$ and $r = |\vec{r}|$ and $f(r)$ be a differentiable function of the variable r , show that

$$\nabla f(r) = \frac{f'(r)}{r} \vec{r}$$

Ans: $\vec{r} = xi + yj + zk$

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$r^2 = x^2 + y^2 + z^2$$

Diff w.r.t x $2r \frac{\partial r}{\partial x} = 2x \quad \therefore \frac{\partial r}{\partial x} = \frac{x}{r}$

likewise $\frac{\partial r}{\partial y} = \frac{y}{r} \quad \frac{\partial r}{\partial z} = \frac{z}{r}$

$$\begin{aligned} \nabla f(r) &= \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) f(r) \\ &= i \frac{\partial}{\partial x} f(r) + j \frac{\partial}{\partial y} f(r) + k \frac{\partial}{\partial z} f(r) \\ &= i f'(r) \frac{x}{r} + j f'(r) \frac{y}{r} + k f'(r) \frac{z}{r} \\ &= \underline{\underline{\frac{f'(r)}{r} \vec{r}}} \end{aligned}$$

Properties of directional derivative.

(1) If $\nabla f(x, y, z) = 0$, then directional derivative along direction at $P(x, y, z)$ vanishes.

II) If $\nabla f(x,y,z) \neq 0$ at $P(x_0,y_0,z_0)$ then the directional derivative is maximum in direction of ∇f , which is normal to the level surface $f(x,y,z) = k$ and the maximum value is $|\nabla f|$. The directional derivative is minimum in the direction of $-\nabla f$, which is normal to the level surface $f(x,y,z) = k$ and the minimum value is $-|\nabla f|$.

III) $\nabla \phi$ is the normal to the surface $\phi(x,y,z) = c$

Q) Find a ~~unit~~ unit vector normal to the surface

$$x^3 + y^3 + 3xyz = 3 \text{ at the point } (1, 2, -1)$$

$$\text{Ans: normal vector} = \nabla \phi$$

$$\begin{aligned} &= \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) (x^3 + y^3 + 3xyz) \\ &= i[3x^2 + 3yz] + j[3y^2 + 3xz] + k[3xy] \end{aligned}$$

$$\begin{aligned} \nabla \phi|_{(1,2,-1)} &= i[3 + -6] + j[12 - 3] + k[6] \\ &= -3i + 9j + 6k \end{aligned}$$

$$\begin{aligned} \text{unit normal vector} &= \frac{\nabla \phi}{|\nabla \phi|} = \frac{-3i + 9j + 6k}{\sqrt{126}} \\ &= \frac{-i + 3j + 2k}{\sqrt{14}} \end{aligned}$$

$$Q) \text{ Show that } \nabla(\gamma^3) = -3\gamma^{-5}\vec{\gamma}$$

$$\begin{aligned} \nabla(\gamma^3) &= i \frac{\partial}{\partial x} \gamma^3 + j \frac{\partial}{\partial y} \gamma^3 + k \frac{\partial}{\partial z} \gamma^3 \\ &= i(-3\gamma^{-4}) \frac{\partial \gamma}{\partial x} + j(-3\gamma^{-4}) \frac{\partial \gamma}{\partial y} + k(-3\gamma^{-4}) \frac{\partial \gamma}{\partial z} \\ &= -3\gamma^{-4} \left(\frac{x}{\gamma} i + \frac{y}{\gamma} j + \frac{z}{\gamma} k \right) \\ &= -3\gamma^{-4} \frac{\vec{\gamma}}{\gamma} = -3\gamma^{-5} \underline{\vec{\gamma}} \end{aligned}$$

$$Q) \text{ Show that } \nabla \log \gamma = \frac{1}{\gamma^2} \vec{\gamma}$$

$$\begin{aligned} \nabla \log \gamma &= i \frac{\partial}{\partial x} \log \gamma + j \frac{\partial}{\partial y} \log \gamma + k \frac{\partial}{\partial z} \log \gamma \\ &= i \frac{1}{\gamma} \frac{x}{\gamma} + j \frac{1}{\gamma} \frac{y}{\gamma} + k \frac{1}{\gamma} \frac{z}{\gamma} \\ &= \frac{\vec{\gamma}}{\gamma^2} \end{aligned}$$

$$Q) \text{ Show that } \nabla \gamma^n = n \gamma^{n-2} \vec{\gamma}$$

$$\begin{aligned} \nabla \gamma^n &= i \frac{\partial \gamma^n}{\partial x} + j \frac{\partial \gamma^n}{\partial y} + k \frac{\partial \gamma^n}{\partial z} \\ &= n \gamma^{n-1} \left(\frac{x}{\gamma} i + \frac{y}{\gamma} j + \frac{z}{\gamma} k \right) \\ &= n \gamma^{n-2} \vec{\gamma} \end{aligned}$$

$$Q) \text{ Show that } \nabla(\vec{\alpha} \cdot \vec{\gamma}) = \vec{\alpha} \text{ where } \vec{\alpha} \text{ is a constant vector.}$$

