Module I

1. Open loop control system & Closed loop control system
2. Transfer function of LTI systems
3. Mechanical and Electromechanical Systems
4. Force Voltage and Force Current analogy
5. Block diagram representation
6. Block diagram reduction
7. Signal flow graph
8. Mason's gain formula
9. Characteristics equation

SYSTEM

System when a number of elements or components are connected in a sequence to perform a specific function, the group thus formed is called a system.
Example: a lamp (made up of glass, filaments)

CONTROL SYSTEM

In a system when the output quantity is controlled by varying the input quantity the system is called control system
Example: a lamp controlled by a switch

The output quantity is called controlled variable or response

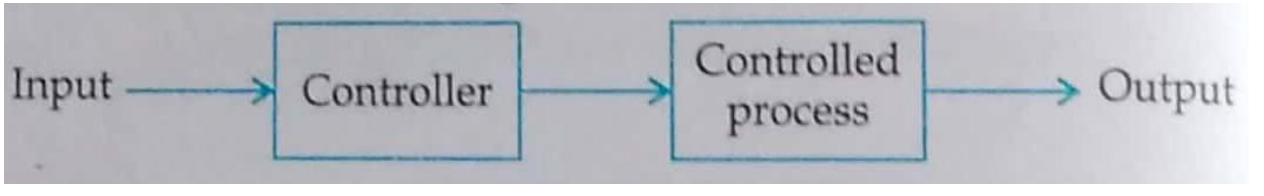
The input quantity is called command signal or excitation

open loop control system
&
closed loop control system

OPEN LOOP SYSTEM

Any physical system which does not automatically correct the variation in its output is called an open loop system or control system in which output quantity has no effect upon the input quantity are called open loop control system.

The output is not a feedback to the input for correction



Example: Automatic Washing Machine

OPEN LOOP SYSTEM

Advantage

1. simple
2. economical
3. easier to construct
4. stable

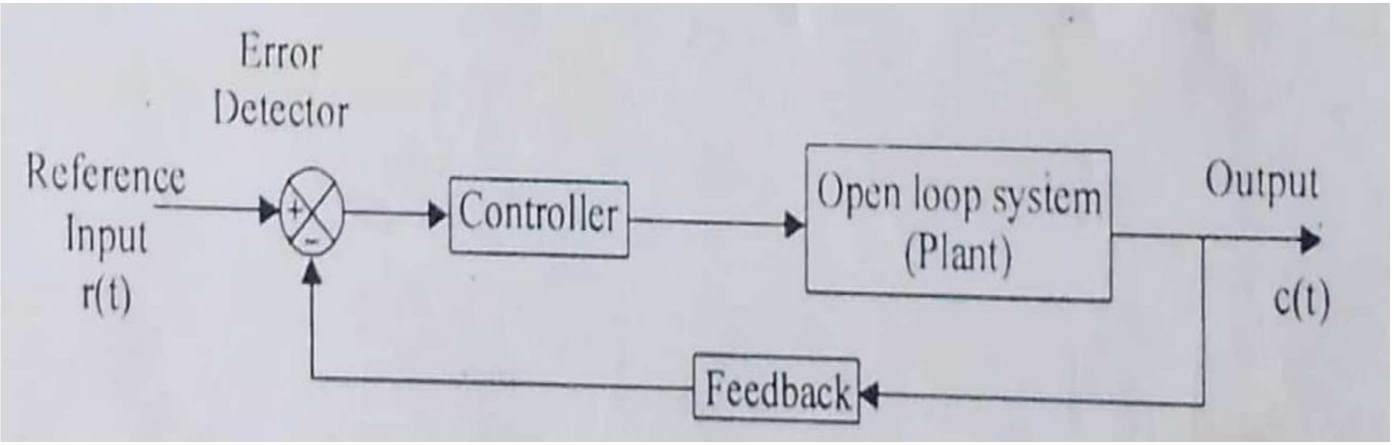
Disadvantage

1. inaccurate
2. unreliable
3. the changes in the output due to external disturbances are not corrected automatically

CLOSED LOOP SYSTEM

(automatic control system)

control systems in which the output has an effect upon the input quantity in order to maintain the desired output value are called closed loop control systems



Example: Air conditioner provided with thermostat

CLOSED LOOP SYSTEM

Advantage

1. accurate
2. the sensitivity of the system may be made small to make the
3. system more stable
4. less affected by noise

Disadvantage

1. complex
2. costly
3. feedback in closed loop system may lead to oscillatory
4. response feedback reduces the overall gain of the system
5. stability is a major problem in closed loop system

Comparison between open loop system and closed loop system

Open loop system	Closed loop system
these are not reliable	these are reliable
it is easier to build	it is difficult to built
if calibration is good they perform accurately	they are accurate because of feedback
operating systems are generally more stable	these are less stable
Optimization is not possible	Optimization is possible

MATHEMATICAL MODEL OF CONTROL SYSTEM

Control system is a collection of physical object connected together to serve an objective

The input output relations of various physical components of a system are governed by differential equation

The mathematical model of a control system constitutes a set of differential equations

The response or output of the system can be studied by solving the differential equations for various input condition

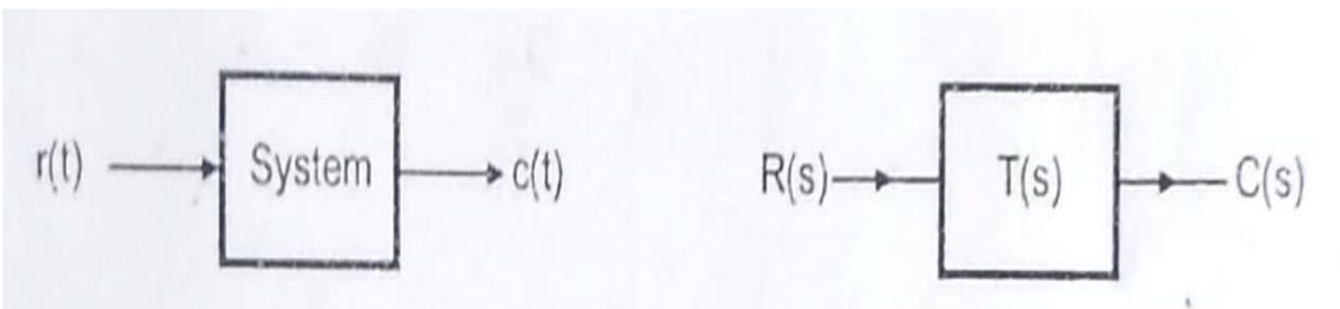
MATHEMATICAL MODEL OF CONTROL SYSTEM

The differential equations of a linear time invariant system can be reshaped into different form for the convenience of analysis

One such model for single input and single output system analysis is called transfer function

TRANSFER FUNCTION

Transfer function is the ratio of Laplace transform of outputs of the system to the Laplace transform of the inputs under the assumption that all initial conditions are zero

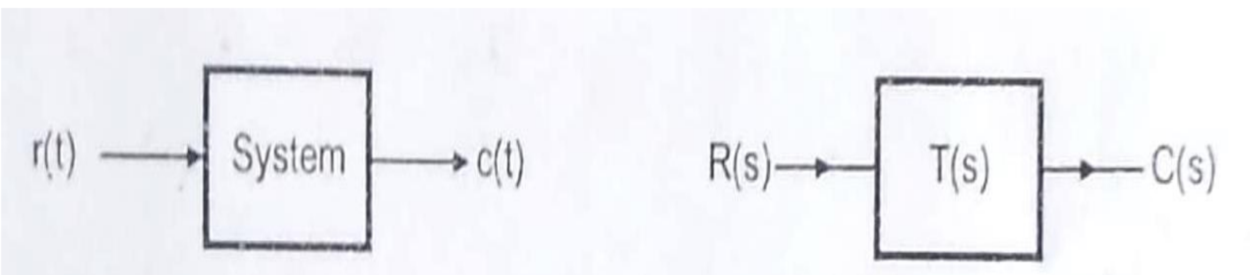


$$T(s) = \frac{\text{Laplace transform of output}}{\text{Laplace transform of input}} = \frac{C(s)}{R(s)}$$

TRANSFER FUNCTION

Transfer function is the ratio of Laplace transform of outputs of the system to the Laplace transform of the inputs under the assumption that all initial conditions are zero

$$\text{Transfer function} = \left.\frac{\text{Laplace Transform of output}}{\text{Laplace Transform of input}}\right|_{\text{with zero initial conditions}}$$



$$T(s) = \frac{\text{Laplace transform of output}}{\text{Laplace transform of input}} = \frac{C(s)}{R(s)}$$

Advantages of Transfer function

1. The response of the system to any input can be determined very easily.
2. It gives the gain of the system.
3. It help in the study of stability of the system
4. Since Laplace transform is used it converts time domain equations to simple algebraic equations
5. Poles and zeroes of a system can be determined from the knowledge of the transfer function of the system.

Disadvantages

1. transfer function cannot be defined for Non linear system
2. transfer function is defined only for linear system
3. from the transfer function physical structure of a system cannot determine
4. initial conditions lose their importance

Characteristic Equation (C.E)

The characteristic equation of a linear system can be obtained by equating the denominator polynomial of the transfer function to zero. The roots of the characteristic equation are the poles of corresponding transfer function.

Poles of a transfer function

The value of 'S' which makes the transfer function infinite after substitution in the denominator of a transfer function are called poles of that transfer function

Zeros of a transfer function

The value of 'S' which make the transfer function zero after substituting in the numerator are called zeros of that transfer function

Consider a linear system having input r(t) & c(t) is the output of the system

The input - output relation can be described by the following n^{th} order differential equation

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + ... a_1 \frac{dc(t)}{dt} + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} .... b_1 \frac{dr(t)}{dt} + b_0 r(t)$$

where 'a' and 'b' are constants

take the Laplace transform

$$(a_n s^n + a_{n-1} s^{n-1} + ... + a_1 s + a_0) C(s) = b_m s^m + b_{m-1} s^{m-1} + ...b_1 s + b_0) R(s)$$

$$TF = \frac{C(s)}{R(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + ...b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + ... + a_1 s + a_0}$$

$$TF = \frac{C(S)}{R(S)} = \frac{K(S-S_a)(S-S_b)..........(S-S_m)(as^2+bs+c)}{(S-S_1)(S-S_2)..........(S-S_n)(ds^2+es+f)}$$

$$\text{Zeros}: S_a, S_b, ...S_m, \frac{-b \pm \sqrt{b^2-ac}}{2a}$$

$$\text{Poles}: S_1, S_2, ...S_n, \frac{-e \pm \sqrt{e^2-df}}{2d}$$

$$C.E: (S-S_1)(S-S_2)..........(S-S_n)(ds^2+es+f) = 0$$

Basic formula

$$L[x(t)]=X(S)$$

$$L[I(t)]=I(S)$$

$$L[\frac{dx(t)}{dt}] = SX(S)\Big|\text{with zero initial condition}$$

$$L[\frac{d^2x(t)}{dt}] = S^2X(S)\Big|\text{with zero initial condition}$$

Find the TF of the given n/w

apply KVL in mesh ①

$$V_i = Ri + L\frac{di}{dt} \quad \text{— ①}$$

apply KVL in mesh ②

$$V_o = L\frac{di}{dt} \quad \text{— ②}$$

take the laplace transform

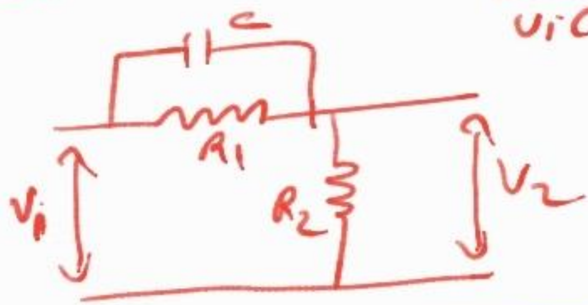
① → $V_i(s) = RI(s) + LSI(s)$

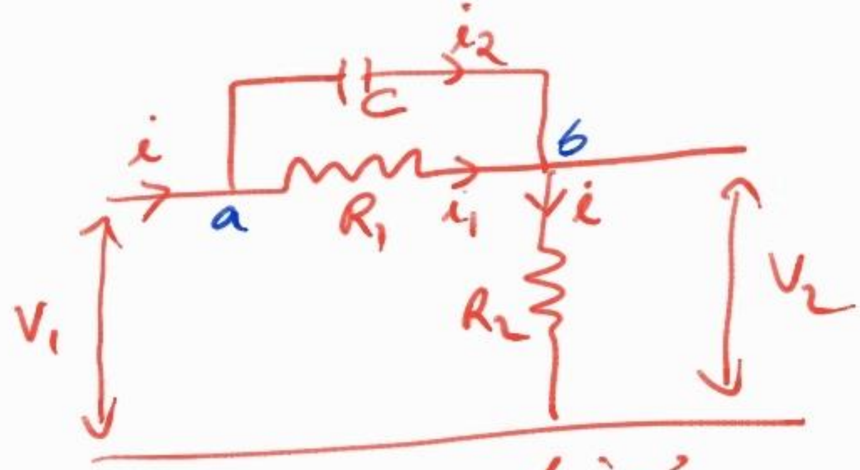
② → $V_o(s) = SLI(s)$

$$\frac{V_o(s)}{V_i(s)} = \frac{SLI(s)}{RI(s)+LSI(s)} = \frac{SLI(s)}{(R+LS)I(s)}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{SL}{R+SL}$$ //

Obtain the TF $\frac{V_2(s)}{V_i(s)}$





KCL at node 'a'

$$i = i_1 + i_2$$

$$i = \frac{V_2}{R_2} \; ; \; i_1 = \frac{V_1 - V_2}{R_1}$$

$$i_2 = C \cdot \frac{d(V_1 - V_2)}{dt}$$

Substitutes the values

$$\frac{V_2}{R_2} = \frac{V_1 - V_2}{R_1} + C\frac{d}{dt}(V_1 - V_2)$$

take laplace transform

$$\frac{V_2(s)}{R_2} = \frac{V_1(s) - V_2(s)}{R_1} + CS(V_1(s) - V_2(s))$$

$$= \frac{V_1(s)}{R_1} - \frac{V_2(s)}{R_1} + CSV_1(s) - CSV_2(s)$$

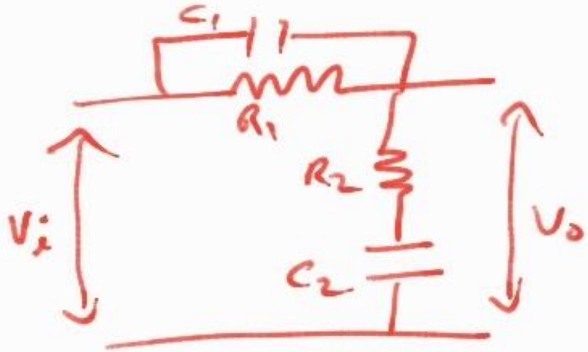
$$\frac{V_2(s)}{R_2} + \frac{V_2(s)}{R_1} + CSV_2(s) = \frac{V_1(s)}{R_1} + CSV_1(s)$$

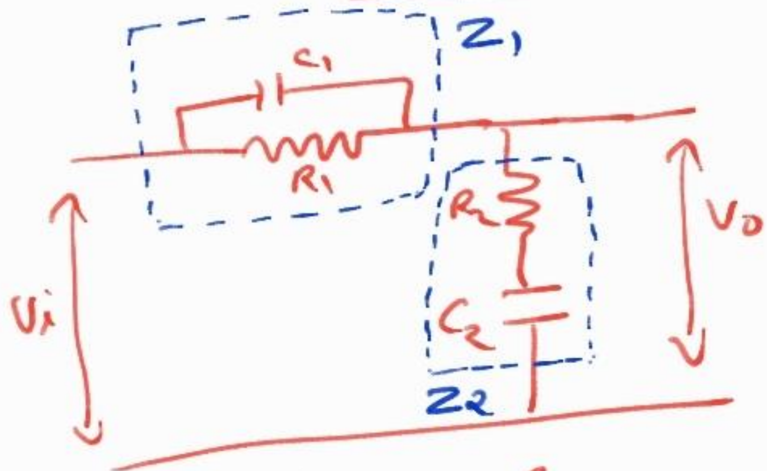
$$V_2(s)\left[\frac{1}{R_2} + \frac{1}{R_1} + CS\right] = V_1(s)\left[\frac{1}{R_1} + CS\right]$$

$$V_2(s)\left[\frac{R_1 + R_2 + SR_1R_2C}{R_1R_2}\right] = V_1(s)\left[\frac{1 + SR_1C}{R_1}\right]$$

$$\frac{V_2(s)}{V_1(s)} = \frac{[1 + SR_1C]R_2}{R_1 + R_2 + SR_1R_2C} = \frac{R_2 + SR_1R_2C}{R_1 + R_2 + R_1R_2CS}$$

Determine the TF





$$Z_1 = \frac{R_1 \cdot \frac{1}{sC_1}}{R_1 + \frac{1}{sC_1}} = \frac{R_1}{R_1 s C_1 + 1}$$

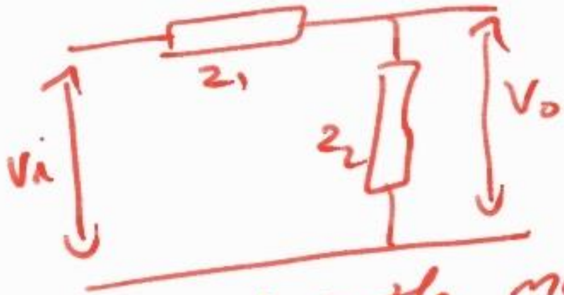
$$Z_2 = R_2 + \frac{1}{sC_2} = \frac{R_2 s C_2 + 1}{sC_2}$$

$$Z_1 = \frac{R_1 \cdot \frac{1}{j\omega C_1}}{R_1 + \frac{1}{j\omega C_1}}$$

$$j\omega = s$$

$$Z_1 = \frac{R_1 \cdot \frac{1}{sC_1}}{R_1 + \frac{1}{sC_1}}$$

$$Z = \frac{R_1 \cdot j\omega L}{R_1 + j\omega L} = \frac{R_1 \cdot sL}{R_1 + sL}$$



KVL at both mesh

$$V_i = Z_1 i + Z_2 i = (Z_1 + Z_2) i$$

$$V_o = Z_2 \cdot i$$

$$\frac{V_o}{V_i} = \frac{Z_2}{Z_1 + Z_2}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{\left[\frac{R_2 s C_2 + 1}{s C_2}\right]}{\left[\frac{R_1}{R_1 s C_1 + 1} + \frac{R_2 s C_2 + 1}{s C_2}\right]}$$

$$= \frac{[R_2 s C_2 + 1][R_1 s C_1 + 1]}{R_1 s C_2 + (R_2 s C_2 + 1)(R_1 s C_1 + 1)}$$

A s/m having input $x(t)$ and output $y(t)$ is represented by,

$$\frac{dy(t)}{dt} + 4y(t) = \frac{dx(t)}{dt} + 5x(t)$$

find TF

take laplace transform

$$sY(s) + 4Y(s) = sX(s) + 5X(s)$$

$$Y(s)[s+4] = X(s)[s+5]$$

$$\frac{Y(s)}{X(s)} = \frac{s+5}{s+4}$$

$$TF = L\left(\frac{o/p}{i/p}\right) = \frac{s+5}{s+4}$$

The TF of the slm is given by

$$TF = \frac{4s+1}{s^2+2s+3}$$

find the differential equation of the slm having i/p x(t) & o/p y(t)

$$TF = \frac{Y(s)}{X(s)} = \frac{4s+1}{s^2+2s+3}$$

Cross multiplying

$$Y(s)\left[s^2+2s+3\right] = X(s)\left[4s+1\right]$$

$$s^2Y(s) + 2sY(s) + 3Y(s) = 4sX(s) + X(s)$$

Take inverse laplace transform

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 3y(t) = 4\frac{dx(t)}{dt} + x(t)$$

Mechanical system

two types

1. translational systems
2. rotational systems

The motion takes place along a straight line is known as translational motion.

The rotational motion of a body can be defined as the motion of a body about a fixed axis.

Mechanical translational system

The model of mechanical translational system can be obtained by using three basic elements mass, spring and dashpot.

The weight of the mechanical system is represented by the element of mass
The elastic deformation of the body can be represented by spring

The friction existing in rotating mechanical system can be represented by dashpot

When a force is applied to a translational mechanical system it is opposed by opposing forces due to mass, friction and elasticity of the system

Force acting on a mechanical body are governed by Newton's second law of motion

LIST OF SYMBOLS USED IN MECHANICAL TRANSLATIONAL SYSTEM

x = Displacement, m

$v = \frac{dx}{dt}$ = Velocity, m/sec

$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$ = Acceleration, m/sec^2

f = Applied force, N (Newtons)

f_m = Opposing force offered by mass of the body, N

f_k = Opposing force offered by the elasticity of the body (spring), N

f_b = Opposing force offered by the friction of the body (dash - pot), N

M = Mass, kg

K = Stiffness of spring, N/m

B = Viscous friction co-efficient, N-sec/m

Note : *Lower case letters are functions of time*

FORCE BALANCE EQUATIONS OF IDEALIZED ELEMENTS

Consider an ideal mass element shown in fig 1.9 which has negligible friction and elasticity. Let a force be applied on it. The mass will offer an opposing force which is proportional to acceleration of the body.

Let, f = Applied force

f_m = Opposing force due to mass

Here, $f_m \propto \frac{d^2x}{dt^2}$ or $f_m = M\frac{d^2x}{dt^2}$

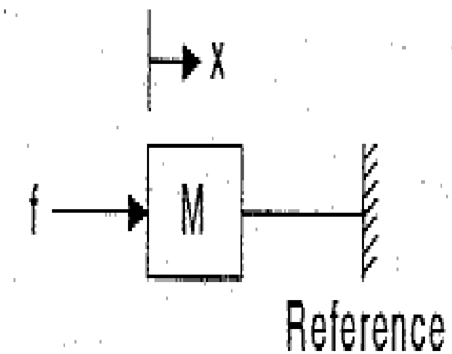


Fig 1.9 : Ideal mass element.

By Newton's second law, $$f = f_m = M\frac{d^2x}{dt^2} \quad .....(1.2)$$

Consider an ideal frictional element dashpot shown in fig 1.10 which has negligible mass and elasticity . Let a force be applied on it. The dash-pot will offer an opposing force which is proportional to velocity of the body.