$$\text{Let } \vec{\alpha} = a_1 i + a_2 j + a_3 k$$

$$\vec{\gamma} = xi + yj + zk$$

$$\vec{\alpha} \cdot \vec{\gamma} = a_1 x + a_2 y + a_3 z$$

$$\nabla(\vec{a} \cdot \vec{r}) = \frac{\partial}{\partial x}(a_1x + a_2y + a_3z) + i \frac{\partial}{\partial y}(a_1x + a_2y + a_3z) + k \frac{\partial}{\partial z}(a_1x + a_2y + a_3z)$$

$$= ia_1 + ja_2 + ka_3 = \underline{\underline{\vec{a}}}$$

Q) Show that $\nabla^2(r^n) = n(n+1)r^{n-2}$ where $r = |\vec{r}|$

$$\nabla^2(r^n) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) r^n$$

$$\frac{\partial^2}{\partial x^2}(r^n) = \frac{\partial}{\partial x}\left(nr^{n-1}\frac{x}{r}\right)$$

$$= \frac{\partial}{\partial x}\left(nr^{n-2}x\right)$$

$$= n\left[r^{n-2} + x(n-2)r^{n-3}\frac{x}{r}\right]$$

$$= n\left[r^{n-2} + (n-2)r^{n-4}x^2\right]$$

$$\text{likewise } \frac{\partial^2}{\partial y^2}(r^n) = n\left[r^{n-2} + (n-2)r^{n-4}y^2\right]$$

$$\frac{\partial^2}{\partial z^2}(r^n) = n\left[r^{n-2} + (n-2)r^{n-4}z^2\right]$$

$$\frac{\partial^2 r^n}{\partial x^2} + \frac{\partial^2 r^n}{\partial y^2} + \frac{\partial^2 r^n}{\partial z^2} = n\left[3r^{n-2} + (n-2)r^{n-4}r^2\right]$$

$$= n\left[3r^{n-2} + (n-2)r^{n-2}\right]$$

$$= nr^{n-2}[3+n-2]$$

$$= n(n+1)\underline{\underline{r^{n-2}}}$$

Q) Show that $\nabla^2 f(r) = \frac{2d'(r)}{r} + d''(r)$

$$\nabla^2 f(r) = \frac{\partial^2}{\partial x^2}f(r) + \frac{\partial^2}{\partial y^2}f(r) + \frac{\partial^2}{\partial z^2}f(r)$$

$$\frac{\partial^2}{\partial x^2}f(r) = \frac{\partial}{\partial x}\left[f'(r) \cdot \frac{x}{r}\right]$$

$$\begin{aligned}
 &= \frac{\partial}{\partial x} \left[\frac{d^1(r)x}{r} \right] \\
 &= \frac{r \frac{\partial}{\partial x} [d^1(r)x] - d^1(r)x \frac{\partial r}{\partial x}}{r^2} \\
 &= \frac{r \left[d^1(r) + x d''(r) \frac{x}{r} \right] - d^1(r) \frac{x^2}{r}}{r^2} \\
 &= \frac{d^1(r)}{r} + \frac{x^2 d''(r)}{r^2} - \frac{d^1(r)x^2}{r^3}
 \end{aligned}$$

$$III_{xy} \frac{\partial^2 f(r)}{\partial y^2} = \frac{d^1(r)}{r} + \frac{y^2 d''(r)}{r^2} - \frac{d^1(r)y^2}{r^3}$$

$$\frac{\partial^2 f(r)}{\partial y^2} = \frac{d^1(r)}{r} + \frac{y^2 d''(r)}{r^2} - \frac{d^1(r)y^2}{r^3}$$

$$\frac{\partial^2 f(r)}{\partial x^2} + \frac{\partial^2 f(r)}{\partial y^2} + \frac{\partial^2 f(r)}{\partial z^2} = \frac{3f^1(r)}{r} + \frac{f''(r)r^2}{r^2} - \frac{f^1(r)r^2}{r^3}$$

$$= \frac{3f^1(r)}{r} - \frac{f^1(r)}{r} + \frac{f''(r)}{r}$$

$$= \frac{2f^1(r)}{r} + \frac{f''(r)}{r}$$

Q) Find the directional derivative of the function $f = x^2 - y^2 + 2z^2$
 35 at the point P(1, 2, 3) in the direction of the line PQ
 where Q is the point (5, 0, 4)

$$\nabla f = \left(\frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) (x^2 - y^2 + 2z^2)$$

$$= 2x^1 i - 2y^1 j + 4z^1 k \quad \nabla f(1, 2, 3) = 2i^0 - 4j^1 + 12k$$

$$\overrightarrow{PQ} = (5i + 0j + 4k) - (1i^0 + 2j^1 + 3k) = 4i^0 - 2j^1 + k$$

$$\begin{aligned}
 D \cdot D &= \nabla f \cdot \hat{PQ} \\
 &= (2i - 4j + 12k) \cdot \frac{(4i - 2j + 10k)}{\sqrt{21}} \\
 &= \underline{\underline{\frac{28}{\sqrt{21}}}}
 \end{aligned}$$