Let, f = Applied force

f_b = Opposing force due to friction

Here, $f_b \propto \frac{dx}{dt}$ or $f_b = B\frac{dx}{dt}$

By Newton's second law, $\boxed{f = f_b = B\frac{dx}{dt}}$ .....(1.3)

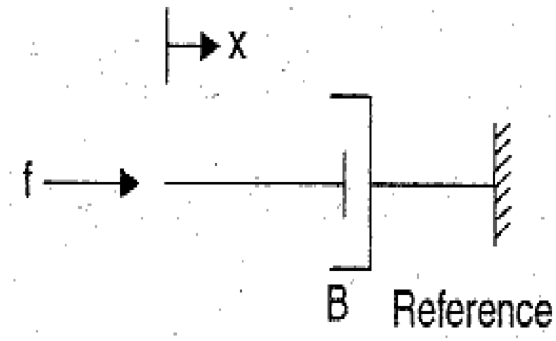


***Fig 1.10 :** Ideal dashpot with one end fixed to reference.*

When the dashpot has displacement at both ends as shown in fig 1.11, the opposing force is proportional to differential velocity.

$$f_b \propto \frac{d}{dt}(x_1 - x_2) \quad \text{or} \quad f_b = B\frac{d}{dt}(x_1 - x_2)$$

$$\therefore \boxed{f = f_b = B\frac{d}{dt}(x_1 - x_2)} \quad .....(1.4)$$

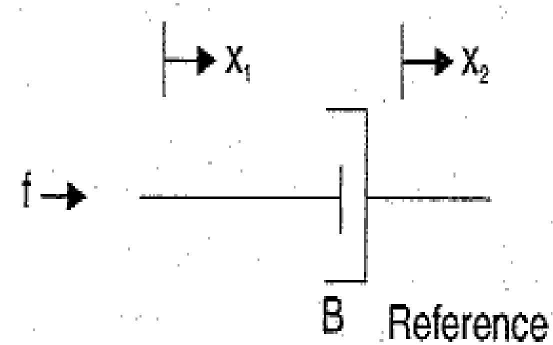


***Fig 1.11 :** Ideal dashpot with displacement at both ends.*

Consider an ideal elastic element spring shown in fig 1.12, which has negligible mass and friction. Let a force be applied on it. The spring will offer an opposing force which is proportional to displacement of the body.

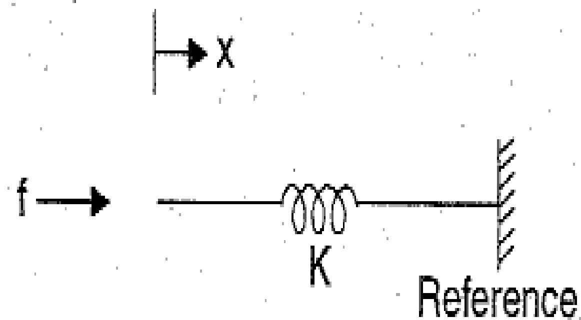


Fig 1.12 : Ideal spring with one end fixed to reference.

Let, f = Applied force

f_k = Opposing force due to elasticity

Here $f_k \propto x$ or $f_k = K x$

By Newton's second law, $\boxed{f = f_k = Kx}$ .....(1.5)

When the spring has displacement at both ends as shown in g 1.13 the opposing force is proportional to differential displacement.

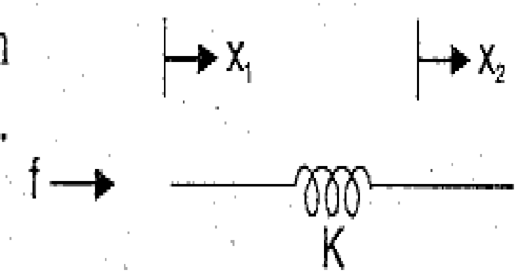


Fig 1.13 : Ideal spring with displacement at both ends.

$f_k \propto (x_1 - x_2)$ or $f_k = K(x_1 - x_2)$

$\therefore \boxed{f = f_k = K(x_1 - x_2)}$ .....(1.6)

Guidelines to determine the transfer function of mechanical translational system

1. consider each mass separately

2. draw the free body diagram

3. write the differential equations

4. take the Laplace transform of differential equations

5. rearrange the s-domain equation to eliminate the unwanted variables and obtain the ratio between output variable and input variable

Mechanical rotational systems

The model of mechanical rotational systems can be obtained by using three elements moment of inertia [J] of mass, dash-pot with rotational frictional Coefficient [B] and torsional spring with stiffness [K]

The weight of the rotational mechanical system is represented by the moment of inertia of the mass

The elastic deformation of the body can be represented by a spring

The friction existing in rotational mechanical system can be represented by dash-pot

LIST OF SYMBOLS USED IN MECHANICAL ROTATIONAL SYSTEM

θ = Angular displacement, rad

$\frac{d\theta}{dt}$ = Angular velocity, rad/sec

$\frac{d^2\theta}{dt^2}$ = Angular acceleration, rad/sec^2

T = Applied torque, N-m

J = Moment of inertia, Kg-m^2/rad

B = Rotational frictional coefficient, N-m/(rad/sec)

K = Stiffness of the spring, N-m/rad

TORQUE BALANCE EQUATIONS OF IDEALISED ELEMENTS

Consider an ideal mass element shown in fig 1.14 which has negligible friction and elasticity. The opposing torque due to moment of inertia is proportional to the angular acceleration.

Let, T = Applied torque.

T_j = Opposing torque due to moment of inertia of the body.

Here $T_j \propto \frac{d^2\theta}{dt^2}$ or $T_j = J\frac{d^2\theta}{dt^2}$

By Newton's second law,

$$T = T_j = J\frac{d^2\theta}{dt^2} \quad .....(1.7)$$

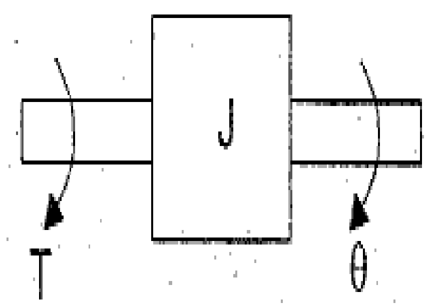


Fig 1.14 : Ideal rotational mass element.

Consider an ideal frictional element dash pot shown in fig 1.15 which has negligible moment of inertia and elasticity. Let a torque be applied on it. The dash pot will offer an opposing torque which is proportional to the angular velocity of the body.

Let, T = Applied torque.

T_b = Opposing torque due to friction.

$$T_b \propto \frac{d\theta}{dt} \quad \text{or} \quad T_b = B\frac{d\theta}{dt}$$

By Newton's second law,

$$\boxed{T = T_b = B\frac{d\theta}{dt}} \qquad .....(1.8)$$

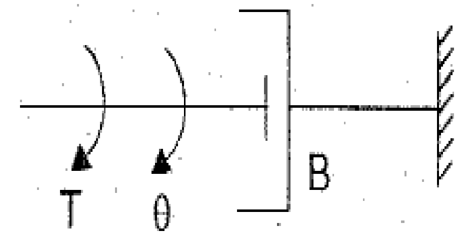


Fig 1.15 : Ideal rotational dash-pot with one end fixed to reference.

When the dash pot has angular displacement at both ends as shown in fig 1.16, the opposing torque is proportional to the differential angular velocity.

$$T_b \propto \frac{d}{dt}(\theta_1 - \theta_2) \quad \text{or} \quad T_b = B\frac{d}{dt}(\theta_1 - \theta_2)$$

$$\boxed{\therefore T = T_b = B\frac{d}{dt}(\theta_1 - \theta_2)} \qquad .....(1.9)$$

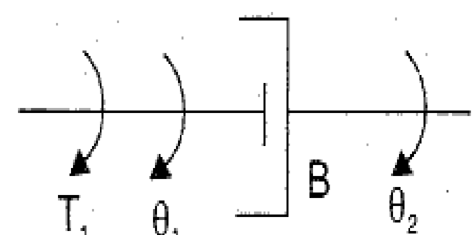


Fig 1.16 : Ideal dash-pot with angular displacement at both ends.

Consider an ideal elastic element, torsional spring as shown in fig 1.17, which has negligible moment of inertia and friction. Let a torque be applied on it. The torsional spring will offer an opposing torque which is proportional to angular displacement of the body.

Let, T = Applied torque.

T_k = Opposing torque due to elasticity.

$T_k \propto \theta$ or $T_k = K\theta$

By Newton's second law, $\boxed{T = T_k = K\theta}$ .....(1.10)

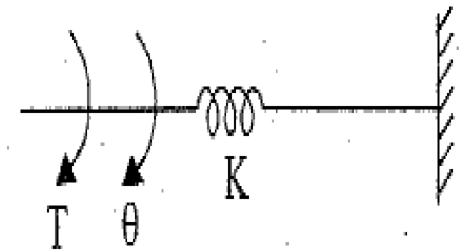


Fig 1.17 : Ideal spring with one end fixed to reference.

When the spring has angular displacement at both ends as shown in fig 1.18 the opposing torque is proportional to differential angular displacement.

$T_k \propto (\theta_1 - \theta_2)$ or $T_k = K(\theta_1 - \theta_2)$

$\therefore \boxed{T = T_k = K(\theta_1 - \theta_2)}$ .....(1.11)

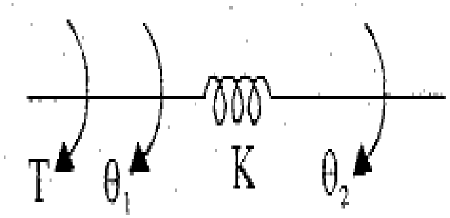


Fig 1.18 : Ideal spring with angular displacement at both ends.

Guidelines to determine the transfer function of mechanical rotational system

1. consider each moment of inertia separately

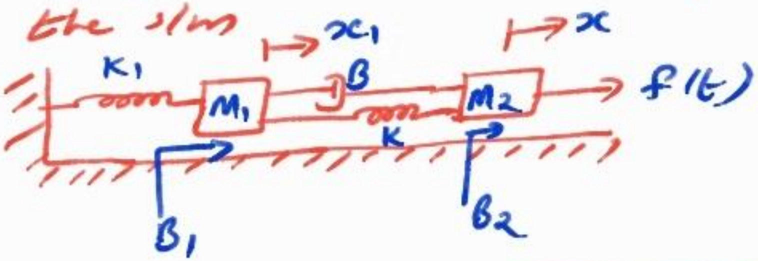
2. draw the free body diagram

3. write the differential equations

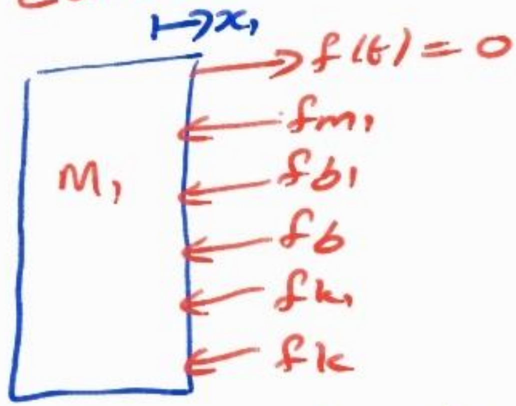
4. take the Laplace transform of differential equations

5. rearrange the s-domain equation to eliminate the unwanted variables and obtain the ratio between output variable and input variable

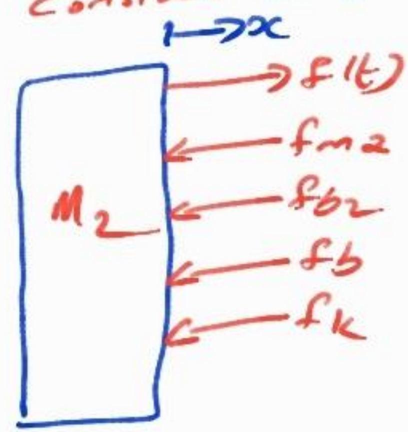
Write the differential equations governing the s/m



Consider mass M_1



Consider mass M_2



By Newton's second law

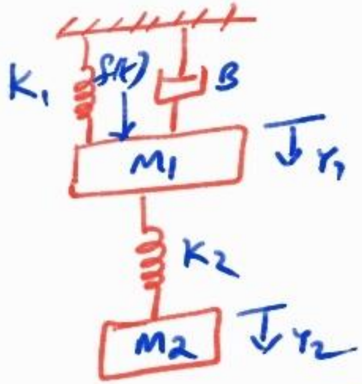
$$f_{m1} + f_{b1} + f_b + f_{k1} + f_k = 0 \quad \rightarrow (1)$$

$$f(t) = f_{m2} + f_{b2} + f_b + f_k \quad \rightarrow (2)$$

$$(1) \Rightarrow M_1 \frac{d^2 x_1}{dt^2} + B_1 \frac{dx_1}{dt} + B \frac{d}{dt}(x_1 - x) + K_1 x_1 + K(x_1 - x) = 0$$

$$(2) \Rightarrow M_2 \frac{d^2 x}{dt^2} + B_2 \frac{dx}{dt} + B \frac{d}{dt}(x - x_1) + K(x - x_1) = f(t)$$

Determine the TF $\frac{Y_2(s)}{F(s)}$ of the s/m



Differential equation

① ⟹ $M_1 \frac{d^2 y_1}{dt^2} + B \frac{dy_1}{dt} + k_1 y_1 + k_2 (y_1 - y_2) = f(t)$

② ⟹ $M_2 \frac{d^2 y_2}{dt^2} + k_2 (y_2 - y_1) = 0$

take the laplace transform

$M_1 s^2 Y_1(s) + Bs Y_1(s) + k_1 Y_1(s) + k_2 (Y_1(s) - Y_2(s)) = F(s)$

$Y_1(s) [M_1 s^2 + Bs + k_1 + k_2] - Y_2(s) k_2 = F(s)$ → ⓐ

$M_2 s^2 Y_2(s) + k_2 (Y_2(s) - Y_1(s)) = 0$

$Y_2(s) [M_2 s^2 + k_2] - Y_1(s) k_2 = 0$

$$Y_1(s) = Y_2(s)\frac{[M_2 s^2 + K_2]}{K_2} \rightarrow (b)$$

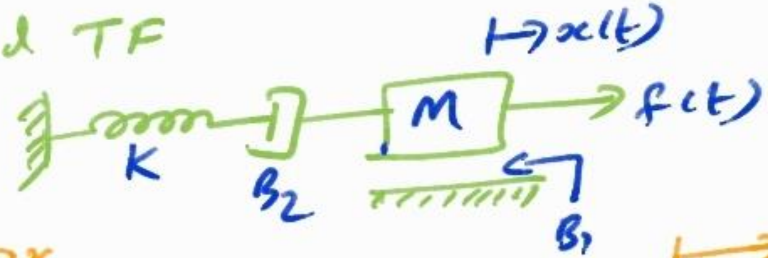
Substitute 'b' in 'a'

$$Y_2(s)\left[\frac{M_2 s^2 + K_2}{K_2}\right]\left[M_1 s^2 + Bs + K_1 + K_2\right] - Y_2(s) K_2 = F(s)$$

$$Y_2(s)\frac{\left[(M_2 s^2 + K_2)(M_1 s^2 + Bs + K_1 + K_2) - K_2^2\right]}{K_2} = F(s)$$

$$\frac{Y_2(s)}{F(s)} = \frac{K_2}{\left[(M_2 s^2 + K_2)(M_1 s^2 + Bs + K_1 + K_2) - K_2^2\right]}$$

find TF



$$M\frac{d^2x}{dt^2} + B_1\frac{dx}{dt} + B_2\frac{d}{dt}(x-x_1) = F(t) \quad \to (1)$$

$$B_2\frac{d}{dt}(x_1-x) + Kx_1 = 0 \quad - (2)$$

$$(1) \Rightarrow Ms^2X(s) + B_1 s X(s) + B_2 s\left(X(s) - X_1(s)\right) = F(s)$$

$$X(s)\left[Ms^2 + B_1 s + B_2 s\right] - B_2 s X_1(s) = F(s) \quad \to (3)$$

$$(2) \Rightarrow B_2 s\left[X_1(s) - X(s)\right] + K X_1(s) = 0$$

$$X_1(s)\left[B_2 s + K\right] - B_2 s X(s) = 0$$

$$x_1(s) = \frac{B_2 s}{B_2 s + K} x(s)$$

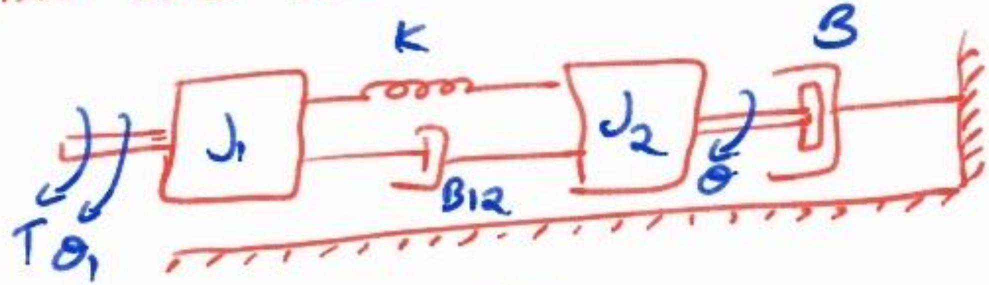
Substitute the value in ③

$$x(s)\left[Ms^2 + B_1 s + B s\right] - B_2 s\left[\frac{B_2 s}{B_2 s + K}\right] x(s) = F(s)$$

$$x(s)\left[\frac{(Ms^2 + B_1 s + B s)(B_2 s + K) - B_2^2 s^2}{B_2 s + K}\right] = F(s)$$

$$\frac{x(s)}{F(s)} = \frac{B_2 s + K}{(Ms^2 + B_1 s + B s)(B_2 s + K) - B_2^2 s^2}$$

1. Write the differential equations
2. find the TF

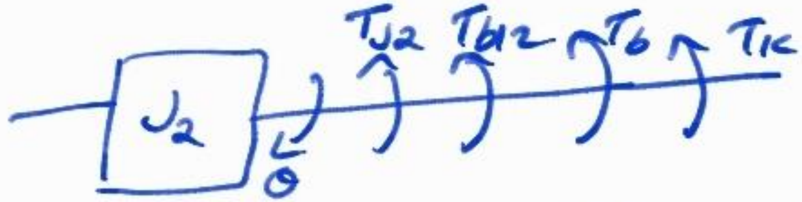


$$J_1 \frac{d^2\theta_1}{dt^2} + B_{12}\frac{d(\theta_1-\theta)}{dt} + K(\theta_1-\theta) = T$$

$$\Leftrightarrow J_1 s^2 \theta_1(s) + B_{12} s[\theta_1(s)-\theta(s)] + K[\theta_1(s)-\theta(s)] = T(s)$$

$$\theta_1(s)\left[J_1 s^2 + sB_{12} + K\right] - \theta(s)\left[B_{12}s + K\right] = T(s)$$

$\hookrightarrow$ (1)



$$J_2 \frac{d^2\theta}{dt^2} + B_{12}\frac{d(\theta-\theta_1)}{dt} + B\frac{d\theta}{dt} + K(\theta-\theta_1) = 0$$

$\mathcal{L} \Rightarrow$

$$J_2 s^2\theta(s) + B_{12}s[\theta(s)-\theta_1(s)] + Bs\theta(s) + K(\theta(s)-\theta_1(s)) = 0$$

$$\theta(s)[J_2 s^2 + B_{12}s + Bs + K] - \theta_1(s)[sB_{12}+K] = 0$$

$$\theta_1(s) = \theta(s)\frac{(J_2 s^2 + B_{12}s + Bs + K)}{sB_{12} + K}$$

Substitute $\theta_1(s)$ in equ (1)

Then

$$\frac{\theta(s)}{T(s)} = \frac{sB_{12}+K}{(J_1 s^2 + sB_{12} + K)(J_2 s^2 + sB_{12} + sB) + K) - (sB_{12}+K)^2}$$

ELECTRICAL ANALOGOUS OF MECHANICAL SYSTEMS

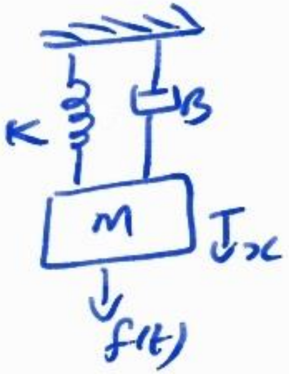
Systems remain analogous as long as the differential equations governing the systems or transfer functions are in ideal form.