Q) Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $g = x^2 + y^2 - 3$ at the point $(2, -1, 2)$

Ans: Angle between two surfaces at a point is the angle between the normals to the surfaces at that point.

$$\text{Let } \phi_1 = x^2 + y^2 + z^2 = 9 \text{ and } \phi_2 = x^2 + y^2 - 3 = 0$$

$$\nabla \phi_1 = 2xi + 2yj + 2zk \quad \nabla \phi_2 = 2xi + 2yj - zk$$

$$\vec{n}_1 = \nabla \phi_1|_{(2, -1, 2)} = 4i - 2j + 4k$$

$$\vec{n}_2 = \nabla \phi_2|_{(2, -1, 2)} = 4i - 2j - k$$

$$\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

Q) Find a unit vector in the direction in which f increases most rapidly at P and find the rate of change of f at P in that direction. $f(x, y) = 4x^3y$ at $P(-1, 1)$

Ans: $\nabla f = i 12x^2y + j 4x^3$

$$\nabla f|_{P(-1, 1)} = 12i + 4j$$

unit vector in the direction in which f increases most rapidly

$$\text{at } P = \frac{\nabla f}{\|\nabla f\|} = \frac{12i + 4j}{\sqrt{160}} = \frac{3i + j}{\sqrt{10}}$$

$$\text{Rate of change of } f \text{ at } P = |\nabla f| = \underline{\underline{4\sqrt{10}}}$$

- Q. 38) Find a unit vector in the direction in which f decreases most rapidly at P and find the rate of change of f at P in that direction. $f(x,y) = 20-x^2-y^2$ at $P(-1,3)$

Ans: $\nabla f = -2x\mathbf{i} - 2y\mathbf{j}$

$$\nabla f)_{P(-1,3)} = 2\mathbf{i} - 6\mathbf{j}$$

unit vector in the direction in which f decreases most rapidly

$$= -\frac{\nabla f}{|\nabla f|} = -\frac{(2\mathbf{i} - 6\mathbf{j})}{\sqrt{40}} = \frac{-\mathbf{i} + 3\mathbf{j}}{\sqrt{10}}$$

$$\text{Rate of change of } f \text{ at } P = -|\nabla f| = -\underline{\underline{\sqrt{10}}}$$

Divergence of a vector point function

The divergence of a differentiable vector point function \vec{F} is denoted by $\text{div } \vec{F}$ and is defined as

$$\begin{aligned}\text{div } \vec{F} &= \nabla \cdot \vec{F} \\ &= \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \cdot \vec{F} \\ &= i \cdot \frac{\partial \vec{F}}{\partial x} + j \cdot \frac{\partial \vec{F}}{\partial y} + k \cdot \frac{\partial \vec{F}}{\partial z}\end{aligned}$$

The divergence of a vector point function is a scalar point function.

If $\vec{F} = F_1 i + F_2 j + F_3 k$ then

$$\begin{aligned}\text{div } \vec{F} &= \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \cdot (F_1 i + F_2 j + F_3 k) \\ &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\end{aligned}$$

Curl of a vector point function

The curl or rotation of a differentiable vector point function is denoted by $\text{curl } \vec{F}$ and is defined as

$$\begin{aligned}\text{curl } \vec{F} &= \nabla \times \vec{F} \\ &= \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \times \vec{F} \\ &= i \times \frac{\partial \vec{F}}{\partial x} + j \times \frac{\partial \vec{F}}{\partial y} + k \times \frac{\partial \vec{F}}{\partial z}\end{aligned}$$

Curl of a vector point function is a vector point function.

If $\vec{F} = F_1 i + F_2 j + F_3 k$

$$\begin{aligned}\text{curl } \vec{F} &= \nabla \times \vec{F} \\ &= \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \times (F_1 i + F_2 j + F_3 k)\end{aligned}$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Q) Compute divergence and curl of the vector field

A1) $\vec{F} = xy^2i + 5xy^2zj - 3x^2z^3k$

Ans: $\operatorname{div} \vec{F} = \nabla \cdot \vec{F}$

$$\begin{aligned} &= \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \cdot (xy^2i + 5xy^2zj - 3x^2z^3k) \\ &= \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(5xy^2z) + \frac{\partial}{\partial z}(-3x^2z^3) \\ &= y^2 + 10xyz - 9x^2z^2 \end{aligned}$$

$\operatorname{curl} \vec{F} = \nabla \times \vec{F}$

$$\begin{aligned} &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 5xy^2z & -3x^2z^3 \end{vmatrix} \\ &= i(0 - 5y^2) - j(-6xz^3 - 2xz) + k(5y^2z - 0) \\ &= -5xy^2i + (6xz^3 + 2xz)j + 5y^2z\underline{k} \end{aligned}$$