Since the electrical systems are two types of inputs either voltage or current source, there are two types of analogies - force voltage analogy/ torque voltage analogy and force current analogy/ torque current analogy

Force / torque voltage analogy - Each junction in the mechanical system response to a closed loop which consists of electrical excitation sources and passive elements analogous to the mechanical driving source and passive elements connected to the junction

Force / torque current analogy - Each junction in the mechanical system corresponds to a node which joins electrical excitation sources and passive elements analogous to the mechanical driving sources and passive elements connected to the junction

Force Voltage Analogy



$$f(t) = M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx \quad \rightarrow (1)$$

$$E = Ri + L\cdot\frac{di}{dt} + \frac{1}{C}\int i\,dt$$

$$= R\cdot\frac{dq}{dt} + L\cdot\frac{d^2q}{dt^2} + \frac{1}{C}q \quad \rightarrow (2)$$

$$\left[i = \frac{dq}{dt}\right]$$

Compare equ (1) & (2)

$f(t) \rightarrow E$

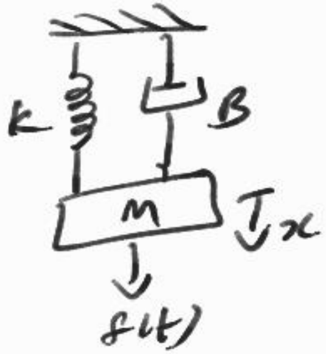
$M \rightarrow L$

$B \rightarrow R$

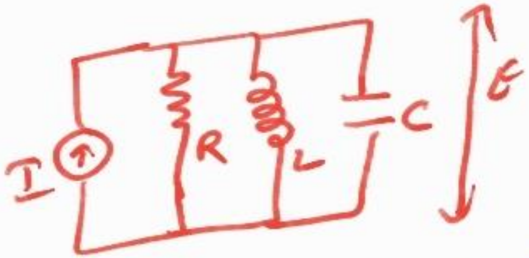
$K \rightarrow \frac{1}{C}$

$x \rightarrow q$

Force Current Analogy



$$f(t) = M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx \quad \to (1)$$



$$I = \frac{E}{R} + \frac{1}{L}\int E\,dt + C\cdot\frac{dE}{dt}$$

$$I = \frac{1}{R}\frac{d\phi}{dt} + \frac{1}{L}\phi + C\cdot\frac{d^2\phi}{dt^2} \quad \to (2)$$

Compare (1) & (2)

$f(t) \to I$

$M \to C$

$B \to \frac{1}{R}$

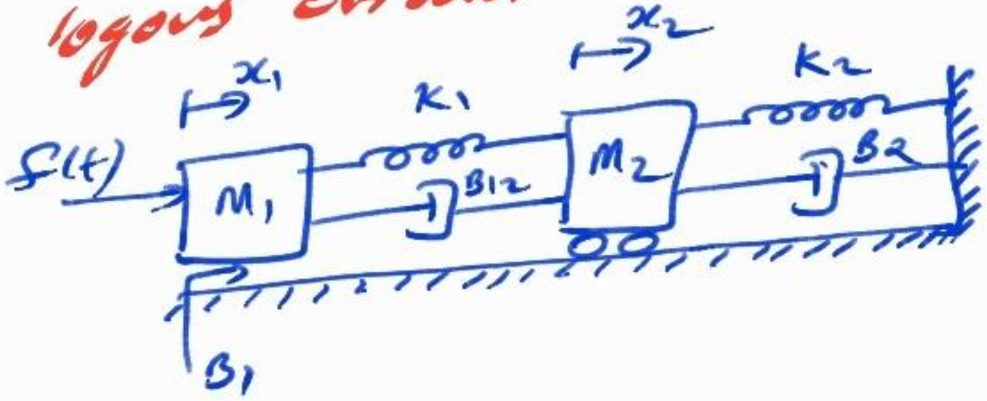
$K \to \frac{1}{L}$

$x \to \phi$

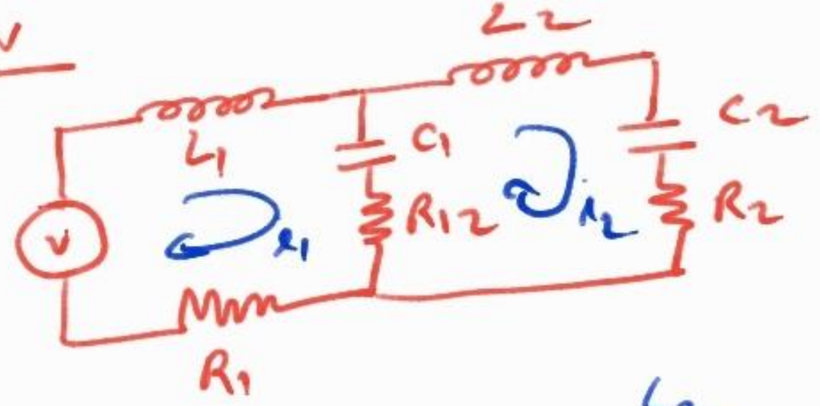
$\frac{dx}{dt} \to \frac{d\phi}{dt} = E$

Mechanical (Translational)	Electrical (Voltage)	Electrical (Current)	Mechanical (Rotational)
F	E	I	T
M	L	C	J
K	$\frac{1}{C}$	$\frac{1}{L}$	K
B	R	$\frac{1}{R}$	B
x	q	ϕ	θ

Draw the FV & FI electrical analogous circuit



FV



$M_1 \to L_1$

$B_1 \to R_1$

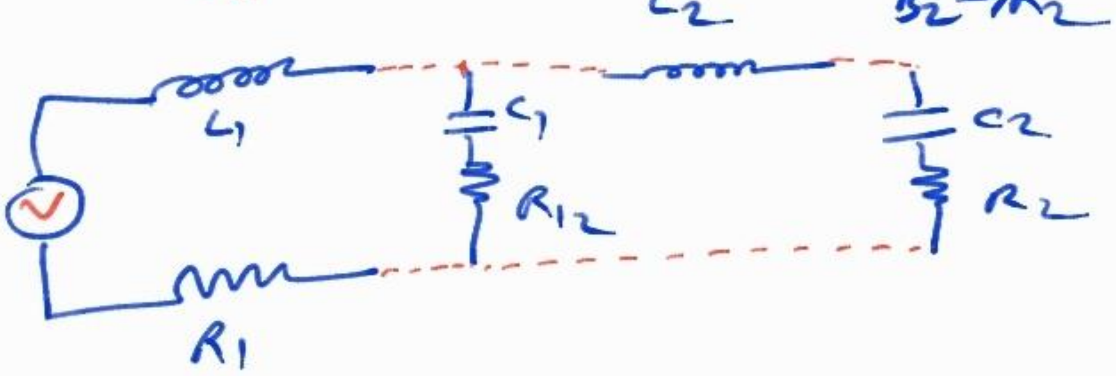
$K_1 \to C_1$

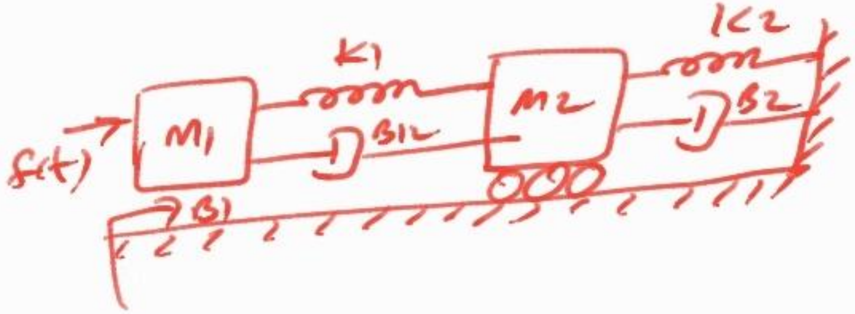
$B_{12} \to R_{12}$

$M_2 \to L_2$

$K_2 \to C_2$

$B_2 \to R_2$





$f \to i$

$M_1 \to C_1$

$M_2 \to C_2$

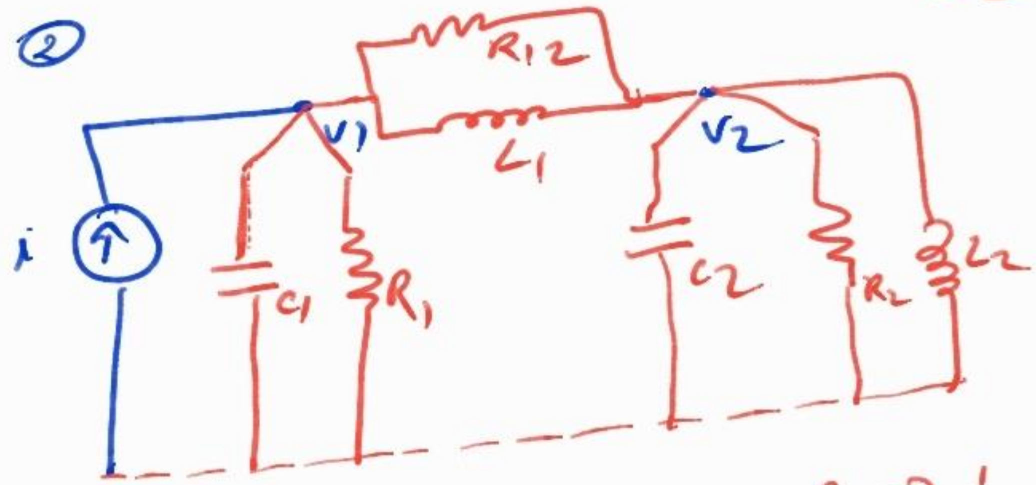
$K_1 \to \frac{1}{L_1}$

$K_2 \to \frac{1}{L_2}$

FI

① $\dot{v}_1$ $\dot{v}_2$

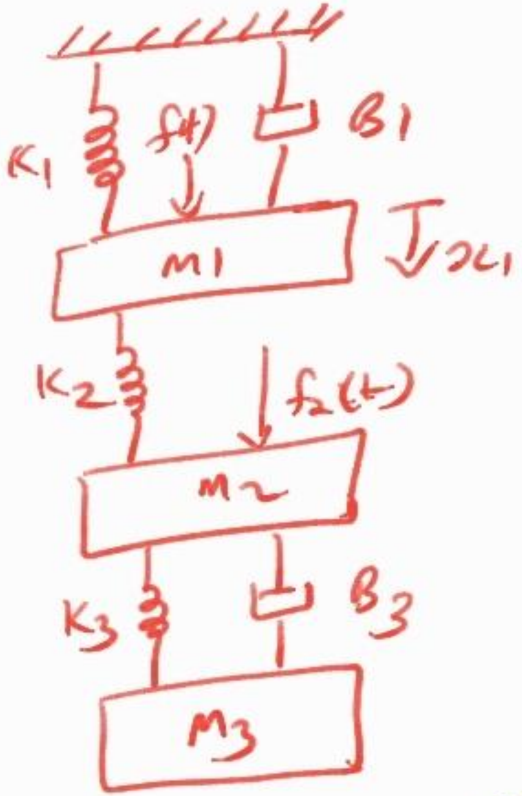
②



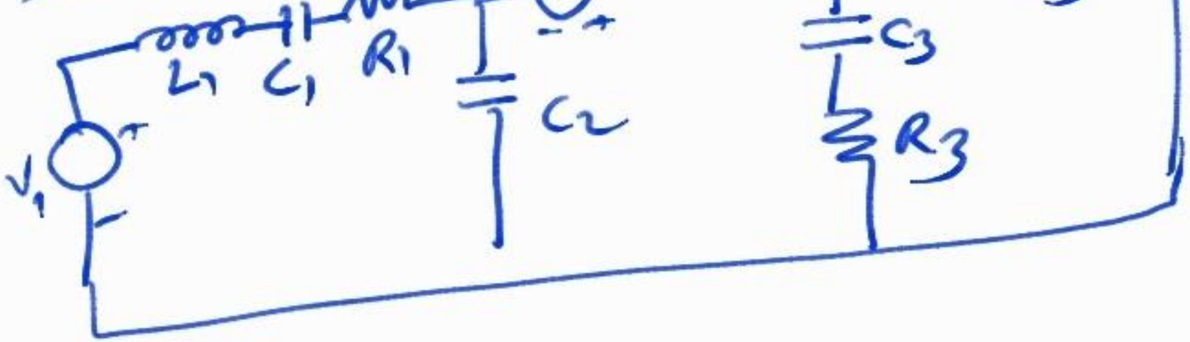
$B_1 \to \frac{1}{R_1}$

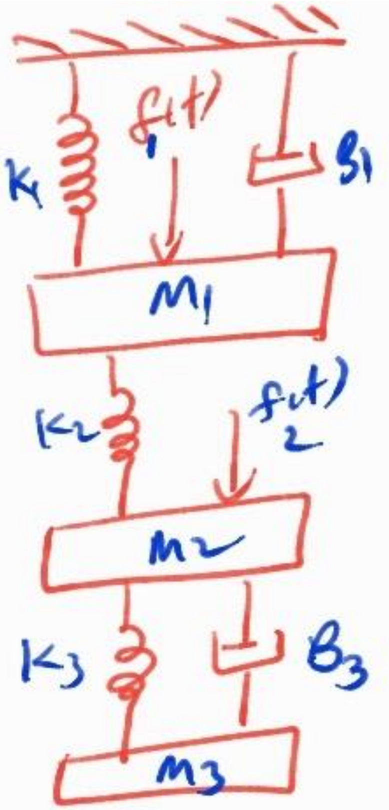
$B_2 \to \frac{1}{R_2}$

$B_{12} \to \frac{1}{R_{12}}$

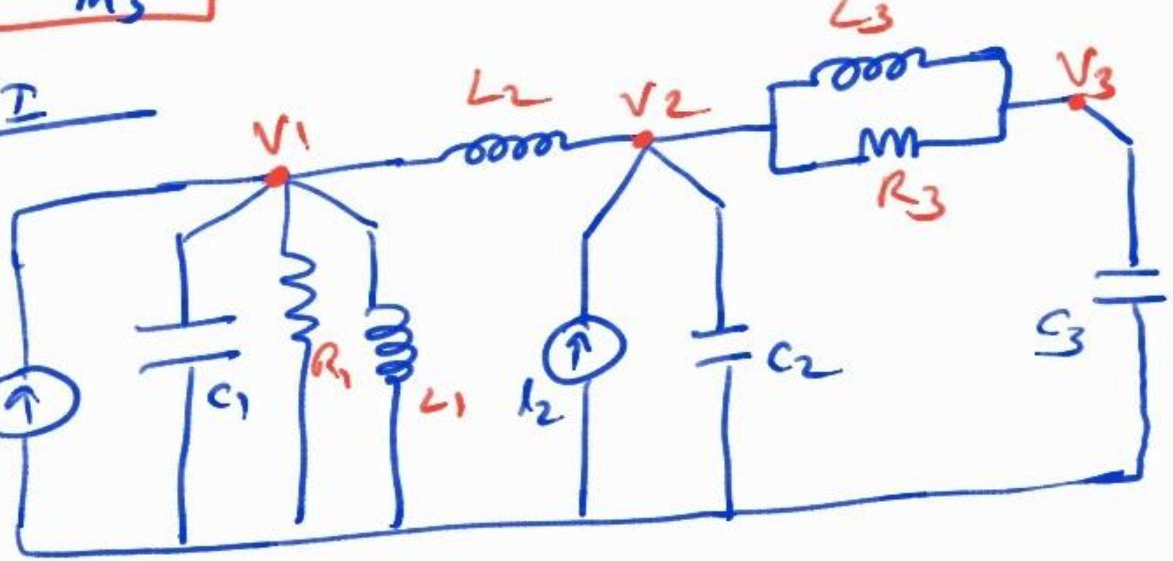


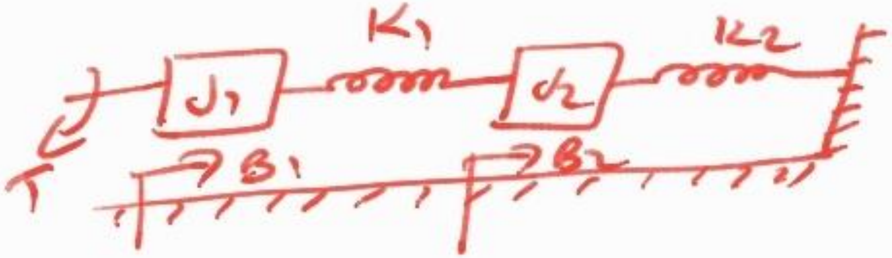
FV



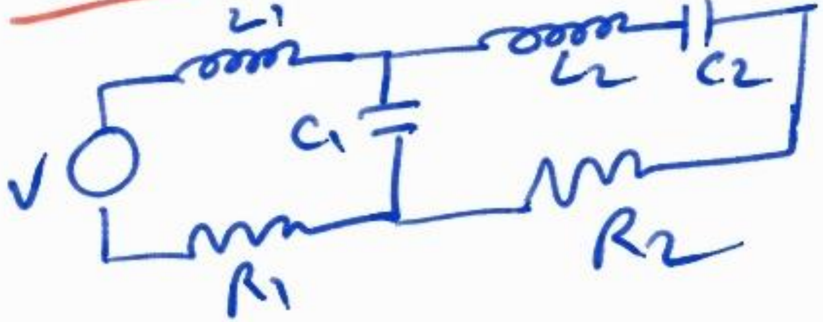


F.I

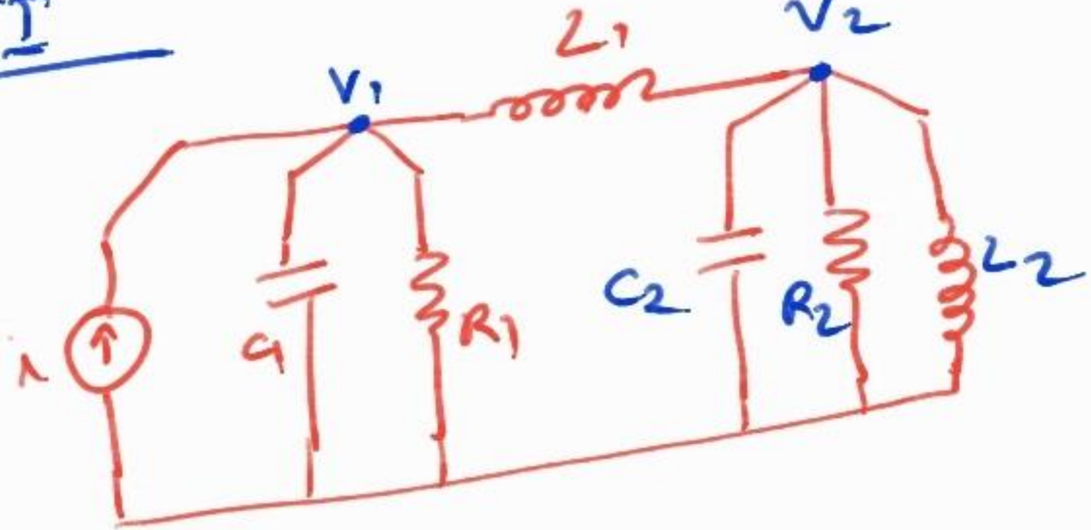




TV



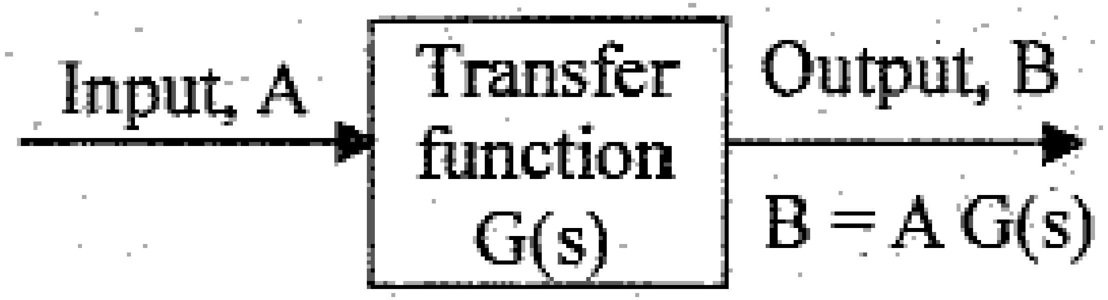
TI



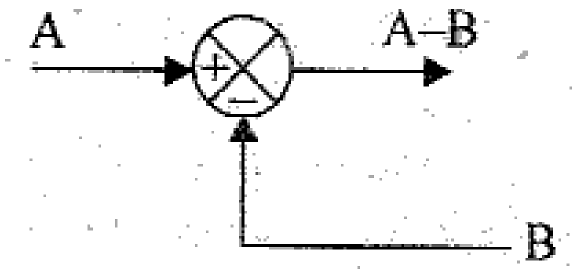
BLOCK DIAGRAM

A block diagram of a system is a pictorial representation of the functions performed by each component and of the flow of signals.
The elements of a block diagram are block, branch point and summing point.

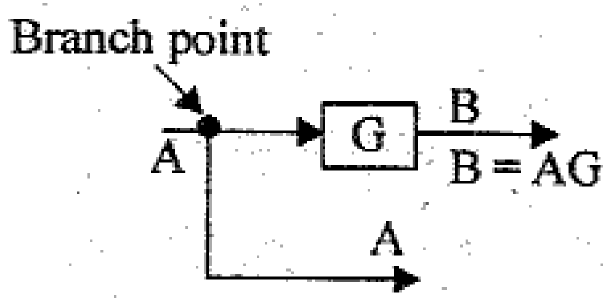
Block - is a symbol for the mathematical operation on the input signal to the block that produces the output
The transfer function of the components are usually ended in the corresponding blocks.



Summing point - is used to add two or more signals in the system
'+' or '-' sign at each arrowhead indicates whether the signal is to be added or subtracted.



Branch points – is a point from which the signal from a block goes concurrently to other blocks or summing points.



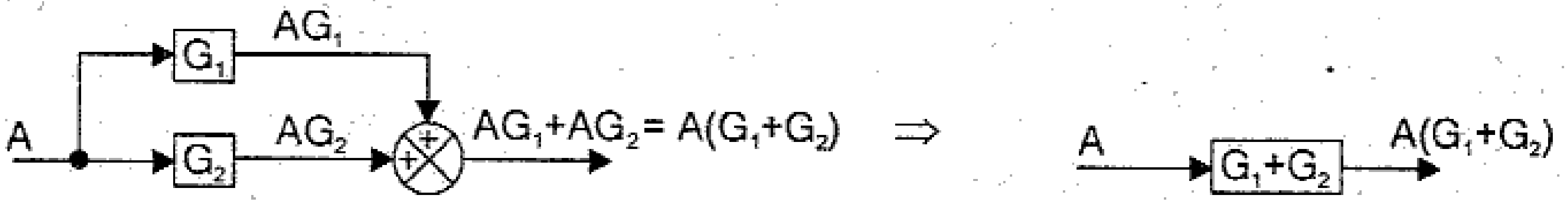
BLOCK DIAGRAM REDUCTION

The block diagram can be reduced to find the overall transfer function of the system.