Q2) Find $\operatorname{div} \vec{F}$ and $\operatorname{curl} \vec{F}$ of $\vec{F} = x^2y^2i + 2y^3z^2j + 3z^2k$

Ans: $\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(x^2y^2) + \frac{\partial}{\partial y}(2y^3z^2) + \frac{\partial}{\partial z}(3z^2)$

$$= 2xy^2 + 6y^2z + 3\underline{k}$$

$$\begin{aligned} \nabla \times \vec{F} &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y^2 & 2y^3z^2 & 3z^2 \end{vmatrix} = i(0 - 2y^3) - j(0 - 0) \\ &\quad + k(0 - x^2) \\ &= -2y^3i - x^2\underline{k} \end{aligned}$$

$$= i \left(-\frac{1}{r^2} \frac{x}{r} \right) + j \left(-\frac{1}{r^2} \frac{y}{r} \right) + k \left(-\frac{1}{r^2} \frac{z}{r} \right)$$

$$= - \frac{\vec{r}}{r^3}$$

$$\operatorname{div} \operatorname{grad} \left(\frac{1}{r} \right) = \operatorname{div} \left(-\frac{\vec{r}}{r^3} \right) = 0 \quad \text{From above prob.}$$

Line integrals

Any integral which is to be evaluated along a curve is called a line integral.

Q8 Consider the line integrals $\int_C (x+y) dx$ and $\int_C (x+y) dy$ where C is the curve represented by $x = 2t$, $y = 3t^2$, $0 \leq t \leq 1$.

$$\text{Ans. } \int_C (x+y) dx = \int_0^1 (2t + 3t^2) 2 dt \quad x = 2t \\ \frac{dx}{dt} = 2$$

$$= 2 \left[t^2 + t^3 \right]_0^1 = 4$$

$$\int_C (x+y) dy = \int_0^1 (2t + 3t^2) 6t dt \quad y = 3t^2 \\ \frac{dy}{dt} = 6t$$

$$= \int_0^1 (12t^2 + 18t^3) dt$$

$$= \left[4t^3 + \frac{9t^4}{2} \right]_0^1$$

$$= 4 + \frac{9}{2} = \underline{\underline{\frac{17}{2}}}$$

Potential of vector field along a curve

Let \vec{F} be a continuous vector function defined at each point in a region in space and let C be a smooth curve in this region. The line integral of the vector field $\vec{F} = f_1 i + f_2 j + f_3 k$ along C is defined as

$$\int_C \vec{F} \cdot d\vec{r} = \int_C f_1 dx + f_2 dy + f_3 dz$$

where $\vec{r} = xi + yj + zk$ and $d\vec{r} = i dx + j dy + k dz$

$$\text{or } \int_C \vec{F} \cdot d\vec{r} = \int_1^{t_2} \left(f_1 \frac{dx}{dt} + f_2 \frac{dy}{dt} + f_3 \frac{dz}{dt} \right) dt$$

Q) Evaluate the line integral $\int_C (x dx - y dy + e^y dz)$ where C is given by $x = t^3$, $y = -t$, $z = t^2$, $1 \leq t \leq 2$.

$$\begin{aligned} \text{Ans} \quad \int_C (x dx - y dy + e^y dz) &= \int_1^2 [(t^3) 3t^2 - (-t)(t^2)(-1) + e^{t^2} 2t] dt \\ &= \int_1^2 (3t^5 - t^4 + 2te^{t^2}) dt \\ &= \left[\frac{3t^6}{6} - \frac{t^5}{5} + e^{t^2} \right]_1^2 \\ &= [32 - 4 + e^4] - [\frac{1}{2} - \frac{1}{4} + e] \\ &= \underline{\underline{\frac{111}{4} + e^4 - e}} \end{aligned}$$

Q) Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \sin \pi i + \cos \pi j$ and C is the curve

$$\vec{r}(t) = \pi i + tj, \quad 0 \leq t \leq 2$$

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \int_0^2 (\sin \pi i + \cos \pi j) \cdot (0i + j) dt \\ &= \int_0^2 \cos \pi dt \\ &= -1 [t]_0^2 = \underline{\underline{-2}} \end{aligned}$$

Q) Evaluate $\int_C [-y dx + x dy]$ along $y^2 = 3x$ from $(3,3)$ to $(0,0)$.

$$\int_C [-y dx + x dy] = \int_C \vec{F} \cdot d\vec{r}$$

$$\vec{F} = -y^i + xj, \quad d\vec{r} = dx i + dy j$$

$$\vec{r} = xi + yj$$

$$y^2 = 3x \quad \text{put } x = \frac{y^2}{3} \quad \therefore \quad \underline{\underline{x = \sqrt{3}t}}$$

$$dx = \frac{2y}{3} dt$$

$$\begin{aligned} \int_C [-y dx + x dy] &= \int_0^0 -y \left(\frac{2y}{3} \right) dy + \frac{y^2}{3} dy \\ &= \int_0^0 \left(-\frac{2y^2}{3} + \frac{y^2}{3} \right) dy \\ &= \int_0^0 -\frac{y^2}{3} dy \\ &= -\left[\frac{y^3}{9} \right]_0^0 = -\frac{1}{9} [0 - 27] = 3 // \end{aligned}$$

Q2) $\int_C y^2 dx + x^2 dy$ where C is the path $y=x$ from $(0,0)$ to $(1,1)$

Ans:

$$y = x$$

$$dy = dx$$

$$\begin{aligned} \int_C y^2 dx + x^2 dy &= \int_0^1 x^2 dx + x^2 dx \\ &= 2 \int_0^1 x^2 dx \\ &= 2 \left[\frac{x^3}{3} \right]_0^1 = 2 \times \frac{1}{3} = \underline{\underline{\frac{2}{3}}} \end{aligned}$$

Q3) Evaluate $\int_C (5x+2y) dx + (2x-y) dy$ along the curve C

(i) The line segment from $(0,0)$ to $(1,1)$

(ii) The parabolic arc $y = x^2$ from $(0,0)$ to $(1,1)$

Ans:

(i) Equation of the line joining $(0,0)$ & $(1,1)$ is

$$\frac{dx-0}{1-0} = \frac{y-0}{1-0}$$

$$x = y$$

$$\therefore dx = dy$$

$$\begin{aligned}\therefore \int_0^1 (5x+2y)dx + (2x-y)dy &= \int_0^1 (5x+2x)dx + (2x-x)dx \\ &= \int_0^1 8x dx \\ &= 4[x^2]_0^1 = \underline{\underline{4}}\end{aligned}$$

$$(ii) \quad y = x^2$$

$$dy = 2x dx$$

$$\begin{aligned}\int_0^1 (5x+2y)dx + (2x-y)dy &= \int_0^1 (5x+2x^2)dx + (2x-x^2)2x dx \\ &= \int_0^1 (7x+x^2) dx \quad \int_0^1 (5x+2x^2+4x^2-2x^3) dx \\ &= \left[\frac{7x^2}{2} + \frac{x^3}{3} \right]_0^1 \quad \int_0^1 (5x+6x^2-2x^3) dx \\ &= \frac{7}{2} + \frac{1}{3} = \frac{23}{6} \quad \left[\frac{5x^2}{2} + \frac{6x^3}{3} - \frac{2x^4}{4} \right]_0^1 \\ &= \frac{5}{2} + \frac{6}{3} - \frac{1}{2} \\ &= \underline{\underline{4}}\end{aligned}$$

Φ) $\int \vec{A} \cdot d\vec{r} = (3x^2 + 6y) \hat{i} - 14yz \hat{j} + 20xz^2 \hat{k}$ evaluate $\int \vec{A} \cdot d\vec{r}$ from $(0,0,0)$ to $(1,1,1)$ along the path $x=t$, $y=t^2$, $z=t^3$

$$\begin{aligned}\text{Ans: } \int \vec{A} \cdot d\vec{r} &= \int (3x^2 + 6y) dx - 14yz dy + 20xz^2 dz \\ &= \int_0^1 (3t^2 + 6t^2) dt - 14t^2 t^3 2t dt + 20t^3 t^3 3t^2 dt \\ &= \int_0^1 [9t^2 - 28t^6 + 60t^9] dt \\ &= \left[9 \frac{t^3}{3} - \frac{28t^7}{7} + \frac{60t^{10}}{10} \right]_0^1 = [3 - 4 + 6] = \underline{\underline{5}}\end{aligned}$$

Q) Evaluate $\int_C xyg \, dx - \cos(yz) \, dy + xz \, dz$ over the
line segment from $(1, 1, 1)$ to $(-2, 1, 3)$

Ans: Equation of the line joining $(1, 1, 1)$ to $(-2, 1, 3)$ is

$$\frac{x-1}{-2-1} = \frac{y-1}{1-1} = \frac{z-1}{3-1} = t$$

$$\frac{x-1}{-3} = \frac{y-1}{0} = \frac{z-1}{2} = t$$

$$x = -3t + 1 \quad y = 1, \quad z = 2t + 1$$

$$dx = -3dt \quad dy = 0 \quad dz = 2dt$$

$$\int_C xyg \, dx - \cos(yz) \, dy + xz \, dz = \int_C (-3t+1)(2t+1) - 3dt + (-3t+1)(2t+1) 2dt$$

$$= \int_C -(-6t^2 - 3t + 2t + 1) dt + \text{C}$$

$$= \int_0^1 (-18t^2 - 9t + 6t + 3) dt \quad \int_0^1 (6t^2 + 3t - 2t - 1) dt$$

$$= \int_0^1 (-18t^2 - 3t + 3) dt$$

$$= \int_0^1 (6t^2 + t - 1) dt$$

$$= \left[-6t^3 - \frac{3t^2}{2} + 3t \right]_0^1$$

$$= \left[2t^3 + \frac{t^2}{2} - t \right]_0^1$$

$$= -6 - \frac{3}{2} + 3$$

$$= 2 + \frac{1}{2} - 1$$

$$= \underline{\underline{\frac{3}{2}}}$$

Q) Evaluate $\int \vec{A} \cdot d\vec{r}$ if $\vec{A} = (2y+3)\hat{i} + xz\hat{j} + (yz-x)\hat{k}$
along the line joining the origin and the point $(2, 1, 1)$.