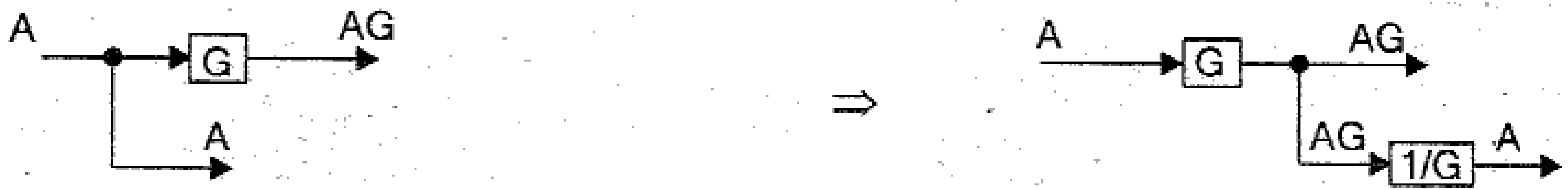
Rules of block diagram algebra

Rule-1 : ***Combining the blocks in cascade***

Rule-2 : ***Combining Parallel blocks (or combining feed forward paths)***

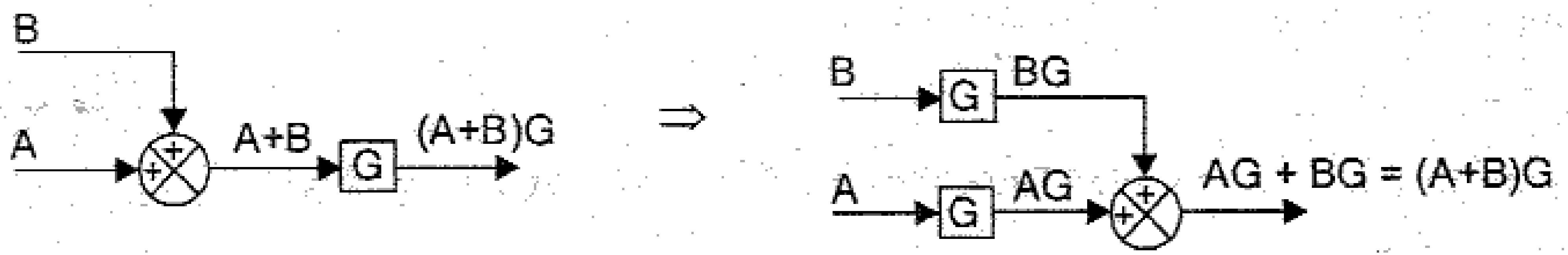


Rule-3 : ***Moving the branch point ahead of the block***



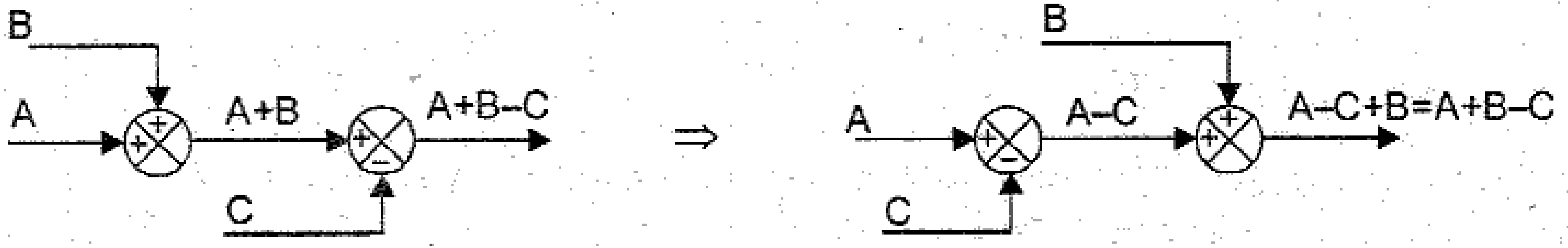
Rule-4 : ***Moving the branch point before the block***

Rule-5 : ***Moving the summing point ahead of the block***

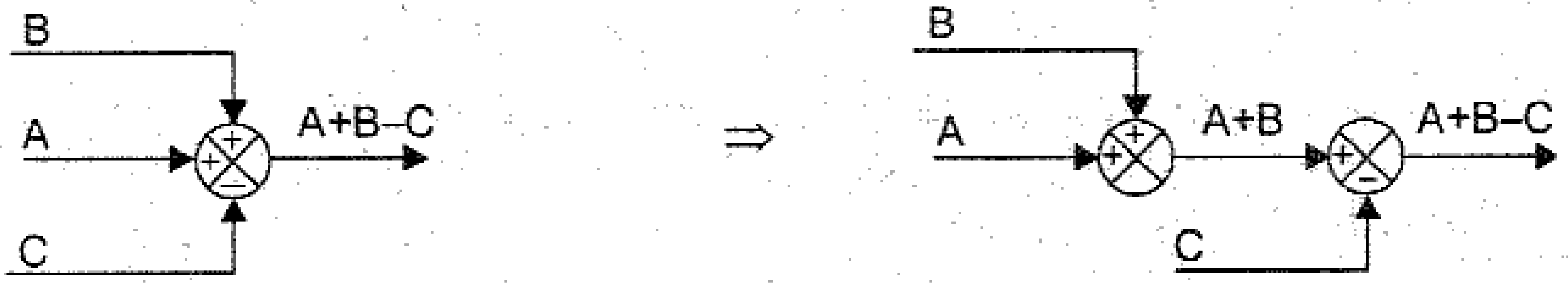


Rule-6 : ***Moving the summing point before the block***

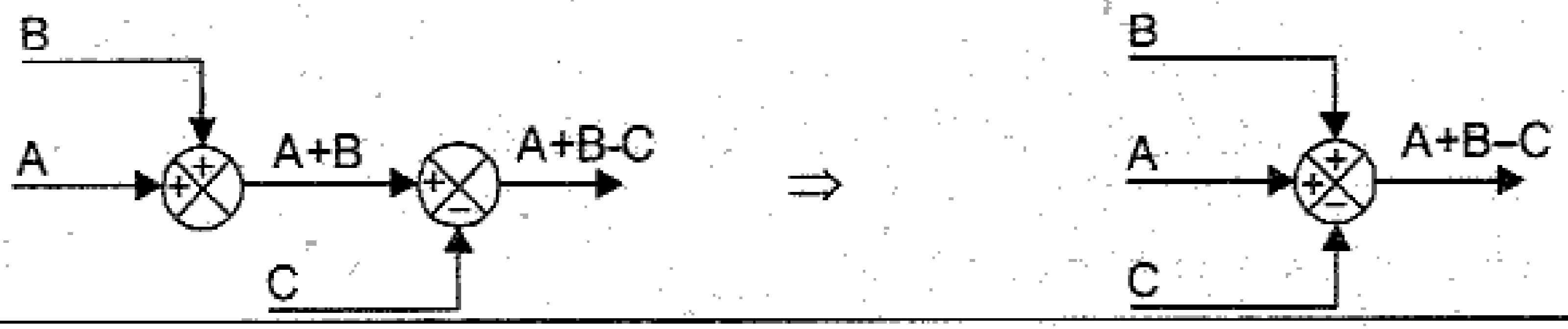
Rule-7 : ***Interchanging summing point***



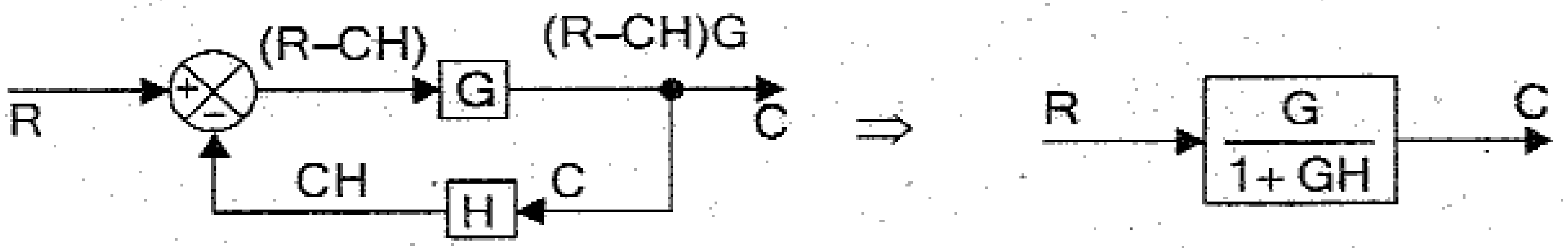
Rule-8 : ***Splitting summing points***



Rule-9 : *Combining summing points*



Rule-10 : *Elimination of (negative) feedback loop*



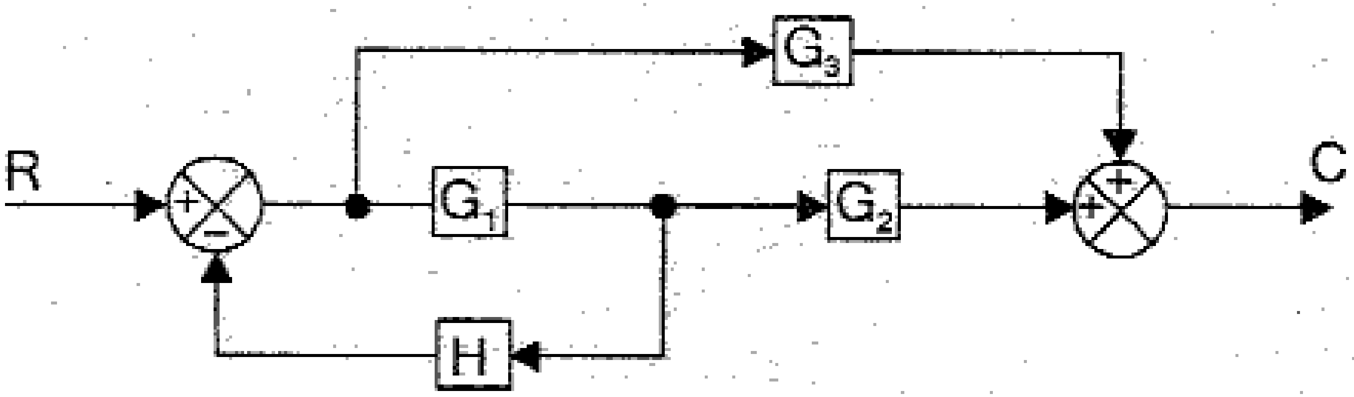
Proof:

$$C = (R - CH)G \quad \Rightarrow \quad C = RG - CHG \quad \Rightarrow \quad C + CHG = RG$$

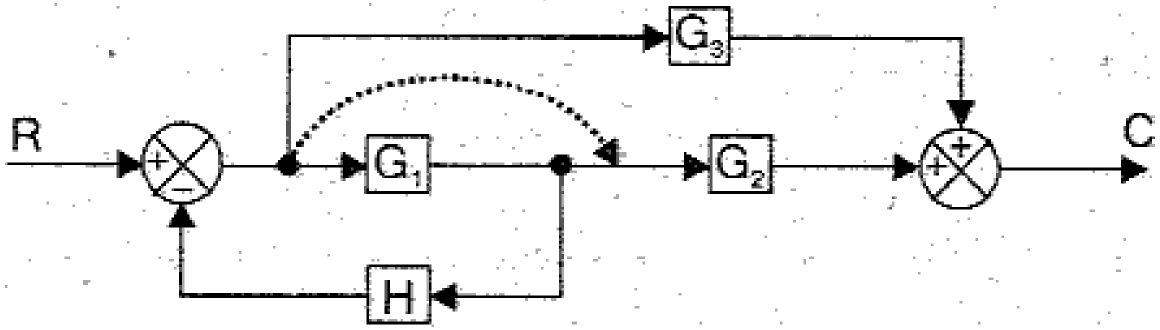
$$\therefore C(1 + HG) = RG \quad \Rightarrow \quad \frac{C}{R} = \frac{G}{1 + GH}$$

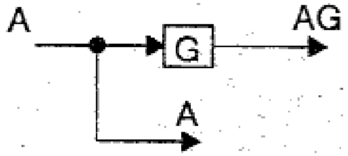
Rule-11 : *Elimination of (positive) feedback loop*

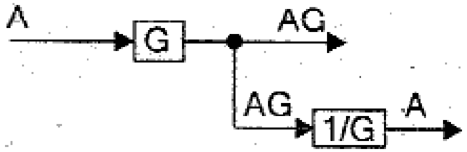
Reduce block diagram and find C(s)/R(s)

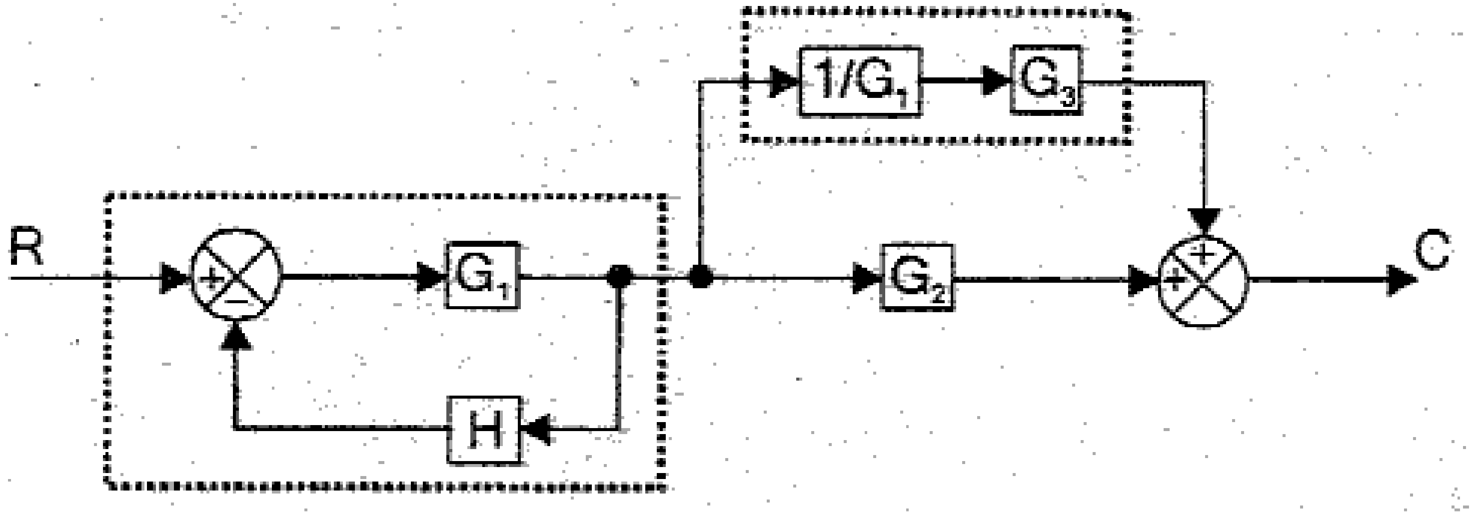


Step 1: Move the branch point after the block.