Ans: Equation of the line joining $(0, 0, 0)$ and $(2, 1, 1)$ is given by

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{1} = t$$

$$\begin{array}{l} x=2t \\ dx=2dt \end{array} \quad \begin{array}{l} y=t \\ dy=dt \end{array} \quad \begin{array}{l} z=t \\ dz=dt \end{array}$$

$$\begin{aligned} \int \vec{F} \cdot d\vec{r} &= \int (2y+3)dx + xydy + (yz-x)dz \\ &= \int_0^1 (2t+3)2dt + 2t^2 dt + (t^2 - 2t) dt \\ &= \int_0^1 (4t+6+2t^2+t^2-2t) dt \\ &= \int_0^1 (3t^2+2t+6) dt \\ &= [t^3 + t^2 + 6t] \Big|_0^1 = 1+1+6 = \underline{\underline{8}} \end{aligned}$$

work done in terms of line integral

Let \vec{F} be a force applied on an object to move it along a smooth arc C then, the work done is given by $\int_C \vec{F} \cdot d\vec{r}$

- Q) Calculate the work done by $\vec{F} = i^0 - 4j^0 + xyzk$ in moving a particle from $(0,0,0)$ to $(1,-1,1)$ along the curve $x=t$, $y=-t^2$, $z=t$ for $0 \leq t \leq 1$.

$$\begin{aligned} \text{Ans: Work done} &= \int_C \vec{F} \cdot d\vec{r} \\ &= \int_C (dx - ydy + xyzdz) \\ &= \int_0^1 dt + t^2 \cdot 2t dt + -t^4 dt \\ &= \int_0^1 (1 - 2t^3 - t^4) dt \\ &= \left[t - \frac{t^4}{2} - \frac{t^5}{5} \right] \Big|_0^1 \\ &= 1 - \frac{1}{2} - \frac{1}{5} = \underline{\underline{\frac{3}{10}}} \end{aligned}$$

Irrational or Conservative vector fields

If $\operatorname{curl} \vec{F} = \vec{0}$ then the vector field is said to be ~~more~~ otherwise irrotational. The vector field is conservative if it is irrotational.

Q64) Show that the vector field $\vec{F} = 2e^{2x} \cos y \mathbf{i}^{\circ} - 2e^{2y} \sin x \mathbf{j}^{\circ}$ is irrotational.

$$\begin{aligned}\operatorname{curl} \vec{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2e^{2x} \cos y & -2e^{2y} \sin x & 0 \end{vmatrix} \\ &= \mathbf{i}[0-0] - \mathbf{j}[0-0] + \mathbf{k}[-2\sin y e^{2x} + 2e^{2x} \sin y] \\ &= 0\mathbf{i}^{\circ} - 0\mathbf{j}^{\circ} + 0\mathbf{k}^{\circ} \\ &= \vec{0}\end{aligned}$$

$\therefore \vec{F}$ is irrotational

Q65) Show that $\vec{F} = x^2 y \mathbf{i}^{\circ} + 6xy^2 \mathbf{j}^{\circ}$ is not a conservative

$$\begin{aligned}\operatorname{curl} \vec{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & 6xy^2 & 0 \end{vmatrix} \\ &= \mathbf{i}[0-0] - \mathbf{j}[0-0] + \mathbf{k}[6y^2 - x^2] \\ &= [6y^2 - x^2] \mathbf{k}^{\circ} \\ &\neq \vec{0}\end{aligned}$$

$\therefore \vec{F}$ is not a conservative field.

Q66) Show that $\vec{F} = (e^{2x} \cos y + yz) \mathbf{i}^{\circ} + (xz - e^{2y} \sin y) \mathbf{j}^{\circ} + (xy + z) \mathbf{k}^{\circ}$ is irrotational and hence find a scalar potential for it.

$$\operatorname{curl} \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (e^x \cos y + yz) & (xz - e^x \sin y) & (xy + z) \end{vmatrix}$$

$$= i [x - (x)] - j [y - y] + k [(y - e^x \sin y) - (e^x \sin y + z)] \\ = 0i - 0j + 0k$$

$\therefore \vec{F}$ is irrotational

Let ϕ be the scalar potential of \vec{F}

$$\text{Then } \vec{F} = \nabla \phi$$

$$i(e^x \cos y + yz)i + (xz - e^x \sin y)j + (xy + z)k = i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z}$$

$$\therefore \frac{\partial \phi}{\partial x} = e^x \cos y + yz \quad \text{--- (1)}$$

$$\frac{\partial \phi}{\partial y} = xz - e^x \sin y \quad \text{--- (2)}$$

$$\frac{\partial \phi}{\partial z} = xy + z \quad \text{--- (3)}$$

$$\text{From (1)} \Rightarrow \phi = e^x \cos y + xyz + f_1(y, z)$$

$$(2) \Rightarrow \phi = xyz + e^x \cos y + f_2(x, z)$$

$$(3) \Rightarrow \phi = xyz + \frac{z^2}{2} + f_3(x, y)$$

$$\phi = e^x \cos y + xyz + \frac{z^2}{2} + c$$

Q6) Show that $\vec{F} = (y^2 \cos x + z^3)i + (2yz \sin x - 4)j + 3x z^2 k$ is irrotational and hence find its scalar potential.

$$\operatorname{curl} \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 \cos x + z^3 & 2yz \sin x - 4 & 3x z^2 \end{vmatrix}$$

$$= i[0 - 0] - j[3z^2 - 3z^2] + k[2y \cos x - 2y \cos x] \\ = 0i - 0j + 0k$$

$\therefore \vec{F}$ is irrotational

Let ϕ be the scalar potential of \vec{F}
 then $\nabla\phi = \vec{F}$

$$\textcircled{e} (y^2 \cos x + y^3) \mathbf{i} + (2y \sin x - 4) \mathbf{j} + 3x y^2 \mathbf{k} = i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z},$$

$$\frac{\partial \phi}{\partial x} = y^2 \cos x + y^3 \quad \textcircled{1}$$

$$\frac{\partial \phi}{\partial y} = 2y \sin x - 4 \quad \textcircled{2}$$

$$\frac{\partial \phi}{\partial z} = 3x y^2 \quad \textcircled{3}$$

$$\textcircled{1} \Rightarrow \phi = y^2 \sin x + y^3 x + f_1(y, z)$$

$$\textcircled{2} \Rightarrow \phi = y^2 \sin x - 4y + f_2(x, z)$$

$$\textcircled{3} \Rightarrow \phi = x y^3 + f_3(x, y)$$

$$\phi = y^2 \sin x + x y^3 - 4y + c$$

Q68) Show that $\vec{F} = (2xy + y^3) \mathbf{i} + x^2 \mathbf{j} + 3x y^2 \mathbf{k}$ is a conservative vector field. Find its scalar potential. Find work done in moving an object in this field from $(1, -2, 1)$ to $(3, 1, 4)$.

Ans: cond $\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + y^3 & x^2 & 3x y^2 \end{vmatrix}$

$$= i [0-0] - j [3y^2 - 3y^2] + k [2x - 2x]$$

$$= \vec{0}$$

$\therefore \vec{F}$ is a conservative field.

$$\vec{F} = \nabla \phi$$

$$\frac{\partial \phi}{\partial x} = 2xy + y^3 \quad \textcircled{1}$$

$$\frac{\partial \phi}{\partial y} = x^2 \quad \textcircled{2}$$

$$\frac{\partial \phi}{\partial z} = 3x y^2 \quad \textcircled{3}$$

$$\begin{aligned} \textcircled{1} \Rightarrow \phi &= x^2y + y^3x + f_1(x, y) \\ \textcircled{2} \Rightarrow \phi &= x^2y + f_2(x, y) \\ \textcircled{3} \Rightarrow \phi &= xy^3 + f_3(x, y) \\ \phi &= x^2y + xy^3 + c \end{aligned}$$

Since \vec{F} is a conservative field, work done = $\int_C \vec{F} \cdot d\vec{r}$

$$\begin{aligned} &= \phi[3, 1, 4] - \phi[1, -2, 1] \\ &= [9 + 192 + c] - [-2 + 1 + c] \\ &= 201 - -1 \\ &= \underline{\underline{202}} \end{aligned}$$

Note

- * Let \vec{F} is a conservative field with scalar potential ϕ so that $\vec{F} = \nabla\phi$.
- * The potential energy of an object at the point (x, y, z) in the vector field \vec{F} is defined as $P(x, y, z) = -\phi(x, y, z)$.
- * The work done in moving an object from A to B is $W = \int_C \vec{F} \cdot d\vec{r} = \phi(B) - \phi(A)$
- * $\int_A^B \vec{F} \cdot d\vec{r}$ is independent of the path joining the points A and B if \vec{F} is irrotational.

Q69) Determine whether $\vec{F} = 4yi + 4xj$ is conservative vector field if so find the potential function and the potential energy.