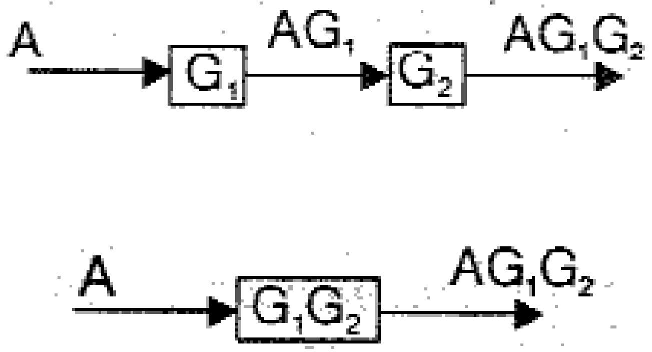




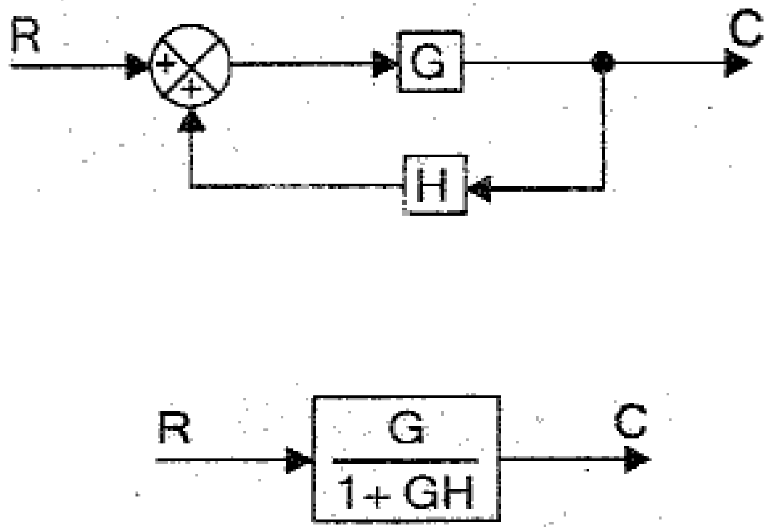




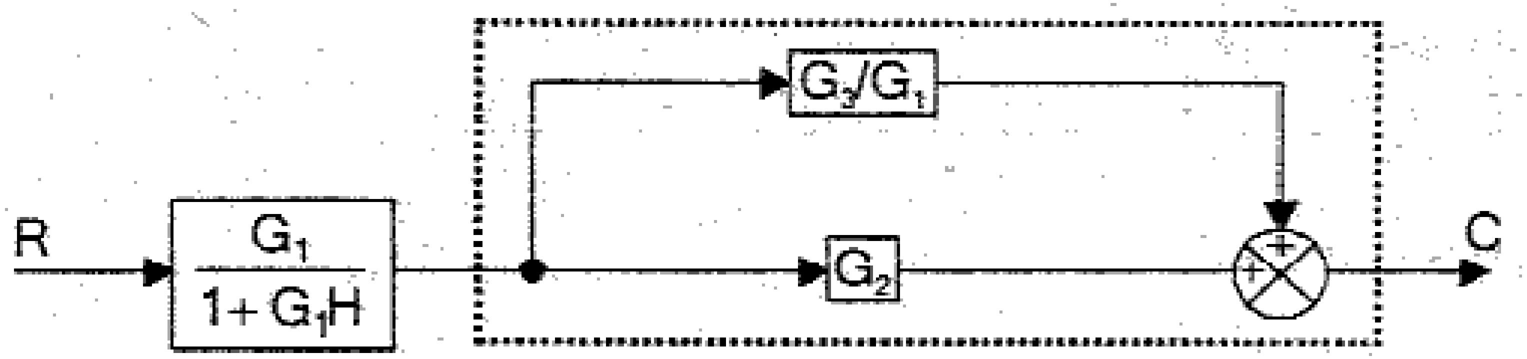
R
+
−
G1
H
1/G1
G3
G2
+
+
C



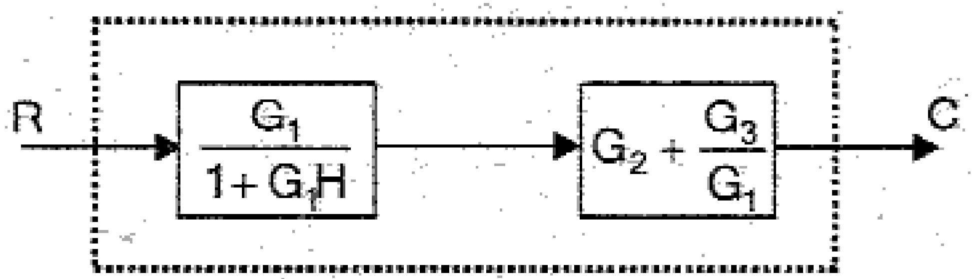
A
G1
AG1
G2
AG1G2
A
G1G2
AG1G2



R
+
+
G
H
C
R
G
1+ GH
C

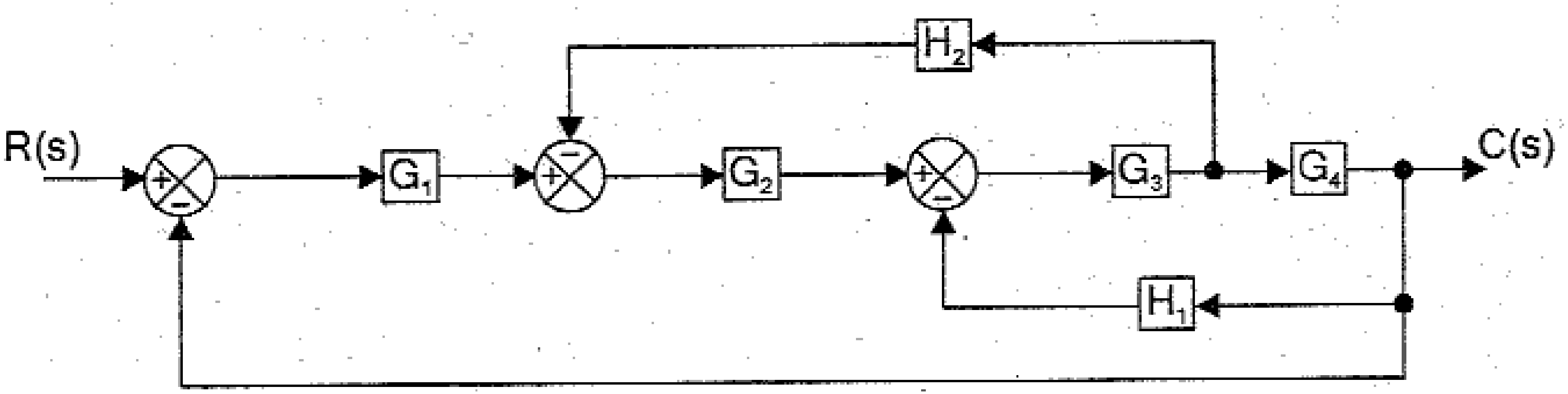


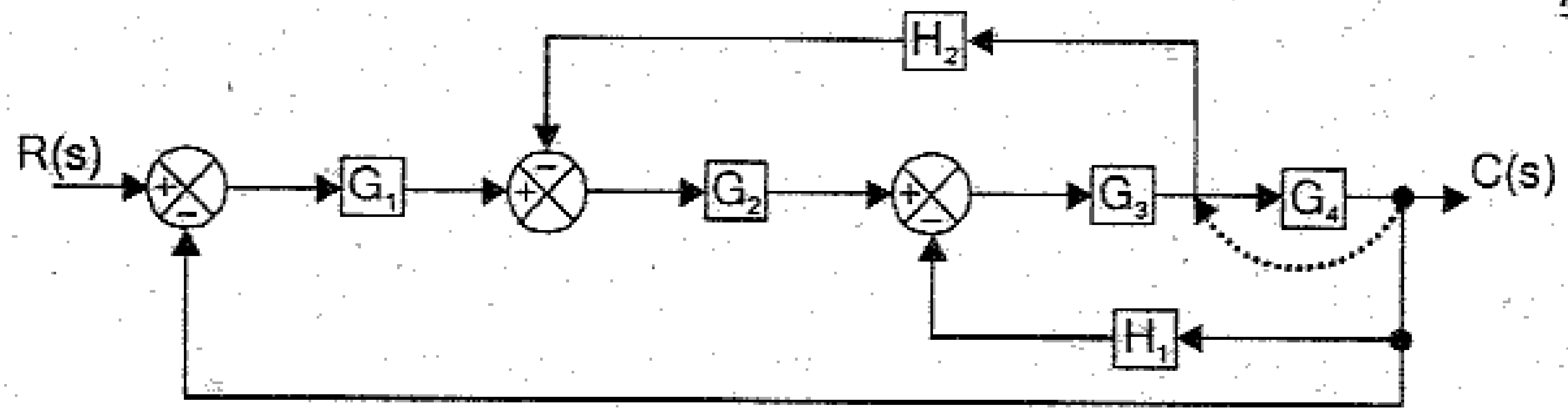
R
G1
1+ G1H
G3/G1
G2
+
+
C

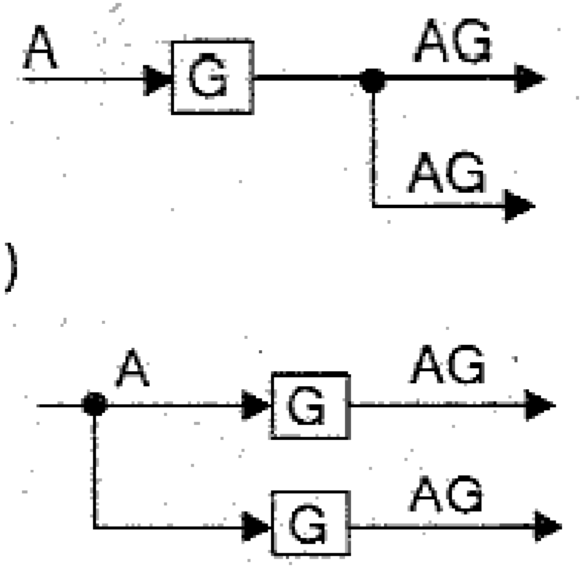


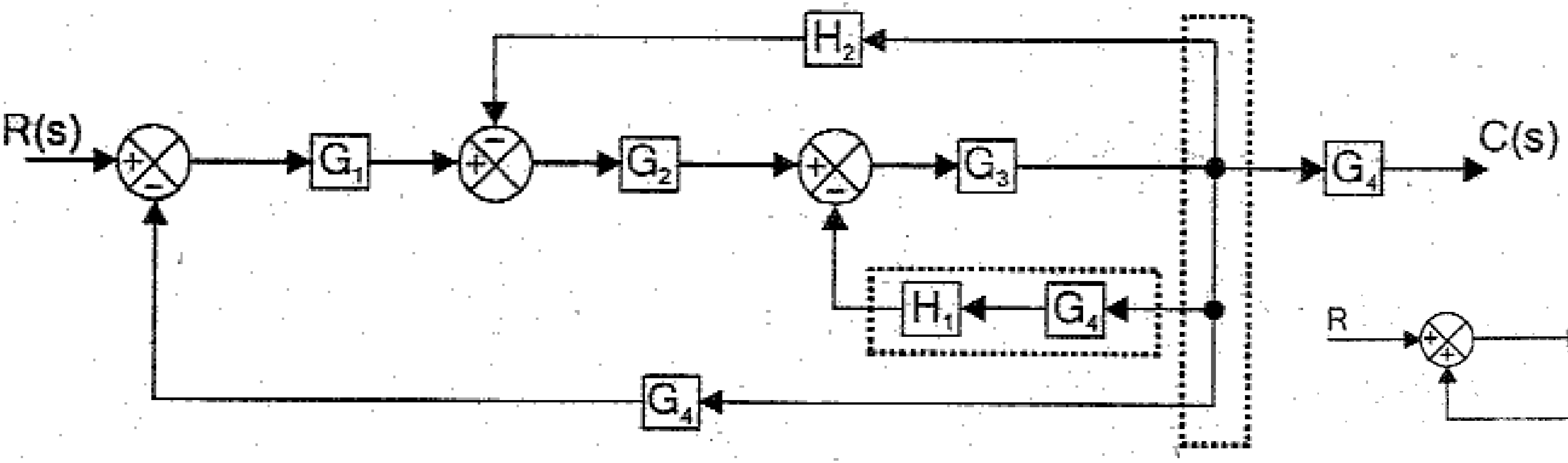
$$\frac{C}{R} = \left(\frac{G_1}{1+G_1H}\right)\left(G_2 + \frac{G_3}{G_1}\right) = \left(\frac{G_1}{1+G_1H}\right)\left(\frac{G_1G_2 + G_3}{G_1}\right) = \frac{G_1G_2 + G_3}{1+G_1H}$$

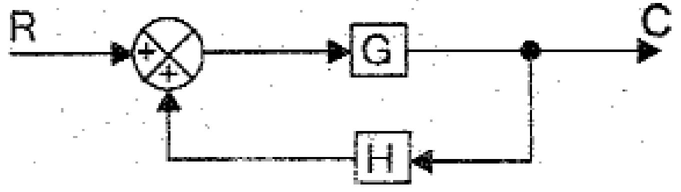
Reduce block diagram and find C(s)/R(s)

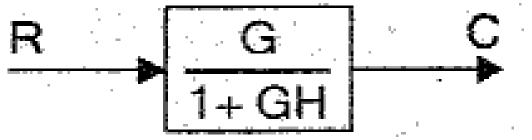


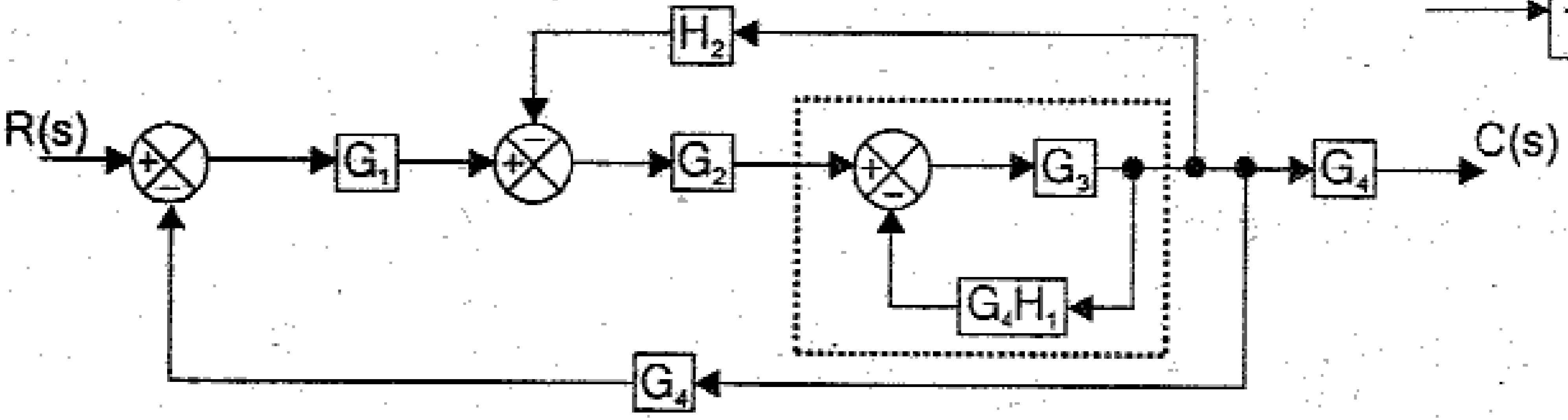


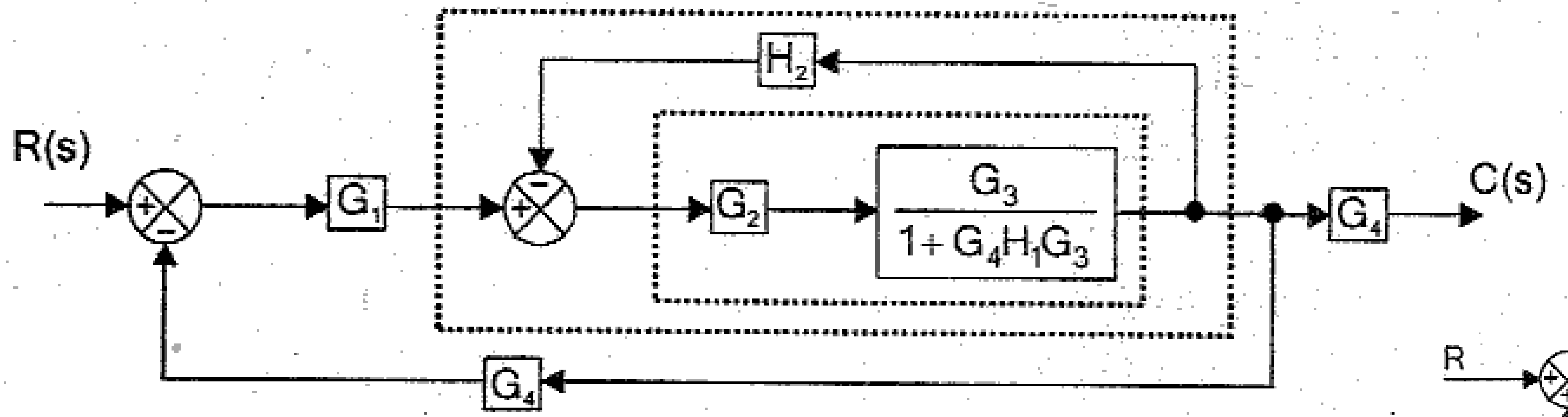


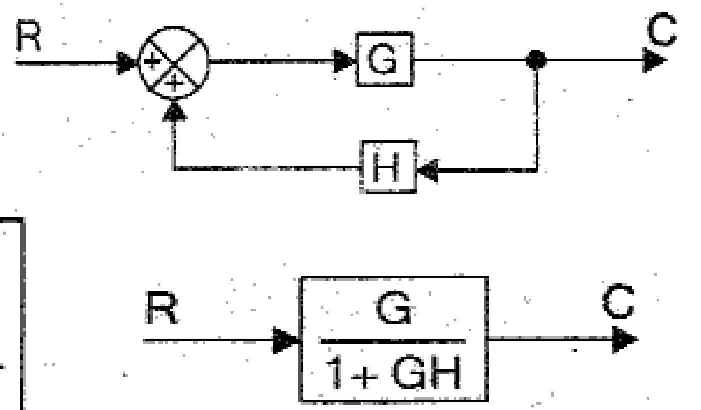

R(s)
G1
G2
G3
G4
C(s)
H2
H1
G4
G4

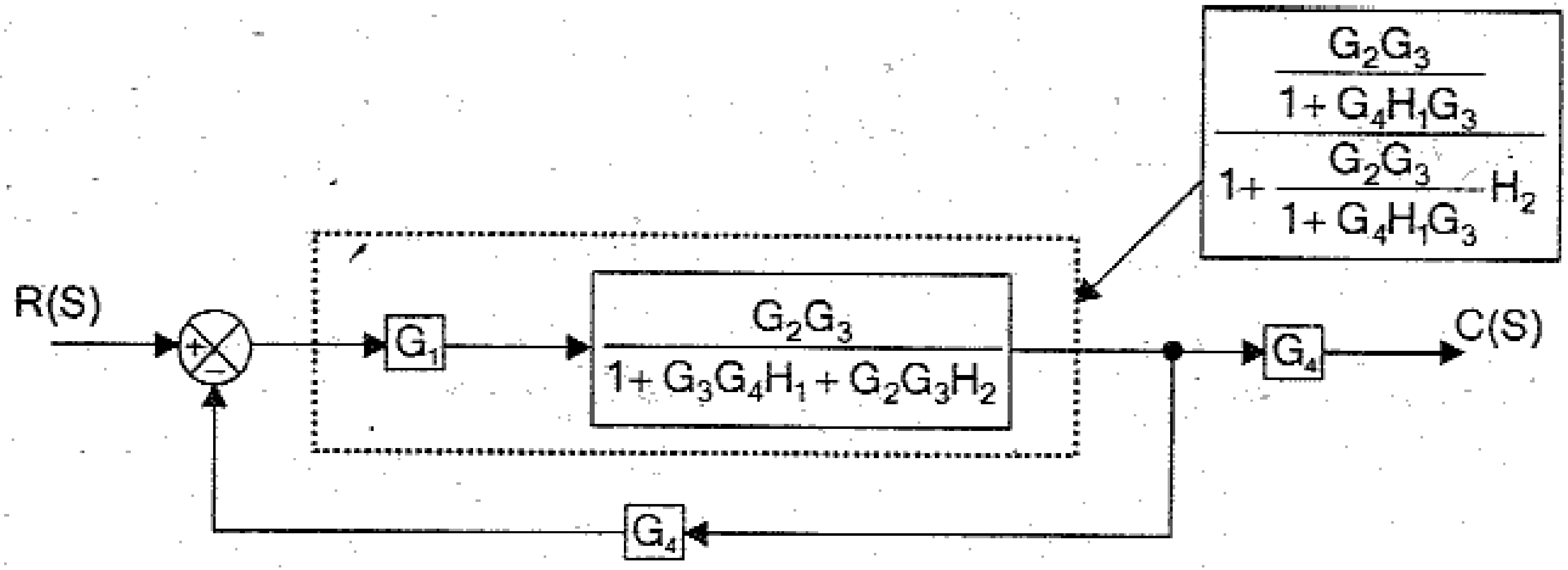

R
G
C
H

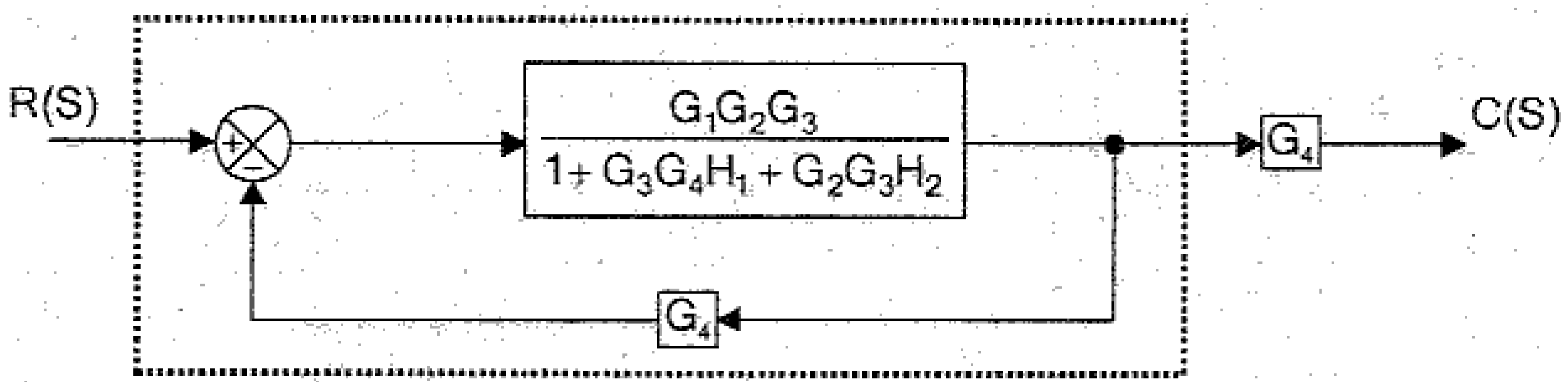

R
G
1+ GH
C

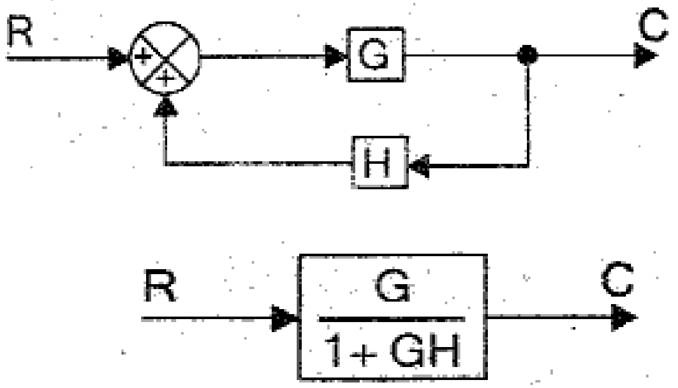

R(s)
G1
G2
G3
G4
C(s)
H2
G4H1
G4

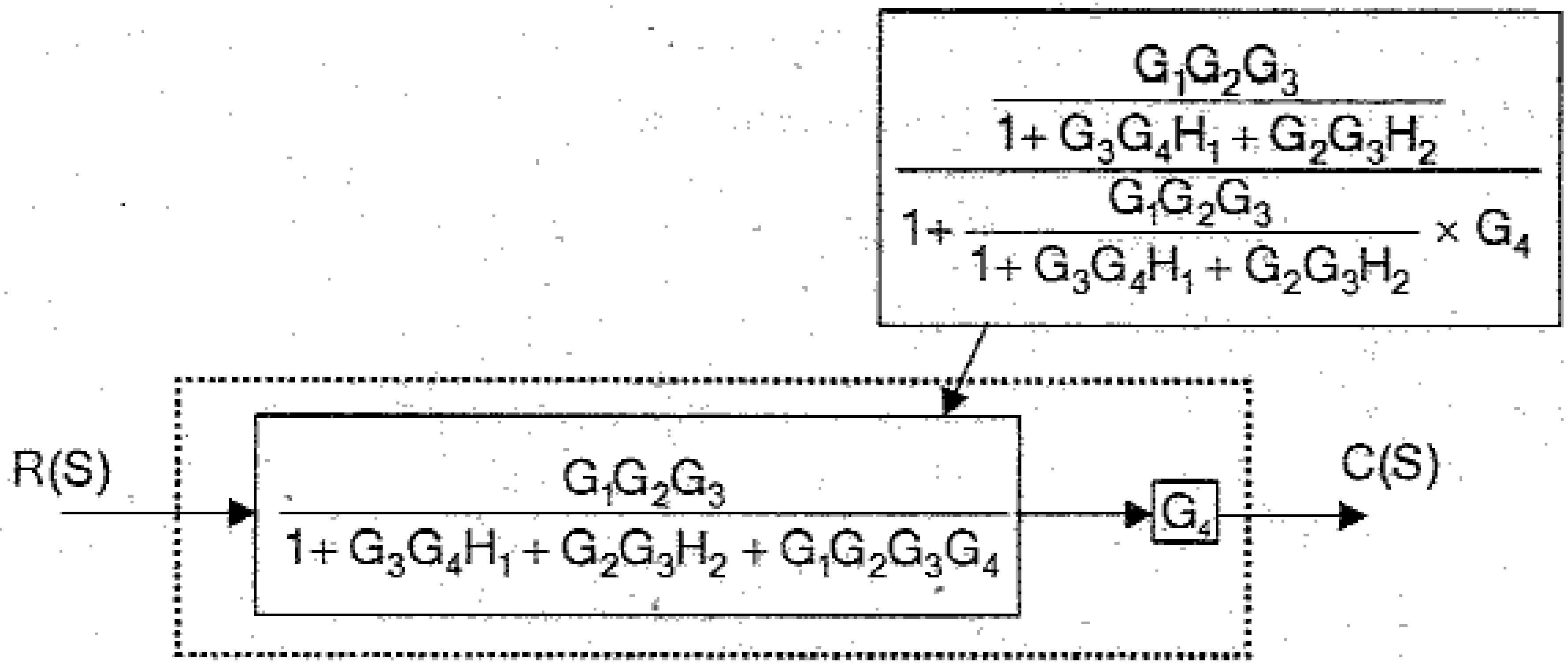

R(s)
G1
H2
G2
G3 / 1+ G4H1G3
G4
C(s)
G4


R
G
C
H
R
G / 1+ GH
C


G2G3 / 1+ G4H1G3 over 1+ G2G3 / 1+ G4H1G3 H2
R(S)
G1
G2G3 / 1+ G3G4H1 + G2G3H2
G4
C(S)
G4

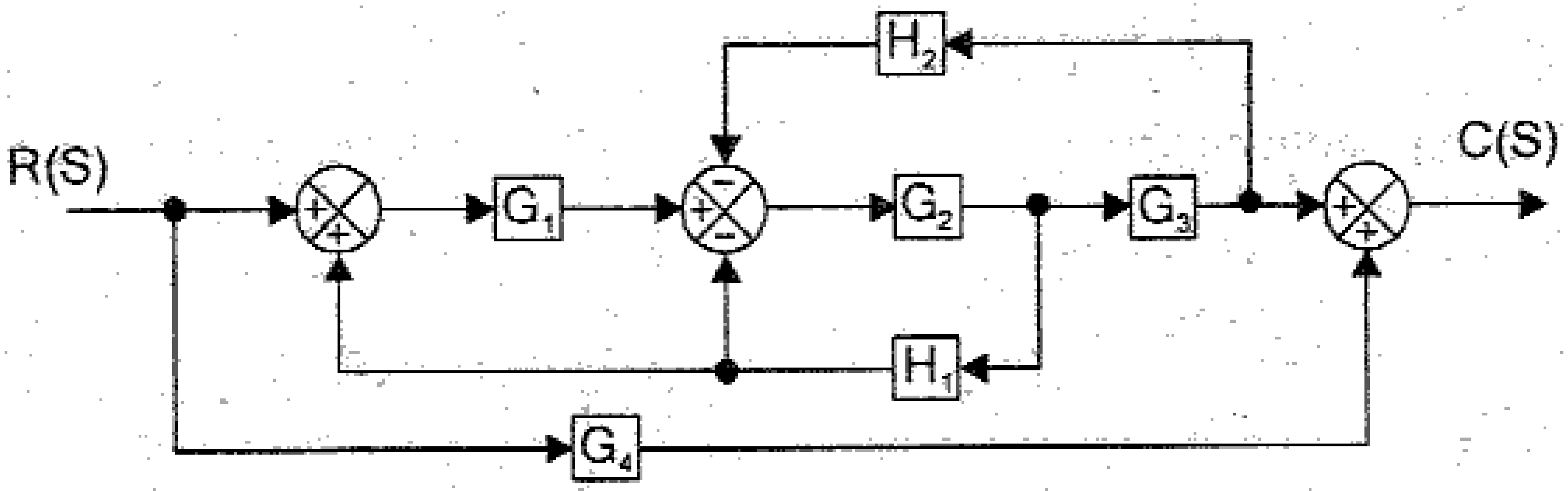

R(S)
+
−
$\frac{G_1G_2G_3}{1+G_3G_4H_1+G_2G_3H_2}$
G_4
G_4
C(S)