Ans. cons $\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4y & 4x & 0 \end{vmatrix}$

$$= i(0) - j(0) + k(4 - 4) = \vec{0}$$

$\therefore \vec{F}$ is a conservative vector field.

$$\vec{F} = \nabla \phi$$

$$4y\mathbf{i} + 4x\mathbf{j} = \frac{\partial \phi}{\partial x} + \mathbf{i} \frac{\partial \phi}{\partial y} + \mathbf{k} \frac{\partial \phi}{\partial z}$$

$$\frac{\partial \phi}{\partial x} = 4y \quad \text{--- (1)}$$

$$\frac{\partial \phi}{\partial y} = 4x \quad \text{--- (2)}$$

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{--- (3)}$$

$$(1) \Rightarrow \phi = 4yx + f_1(y, z)$$

$$(2) \Rightarrow \phi = 4xy + f_2(x, z)$$

$$(3) \Rightarrow \phi = f_3(x, y)$$

$$\phi = 4xy + C$$

$$\text{potential energy} = -\phi$$

$$= -4xy - \underline{\underline{C}}$$

Q6: Q70) Show that $\int_A^B (2xy + y^3) dx + x^2 dy + 3xz^2 dz$ is independent of the path joining the points A and B.

Ans: $\int_A^B \vec{F} \cdot d\vec{r}$ is independent of the path joining the points

If \vec{F} is irrotational. here

$$\vec{F} = (2xy + y^3)\mathbf{i} + (x^2)\mathbf{j} + (3xz^2)\mathbf{k}$$

$$\text{curl } \vec{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + y^3 & x^2 & 3xz^2 \end{vmatrix}$$

$$= i[0-0] - j[3y^2 - 3y^2] + k[2x - 2x]$$

$$= \underline{\underline{0}}$$

Q71) Confirm that $\phi(x,y,z) = x^2 - 3y^2 + 4z^3$ is a potential function
 for $\vec{F} = 2x\mathbf{i} - 6y\mathbf{j} + 12z^2\mathbf{k}$

$$\text{Ans: } \nabla\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) (x^2 - 3y^2 + 4z^3)$$

$$= i(2x) + j(-6y) + k(12z^2)$$

$$= 2x\mathbf{i} - 6y\mathbf{j} + 12z^2\mathbf{k}$$

$$= \vec{F}$$

$\therefore \phi$ is a scalar potential for \vec{F}

Q72) Show that $\int (3x^2e^y dx + x^3e^y dy)$ is independent of the path and hence evaluate the integral from $(0,0)$ to $(3,2)$

Ans: $\int_C \vec{F} \cdot d\vec{r}$ is independent of the paths joining end points if \vec{F} is irrotational.

$$\vec{F} = 3x^2e^y \mathbf{i} + x^3e^y \mathbf{j}$$

$$\text{curl } \vec{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2e^y & x^3e^y & 0 \end{vmatrix}$$

$$= i(0-0) - j(0-0) + k(3x^2e^y - 3x^2e^y)$$

$$= \vec{0}$$

$\therefore \vec{F}$ is irrotational

ϕ be the scalar potential of \vec{F} if $\vec{F} = \nabla\phi$

$$\frac{\partial\phi}{\partial x} = 3x^2e^y \quad \text{--- (1)} \quad \frac{\partial\phi}{\partial y} = x^3e^y \quad \text{--- (2)} \quad \frac{\partial\phi}{\partial z} = 0 \quad \text{--- (3)}$$

$$(1) \Rightarrow \phi = x^3e^y + f_1(y, z)$$

$$(2) \Rightarrow \phi = x^3e^y + f_2(x, z)$$

$$(3) \Rightarrow \phi = f_3(x, y)$$

$$\therefore \phi = x^3e^y + C$$

Since \vec{F} is conservative work done = $\int_C \vec{F} \cdot d\vec{r}$

$$= \phi(3, 2) - \phi(0, 0)$$

$$= \underline{\underline{27e^2}}$$

Evaluation of line integral

Let $\vec{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, $a \leq x \leq b$ be a smooth parametric representation of a smooth curve C and $f(x, y, z)$ continuous function defined at each point on C . Then the line integral is

$$\int_C f(x, y, z) ds = \int_a^b f(x(t), y(t), z(t)) \| \vec{r}'(t) \| dt$$

Q73) Evaluate ~~consider~~ the line integral $\int_C 3xyz ds$ where the curve C has parametrization $x=t$, $y=t^2$, $z=\frac{2}{3}t^3$, $0 \leq t \leq 1$.

Ans:

$$\begin{aligned}\vec{r}(t) &= t\mathbf{i} + t^2\mathbf{j} + \frac{2}{3}t^3\mathbf{k} \\ \vec{r}'(t) &= \mathbf{i} + 2t\mathbf{j} + 2t^2\mathbf{k} \\ \|\vec{r}'(t)\| &= \sqrt{1 + 4t^2 + 4t^4} = \sqrt{4t^4 + 4t^2 + 1} = 2t^2 + 1 \\ \int_C 3xyz ds &= \int_0^1 3t \cdot t^2 \cdot \frac{2}{3}t^3 (2t^2 + 1) dt \\ &= \int_0^1 [4t^8 + 2t^6] dt \\ &= \left[\frac{4t^9}{9} + \frac{2t^7}{7} \right]_0^1 \\ &= \frac{4}{9} + \frac{2}{7} = \underline{\underline{\frac{46}{63}}}\end{aligned}$$