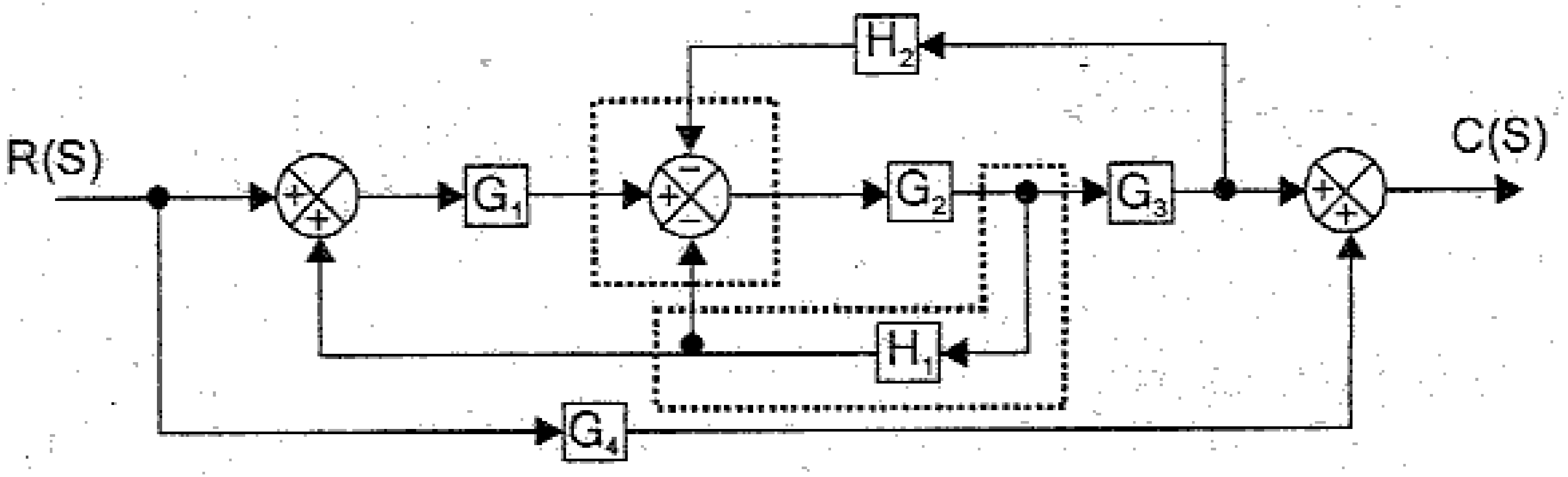

R
+
+
G
C
H
R
$\frac{G}{1+GH}$
C

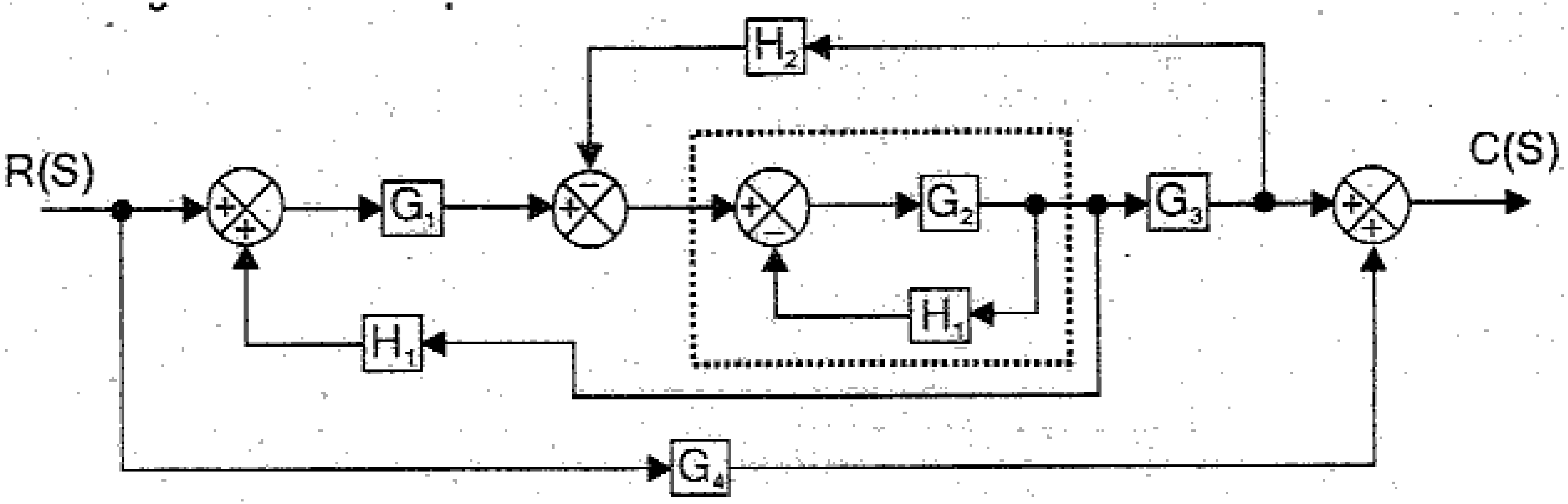

$\frac{\frac{G_1G_2G_3}{1+G_3G_4H_1+G_2G_3H_2}}{1+\frac{G_1G_2G_3}{1+G_3G_4H_1+G_2G_3H_2}\times G_4}$
R(S)
$\frac{G_1G_2G_3}{1+G_3G_4H_1+G_2G_3H_2+G_1G_2G_3G_4}$
G_4
C(S)

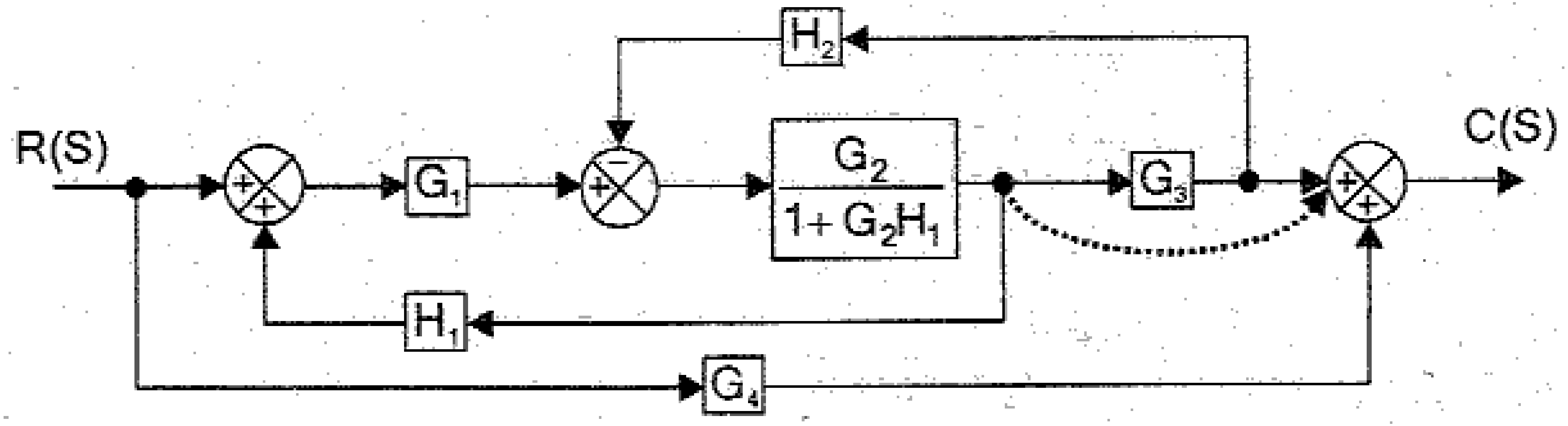
$$\frac{C(s)}{R(s)} = \frac{G_1G_2G_3G_4}{1 + G_3G_4H_1 + G_2G_3H_2 + G_1G_2G_3G_4}$$

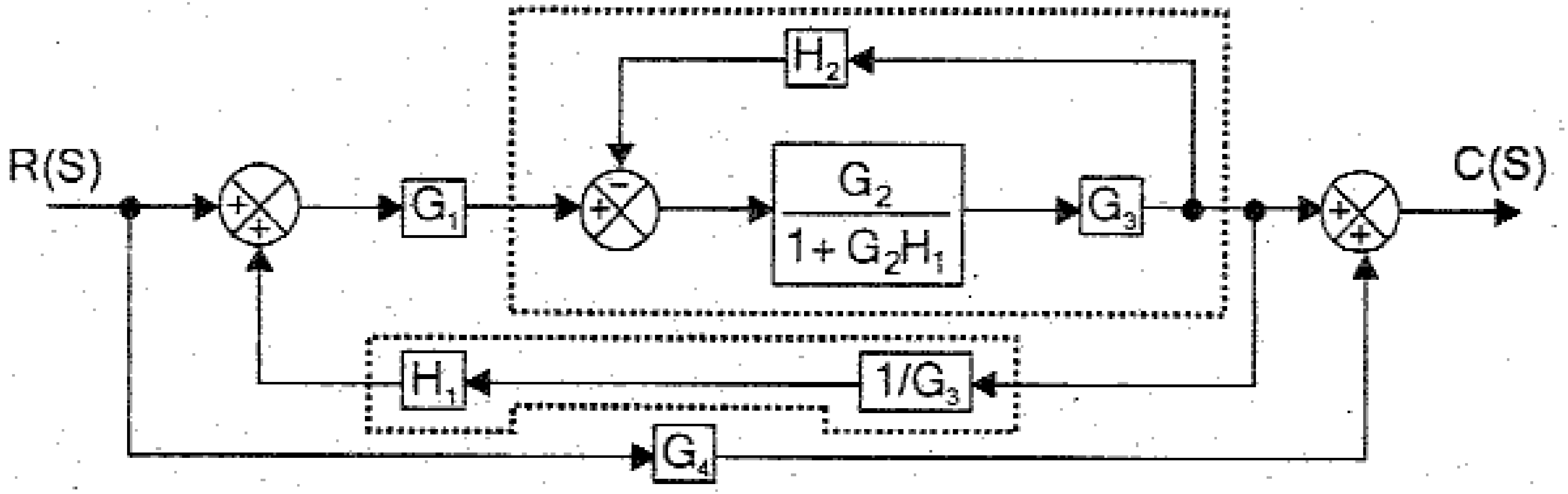
Reduce block diagram and find C(s)/R(s)



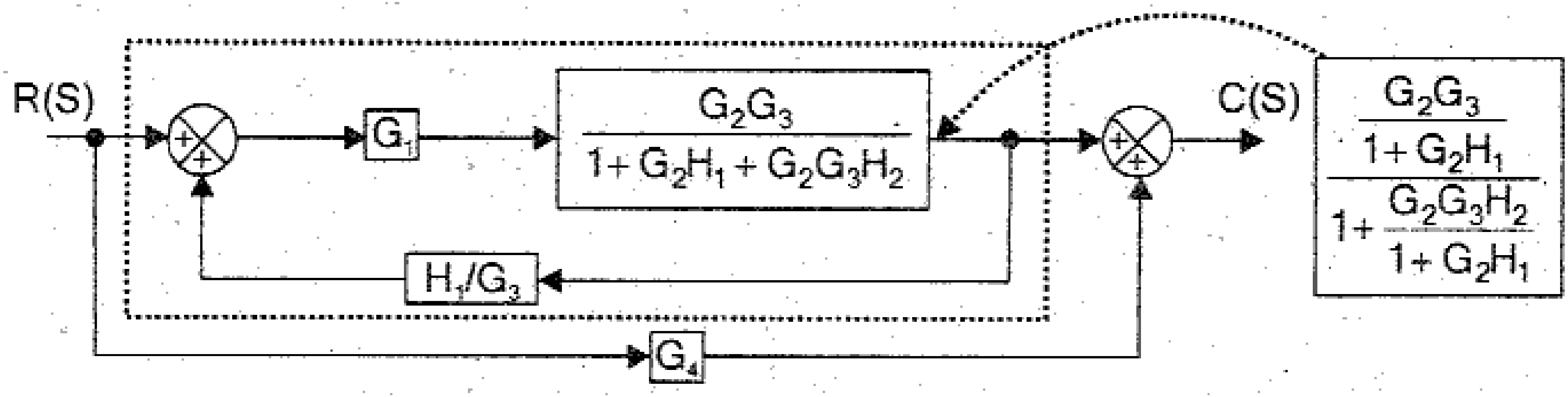



R(S)
C(S)
G_1
G_2
G_3
G_4
H_1
H_1
H_2


R(S)
C(S)
G_1
$\frac{G_2}{1+G_2H_1}$
G_3
G_4
H_1
H_2

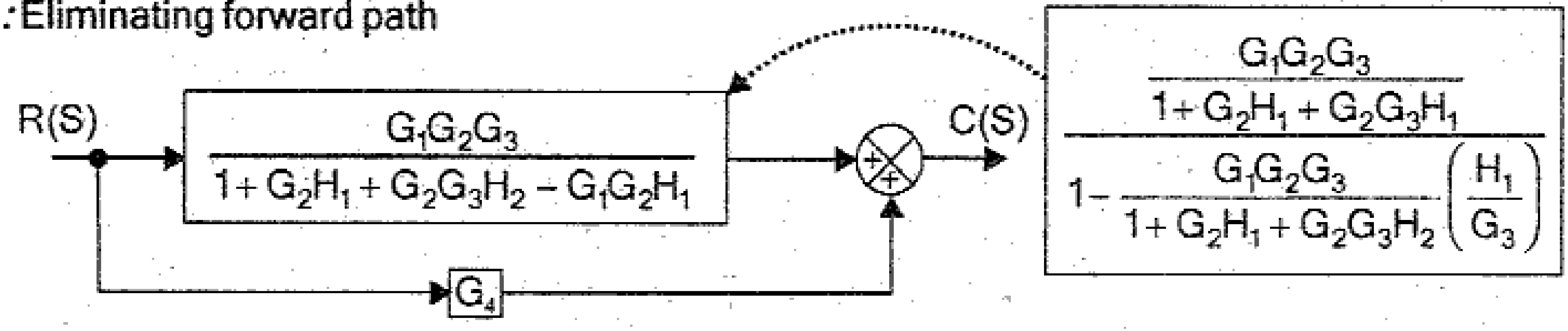


R(S)
G_1
H_2
$\frac{G_2}{1+G_2H_1}$
G_3
C(S)
H_1
$1/G_3$
G_4



R(S)
G_1
$\frac{G_2G_3}{1+G_2H_1+G_2G_3H_2}$
C(S)
H_1/G_3
G_4
$\frac{\frac{G_2G_3}{1+G_2H_1}}{1+\frac{G_2G_3H_2}{1+G_2H_1}}$

∴ Eliminating forward path



$$\frac{C(s)}{R(s)} = \frac{G_1G_2G_3}{1 + G_2H_1 + G_2G_3H_2 - G_1G_2H_1} + G_4$$

Construction of Block Diagram

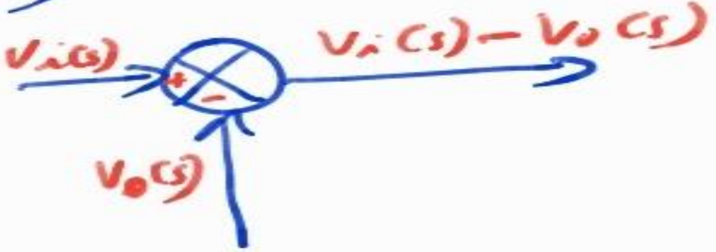
$$i = \frac{V_i - V_0}{R}$$

$$I(s) = \frac{V_i(s) - V_0(s)}{R}$$

$$V_0 = L \cdot \frac{di}{dt}$$

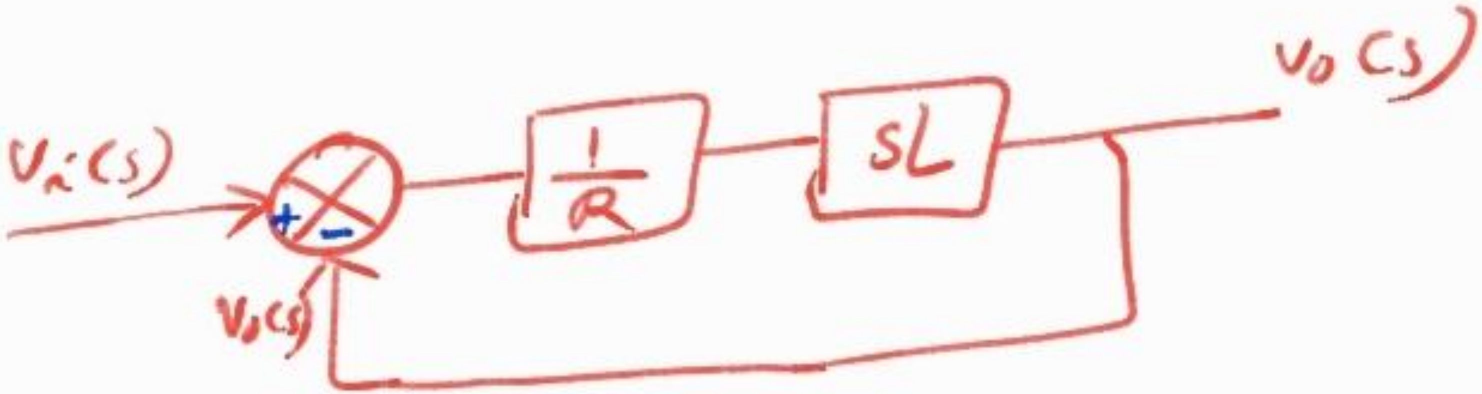
$$V_0(s) = L s I(s)$$

$V_i(s) - V_0(s) \Rightarrow$

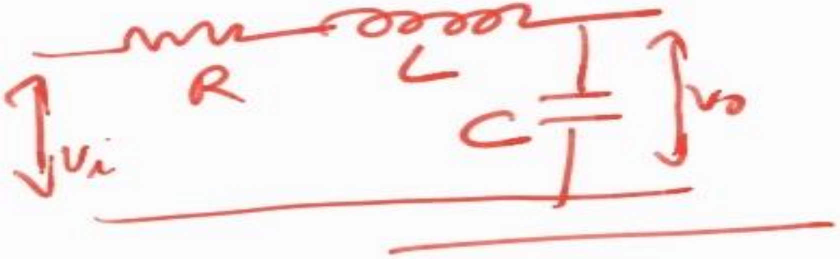


$I(s) =$

Complete the block diagram

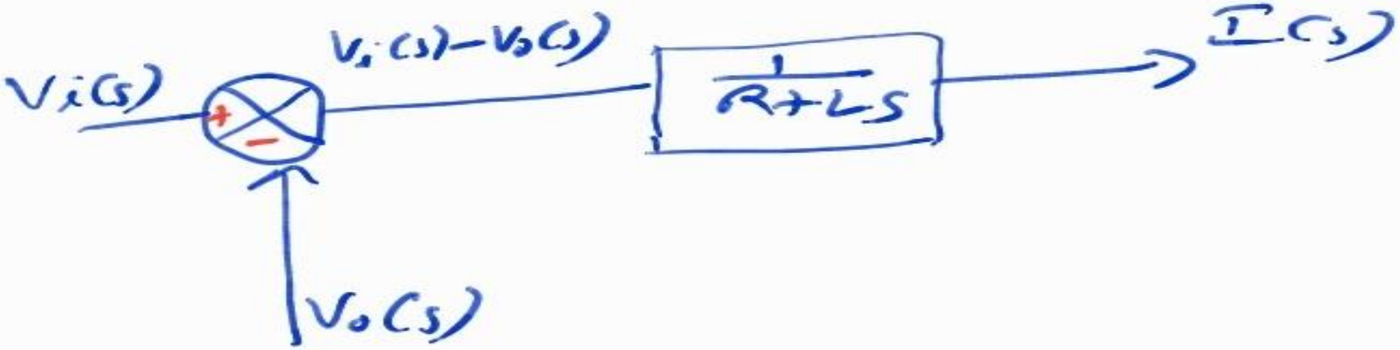


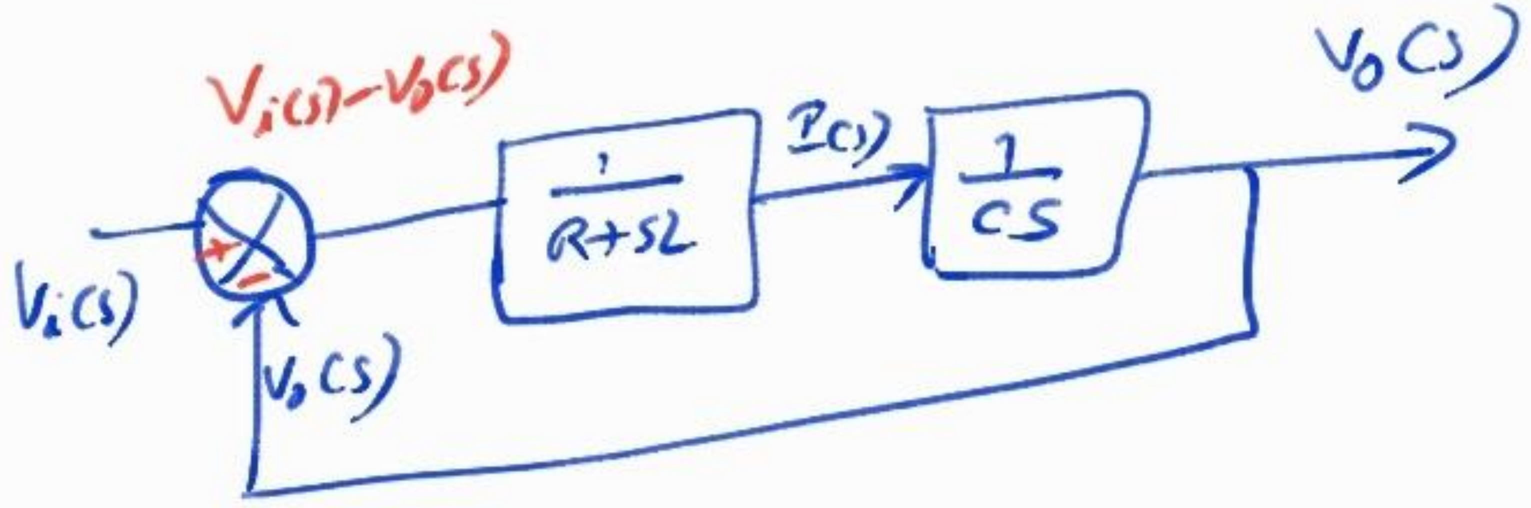
find block diagram



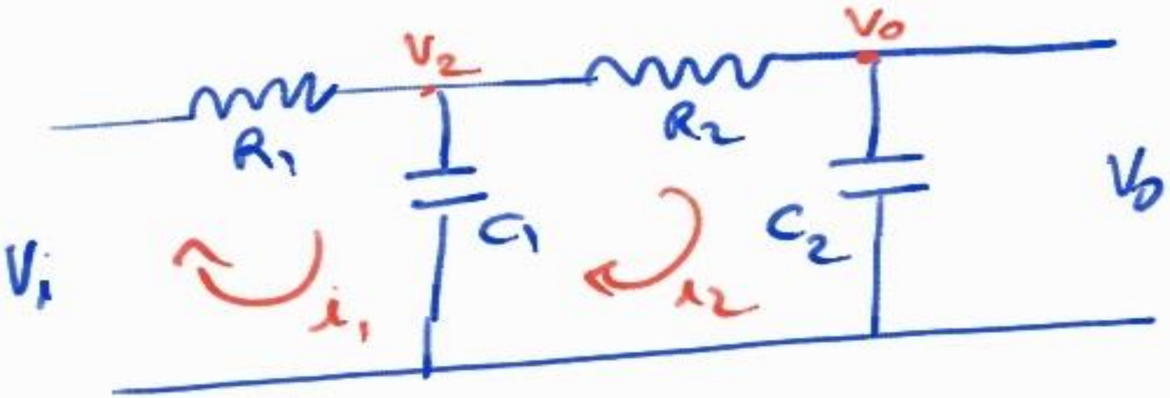
$$I(s) = \frac{V_i(s) - V_o(s)}{R + LS}$$

$$V_o(s) = \frac{1}{CS} I(s) \Rightarrow$$



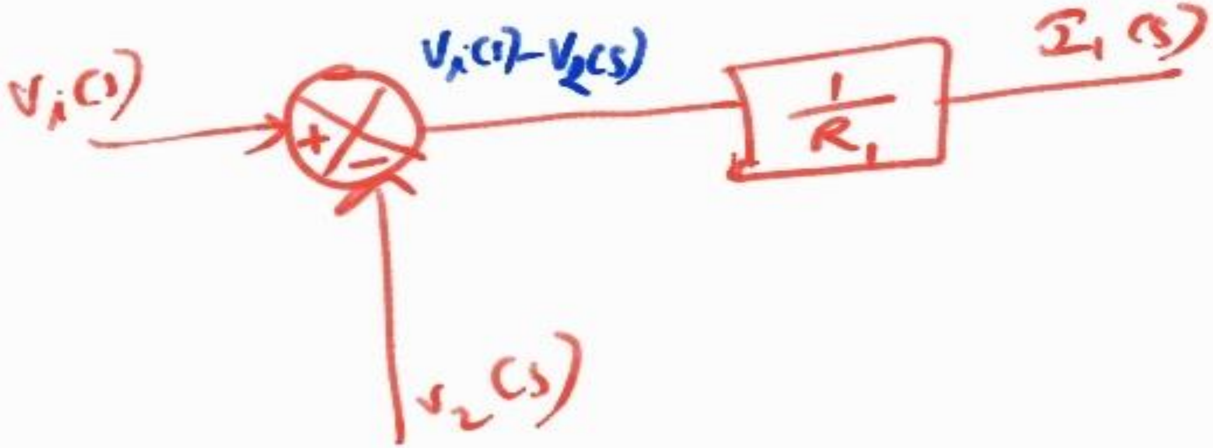


Vi(s)-Vo(s)
Vi(s)
I(s)
1/(R+SL)
1/CS
Vo(s)
Vo(s)



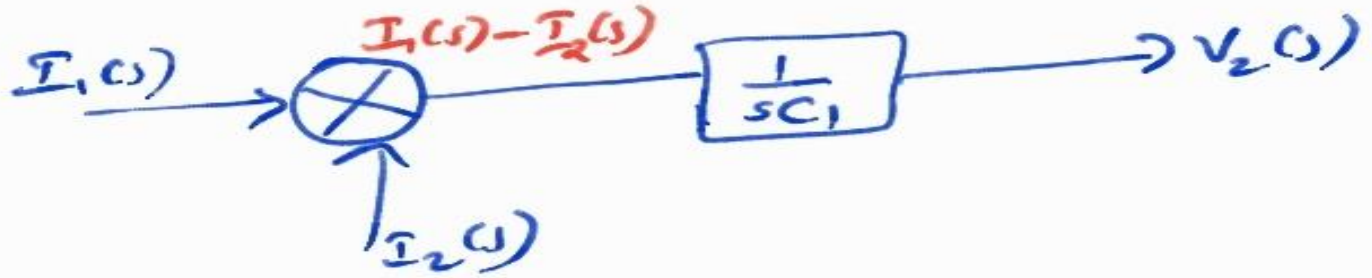
$$i_1 = \frac{V_i - V_2}{R_1}$$

$$I_1(s) = \frac{V_i(s) - V_2(s)}{R_1}$$



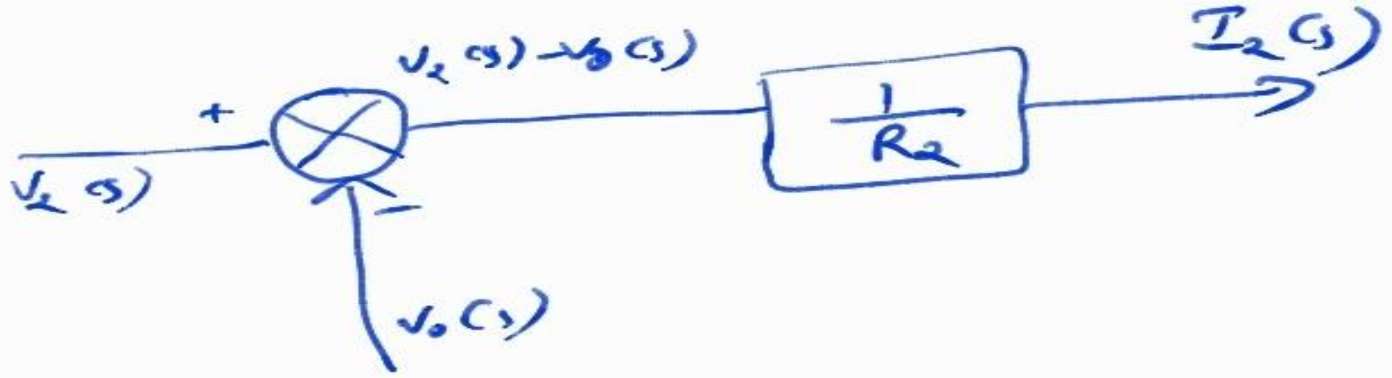
$$V_2 = \frac{1}{C_1}\int (i_1 - i_2)\,dt$$

$$V_2(s) = \frac{1}{C_1 s}\left[I_1(s) - I_2(s)\right]$$



$$i_2 = \frac{1}{R_2}\left[V_2 - V_0\right]$$

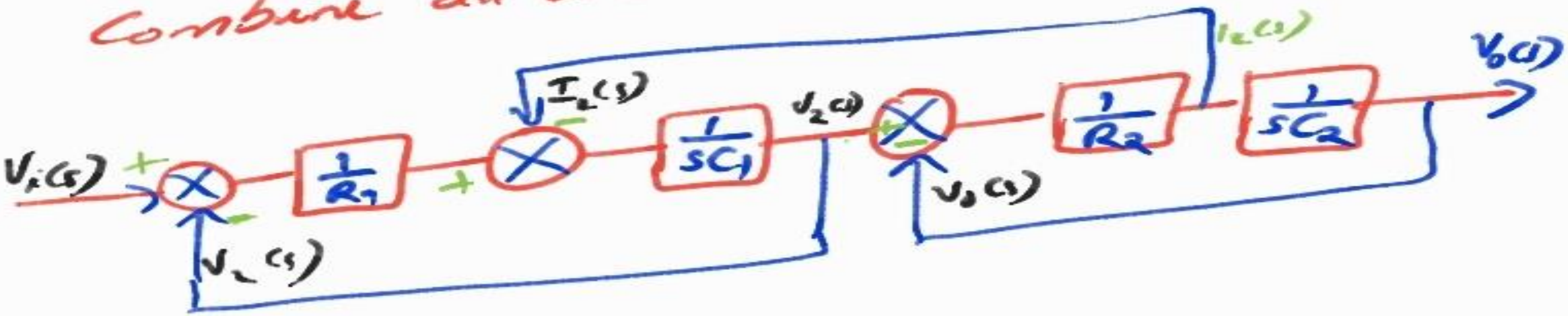
$$I_2(s) = \frac{1}{R_2}\left[V_2(s) - V_0(s)\right]$$



$$V_o = \frac{1}{C_2}\int i_2\,dt$$

$$V_o(s) = \frac{1}{C_2 s} I_2(s)$$

Combine all the blocks



TRANSFER FUNCTION OF ARMATURE CONTROLLED DC MOTOR

The speed of DC motor is directly proportional to armature voltage and inversely proportional to flux in the field winding

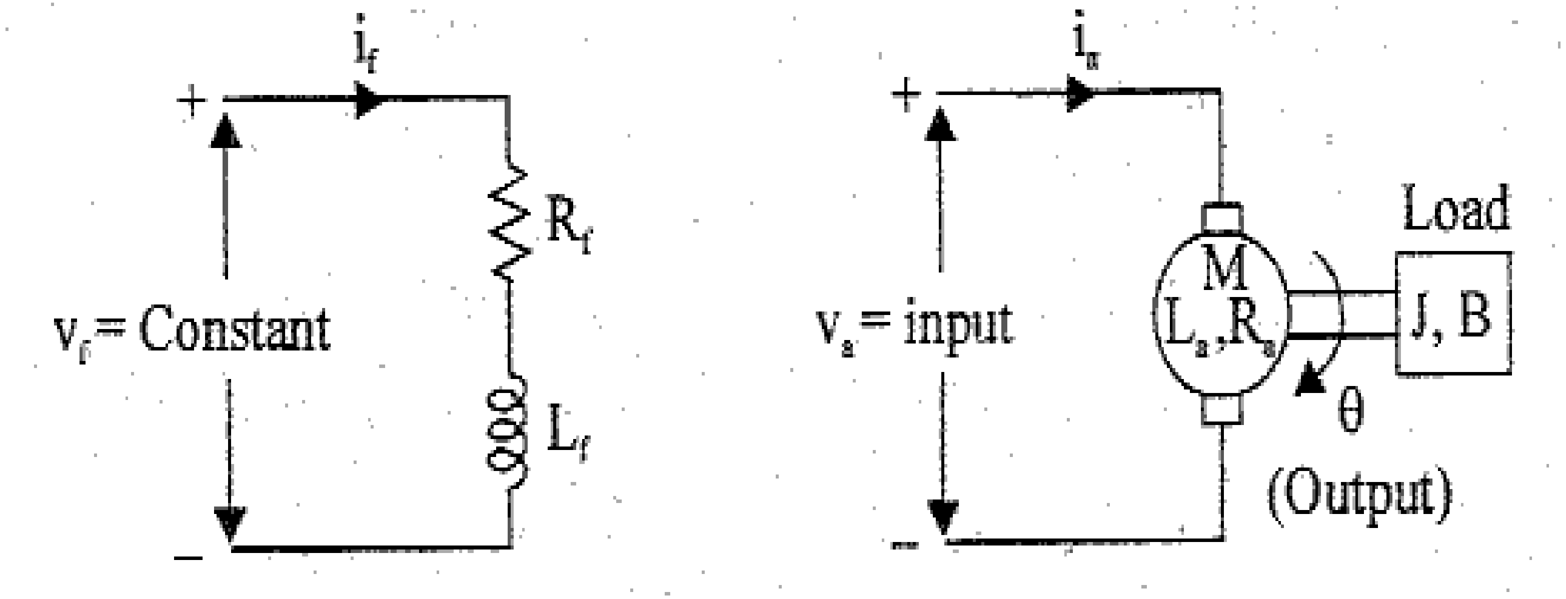
In armature controlled DC motor the desired speed is obtained by varying the armature voltage

This speed control system is an electromechanical control system

The electrical system consists of armature and field circuit but for analysis purpose only the armature circuit is considered because the field is excited by a constant voltage.

The mechanical system consists of the rotating part of the motor and load connected to the shaft of the motor

Armature controlled DC motor



R_a = Armature resistance, Ω

L_a = Armature inductance, H

i_a = Armature current, A

v_a = Armature voltage, V

e_b = Back emf, V

K_t = Torque constant, N-m/A

T = Torque developed by motor, N-m

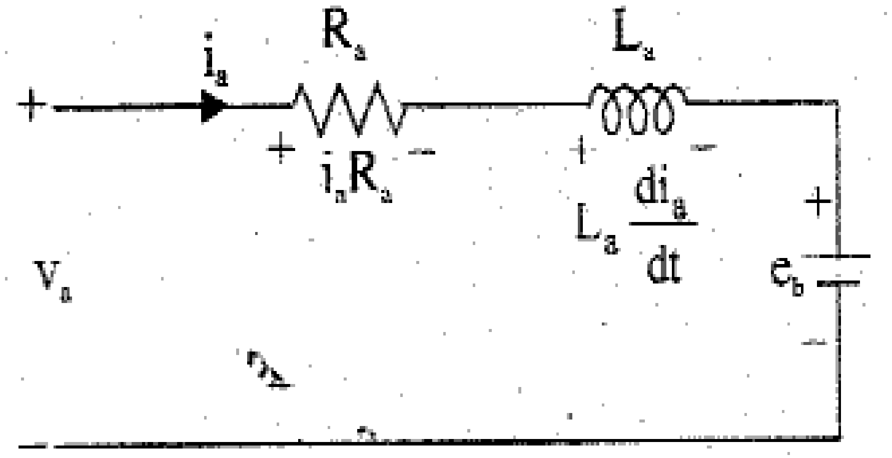
θ = Angular displacement of shaft, rad

J = Moment of inertia of motor and load, Kg-m²/rad

B = Frictional coefficient of motor and load, N-m/(rad/sec)

K_b = Back emf constant, V/(rad/sec)

Armature equivalent circuit



By Kirchoff’s voltage law

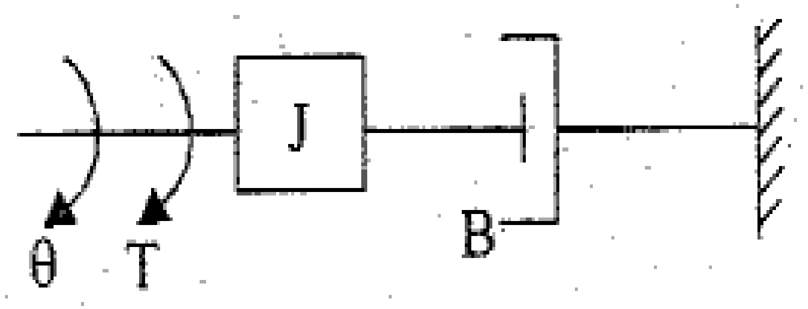
$$i_aR_a + L_a\frac{di_a}{dt} + e_b = v_a \qquad (1)$$

Torque of DC motor is proportional to the product of flux and current

$$T \propto i_a$$

$$\text{Torque, } T = K_t\, i_a \qquad (2)$$

The mechanical system of the motor



The differential equation governing the mechanical system of motor

$$J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = T$$ → 3

The back EMF of DC machine is proportional to speed of shaft

$$e_b \propto \frac{d\theta}{dt}$$

$$\text{Back emf, } e_b = K_b\frac{d\theta}{dt}$$ → 4

Take the Laplace Transform of all equation

$$I_a(s)\,R_a + L_a s\,I_a(s) + E_b(s) = V_a(s) \quad \rightarrow \quad 5$$

$$T(s) = K_t I_a(s) \quad \rightarrow \quad 6$$

$$Js^2\theta(s) + B\,s\,\theta(s) = T(s) \quad \rightarrow \quad 7$$

$$E_b(s) = K_b s\,\theta(s) \quad \rightarrow \quad 8$$

From equation 6 & 7

$$K_t I_a(s) = (Js^2 + Bs)\,\theta(s)$$

$$I_a(s) = \frac{(Js^2 + Bs)}{K_t}\,\theta(s) \quad \rightarrow \quad 9$$

On rearranging $V_a(S)$

$$(R_a + sL_a)\, I_a(s) + E_b(s) = V_a(s) \quad \longrightarrow \quad 10$$

Substitute the values of $I_a(S)$ and $E_b(S)$ in equation 10

$$(R_a + sL_a)\, \frac{(Js^2 + Bs)}{K_t}\theta(s) + K_b s\, \theta(s) = V_a(s)$$

$$\left[\frac{(R_a + sL_a)\,(Js^2 + Bs) + K_b K_t s}{K_t}\right]\theta(s) = V_a(s)$$

The required function is $\frac{\theta(s)}{V_a(s)}$

$$\therefore \frac{\theta(s)}{V_a(s)} = \frac{K_t}{(R_a + sL_a)(Js^2 + Bs) + K_b K_t s}$$

$$= \frac{K_t}{R_a J s^2 + R_a B s + L_a J s^3 + L_a B s^2 + K_b K_t s}$$

$$= \frac{K_t}{s\left[JL_a s^2 + (JR_a + BL_a)s + (BR_a + K_b K_t)\right]}$$

$$= \frac{K_t / JL_a}{s\left[s^2 + \left(\frac{JR_a + BL_a}{JL_a}\right)s + \left(\frac{BR_a + K_b K_t}{JL_a}\right)\right]}$$

TRANSFER FUNCTION OF FIELD CONTROLLED DC MOTOR

The speed of DC motor is directly proportional to armature voltage and inversely proportional to flux

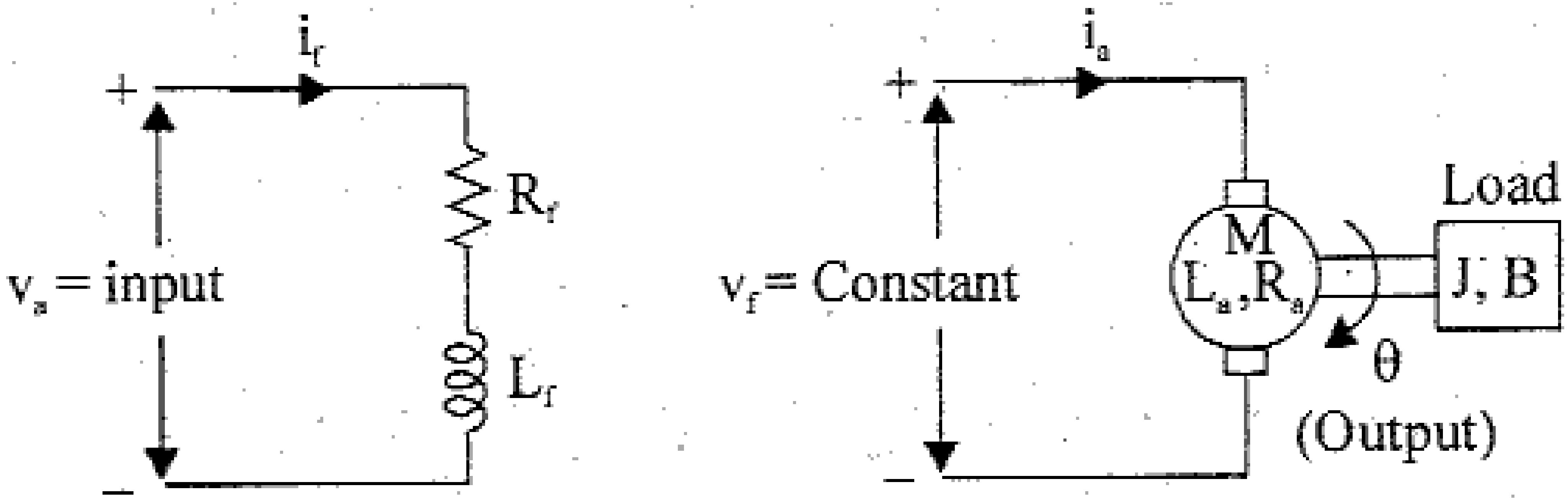
In field controlled DC motor the armature voltage is kept constant and the speed is varied by varying the flux of the machine.

Since flux is directly proportional to field current, the flux is varied by varying field current.

The speed control system is an electromechanical control system

The electrical system consists of armature and field circuit but for analysis purpose only field circuit is considered because the armature is excited by a constant voltage.

The mechanical system consists of the rotating part of the motor and the load connected to the shaft of the motor



i_f
+
R_f
v_a = input
L_f
−
i_a
+
Load
M
L_a , R_a
J, B
v_f = Constant
θ
(Output)
−

R_f = Field resistance, Ω

L_f = Field inductance, H

i_f = Field current, A

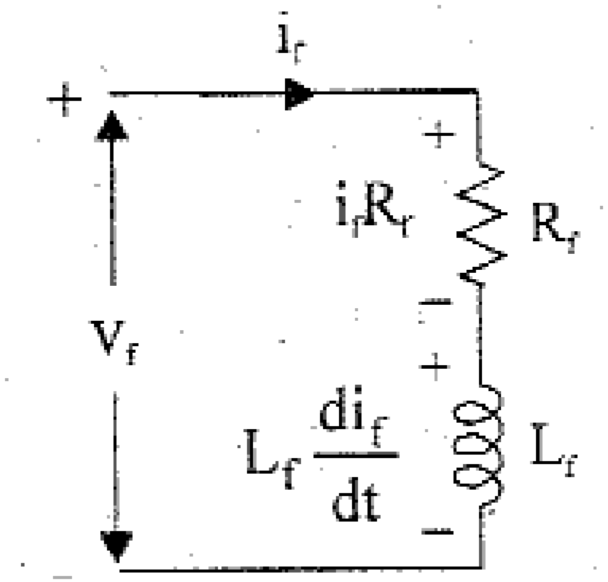
v_f = Field voltage, V

T = Torque developed by motor, N-m

K_{tf} = Torque constant, N-m/A

J = Moment of inertia of rotor and load, Kg-m^2/rad

B = Frictional coefficient of rotor and load, N-m/(rad/sec)



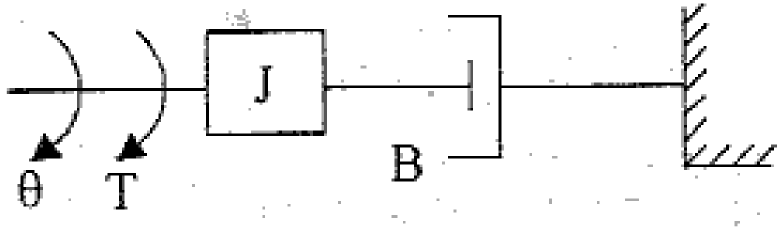
$$R_f i_f + L_f \frac{di_f}{dt} = v_f \qquad (1)$$

The torque of DC motor is proportional to product of flux and armature current.

Since armature current is constant in this system, the torque is proportional to flux alone, but flux is proportional to field current

$$T \propto i_f, \quad \therefore \text{Torque, } T = K_{tf}\, i_f \qquad (2)$$

The mechanical system of the motor



The differential equation governing the mechanical system of the motor

$$J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = T \qquad (3)$$

Take the Laplace transform of all equation

$$R_f I_f(s) + L_f s I_f(s) = V_f(s) \qquad (4)$$

$$T(s) = K_{tf} I_f(s) \qquad (5)$$

$$Js^2\theta(s) + Bs\theta(s) = T(s) \qquad (6)$$

From equation 5 and 6

$$K_{tf}\, I_f(s) = Js^2\theta(s) + Bs\theta(s)$$

$$I_f(s) = s\,\frac{(Js + B)}{K_{tf}}\theta(s) \quad \rightarrow \quad 7$$

Rearranging equation 4

$$(R_f + sL_f)\, I_f(s) = V_f(s) \quad \rightarrow \quad 8$$

Substituting equation 7 in 8

$$(R_f + sL_f)s\,\frac{(Js + B)}{K_{tf}}\,\theta(s) = V_f(s)$$

$$\frac{\theta(s)}{V_f(s)} = \frac{K_{tf}}{s\,(R_f + sL_f)\,(B + sJ)}$$

$$= \frac{K_{tf}}{sR_f\left(1 + \frac{sL_f}{R_f}\right)B\left(1 + \frac{sJ}{B}\right)} = \frac{K_m}{s\,(1 + sT_f)\,(1 + sT_m)}$$

Where

$$K_m = \frac{K_{tf}}{R_f B} = \text{Motor gain constant}$$

$$T_f = \frac{L_f}{R_f} = \text{Field time constant}$$

$$T_m = \frac{J}{B} = \text{Mechanical time constant}$$

SIGNAL FLOW GRAPH

The signal flow graph is used to represent the control system graphically and it was developed by S J mason

A signal flow graph is a diagram that represents a set of simultaneous linear algebraic equations

The advantage in signal flow graph method is that, using mason's gain formula the overall gain of the system can be computed easily.

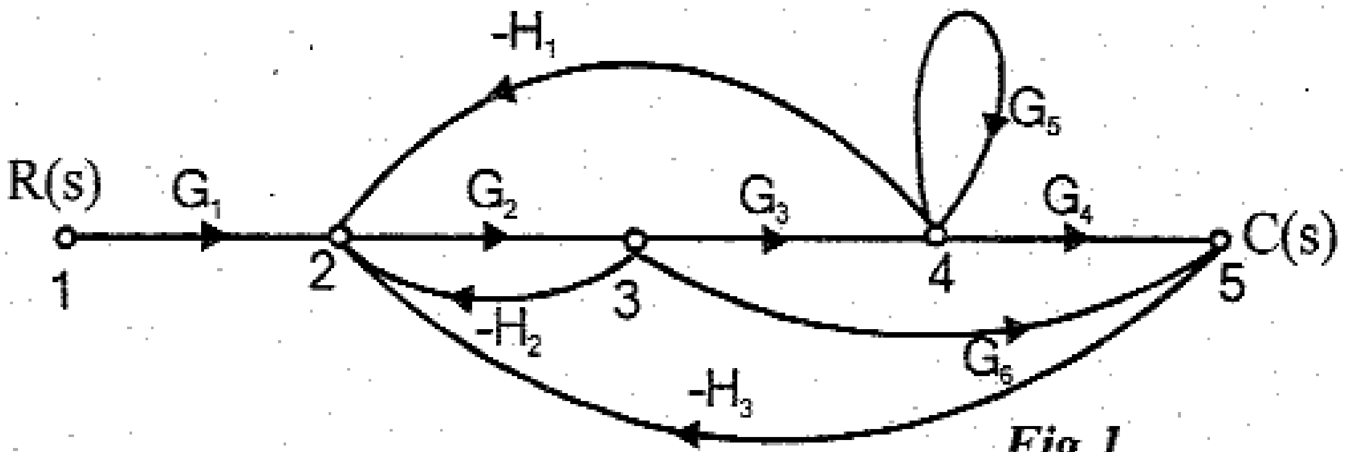


Fig 1

Explanation of terms used in signal flow graph

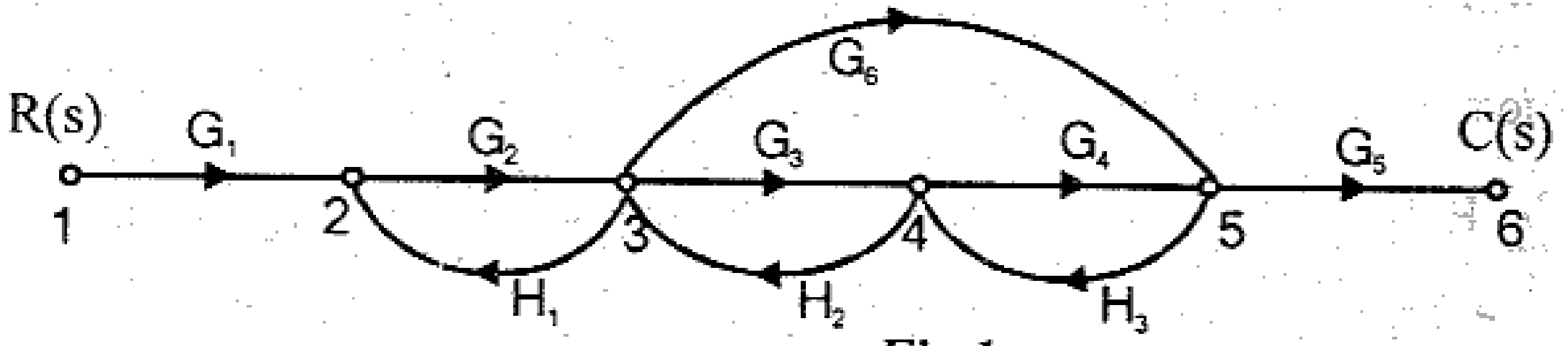


Fig 1

Node: a node is a point or representing a variable or signal

Branch: a branch is directed line segment joining two nodes. The arrow on the branch indicates the direction of signal flow and the gain of a branch is the transmittance

Input node (source): it is a node that has only outgoing branches

Output node (sink): it is a node that has only incoming branches

Explanation of terms used in signal flow graph

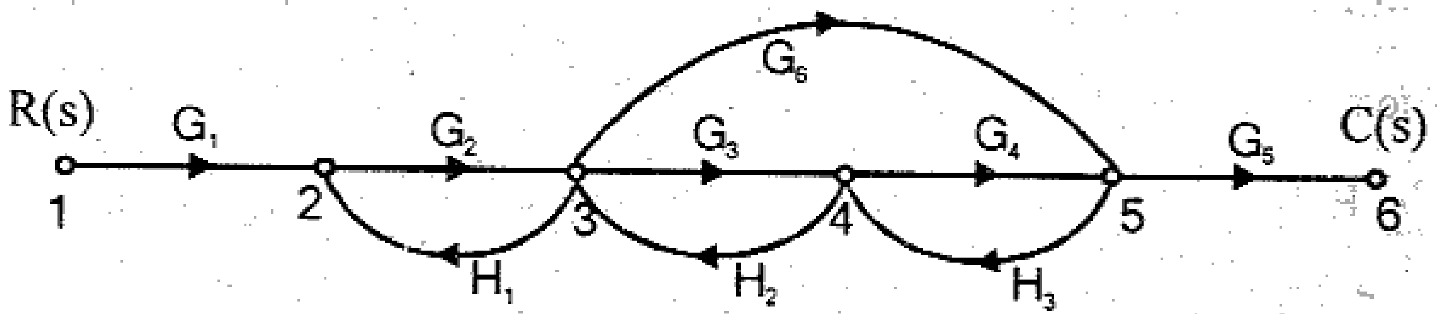


Fig 1

Mixed node: it is a node that has both incoming and outgoing branches

Path: a path is a traversal of connected branches in the direction of the branch arrows. The path should not cross a node more than once

Open path: a open path starts at a node and ends at another node

Closed path: closed path starts and ends at same node

Forward path: it is a path from an input node to an output node that does not cross any node more than once

Explanation of terms used in signal flow graph

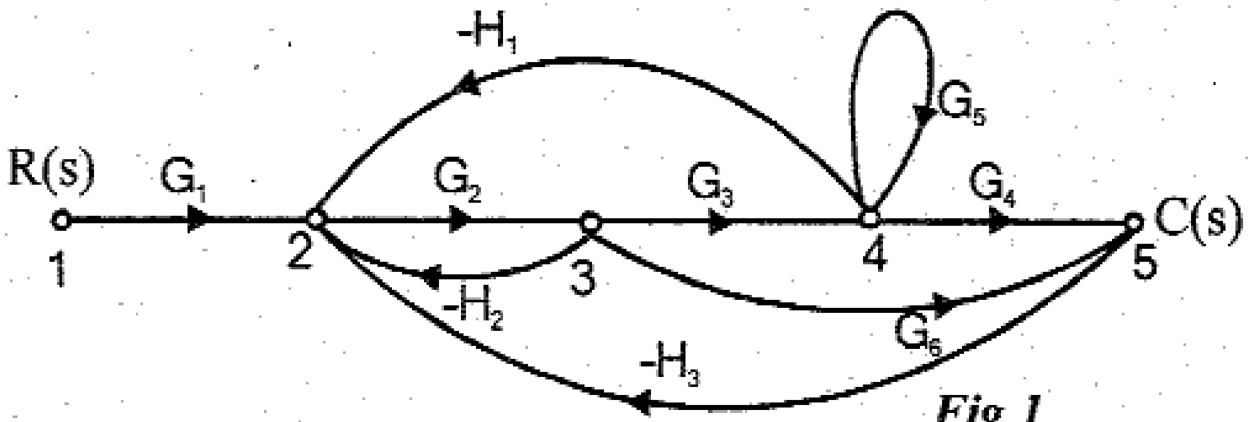


Fig 1

Forward path gain: it is the product of the branch transmittance (gain) of a forward path

Individual loop: it is a closed path starting from a node and after passing through a certain part of a graph arrives at same node without crossing any node more than once

Loop gain: it is the product of branch transmittance (gain) of a loop

Non-touching loop: if the loops does not have a common node, then they are said to be non-touching loops

Properties of signal flow graph

Signal flow graph is applicable to linear time invariant systems

The signal flow is only along the direction of arrows

The value of variable at each node is equal to the algebraic sum of all signals entering at that node

The gain of signal flow graph is given by Mason's gain formula

The signal gets multiplied by the branch gain when it travels along it

The signal flow graph is not be the unique property of the system

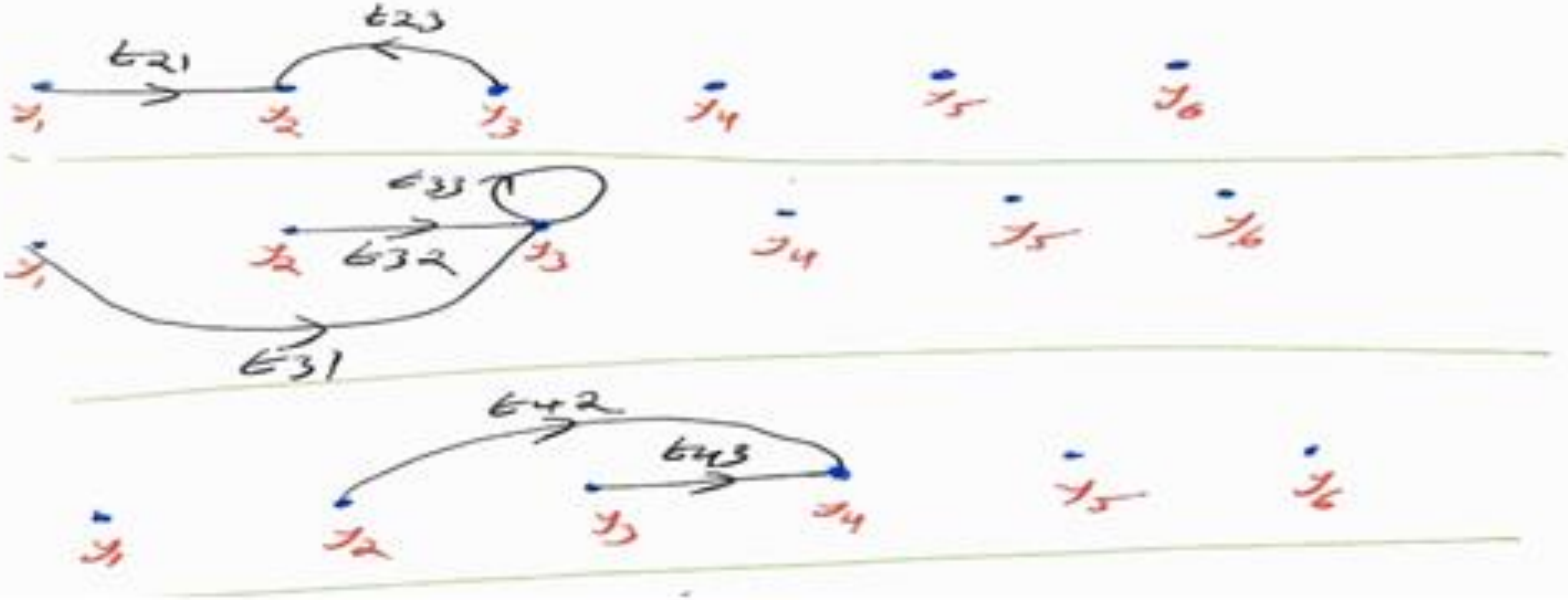
Comparison of block diagram and signal flow graph method

Sl.No	Block diagram	SFG
1	applicable to Linear time invariant systems	applicable to Linear time invariant systems
2	each element is represented by block	each variable is represented by node
3	summing point and take off points are separate	summing and take off points are absent
4	self-loop do not exist	self-loop can be exist
5	it is time consuming method	require less time by using Mason gain formula
6	block diagram is required at each and every step	at each step it is not necessary to draw SFG
7	Only transfer function of the element is shone inside the corresponding block	transfer function is shown along the branches connecting the nodes
8	feedback path is present	feedback loops are used

Construction of signal flow graph from equations

$y_2 = t_{21} y_1 + t_{23} y_3$; $y_3 = t_{32} y_2 + t_{33} y_3 + t_{31} y_1$;

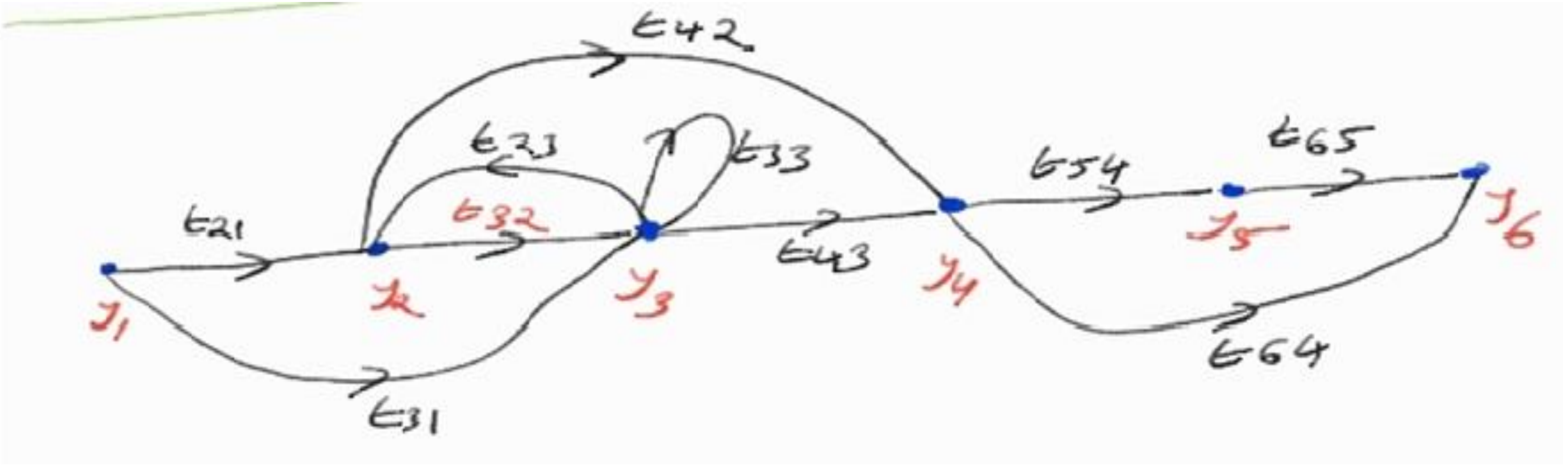
$y_4 = t_{43} y_3 + t_{42} y_2$; $y_5 = t_{54} y_4$; $y_6 = t_{65} y_5 + t_{64} y_4$



Construction of signal flow graph from equations

$y_2 = t_{21} y_1 + t_{23} y_3$; $y_3 = t_{32} y_2 + t_{33} y_3 + t_{31} y_1$;

$y_4 = t_{43} y_3 + t_{42} y_2$; $y_5 = t_{54} y_4$; $y_6 = t_{65} y_5 + t_{64} y_4$

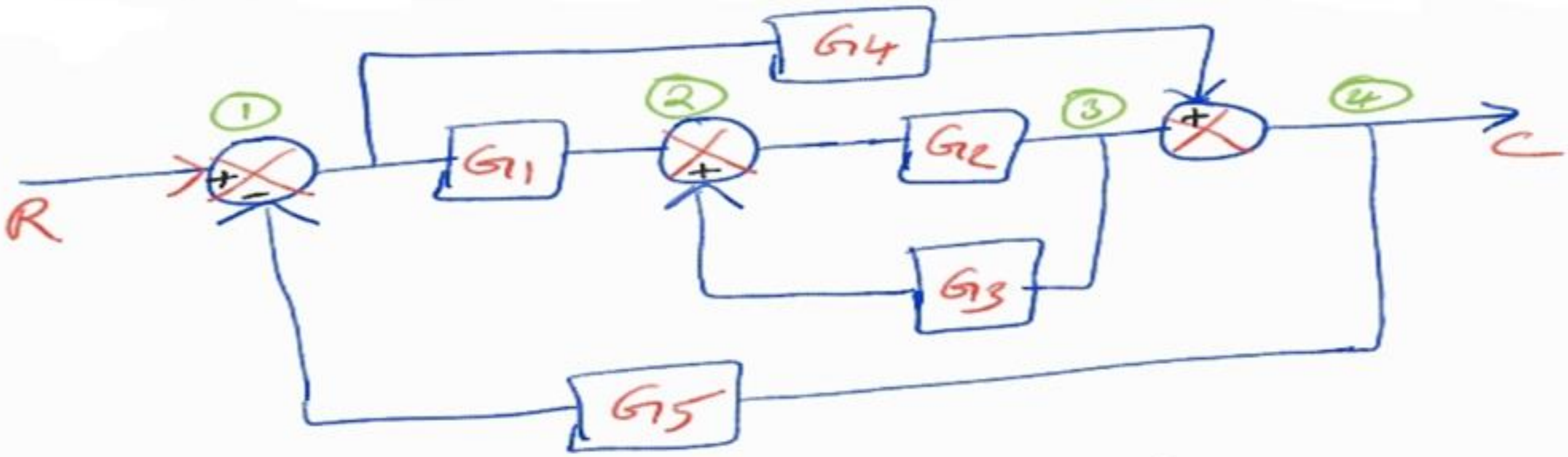


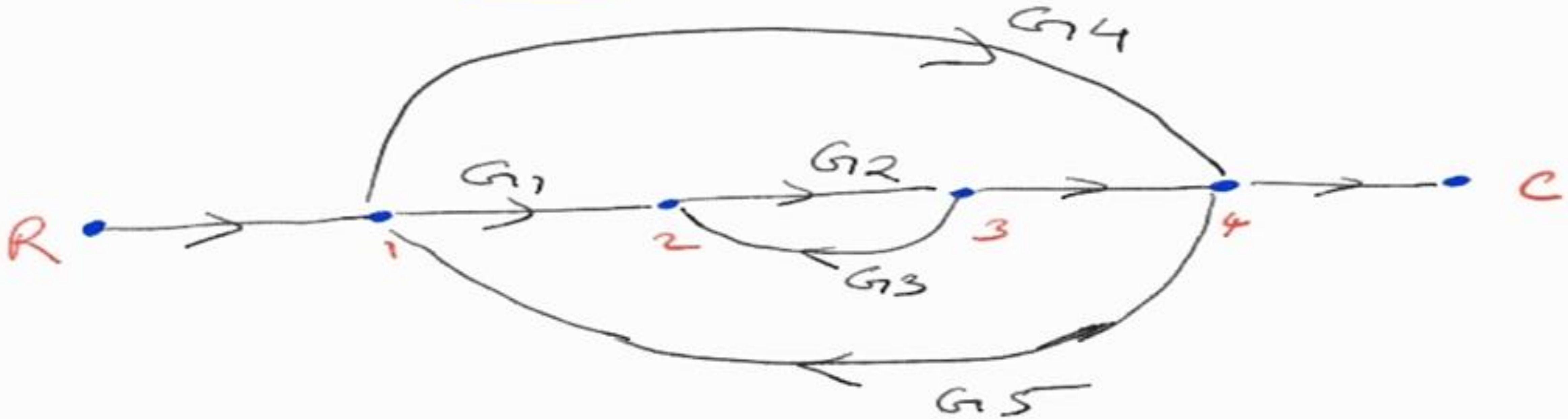
construction of signal flow graph from block diagram

All variables, summing point and take off points are represented by nodes

If a summing point is placed before a take off point in the direction of signal flow, in such case represent the summing point and take off Point by a single node

If a summing point is placed after a takeoff point in the direction of signal flow, in such case, represent the summing point and take off Point by separate nodes connected by a branch having transmittance Unity


G4
1
2
3
4
G1
G2
G3
G5
R
C
+
-
+
+


G4
G1
G2
G3
G5
R
1
2
3
4
C

MASON'S GAIN FORMULA

Mason's gain formula states the overall gain of the system [transfer function]

$$\text{Overall gain, } T = \frac{1}{\Delta} \sum_K P_K \Delta_K$$

T = $T(s)$ = Transfer function of the system

P_K = Forward path gain of K^{th} forward path

K = Number of forward paths in the signal flow graph

Δ = 1 – (Sum of individual loop gains)

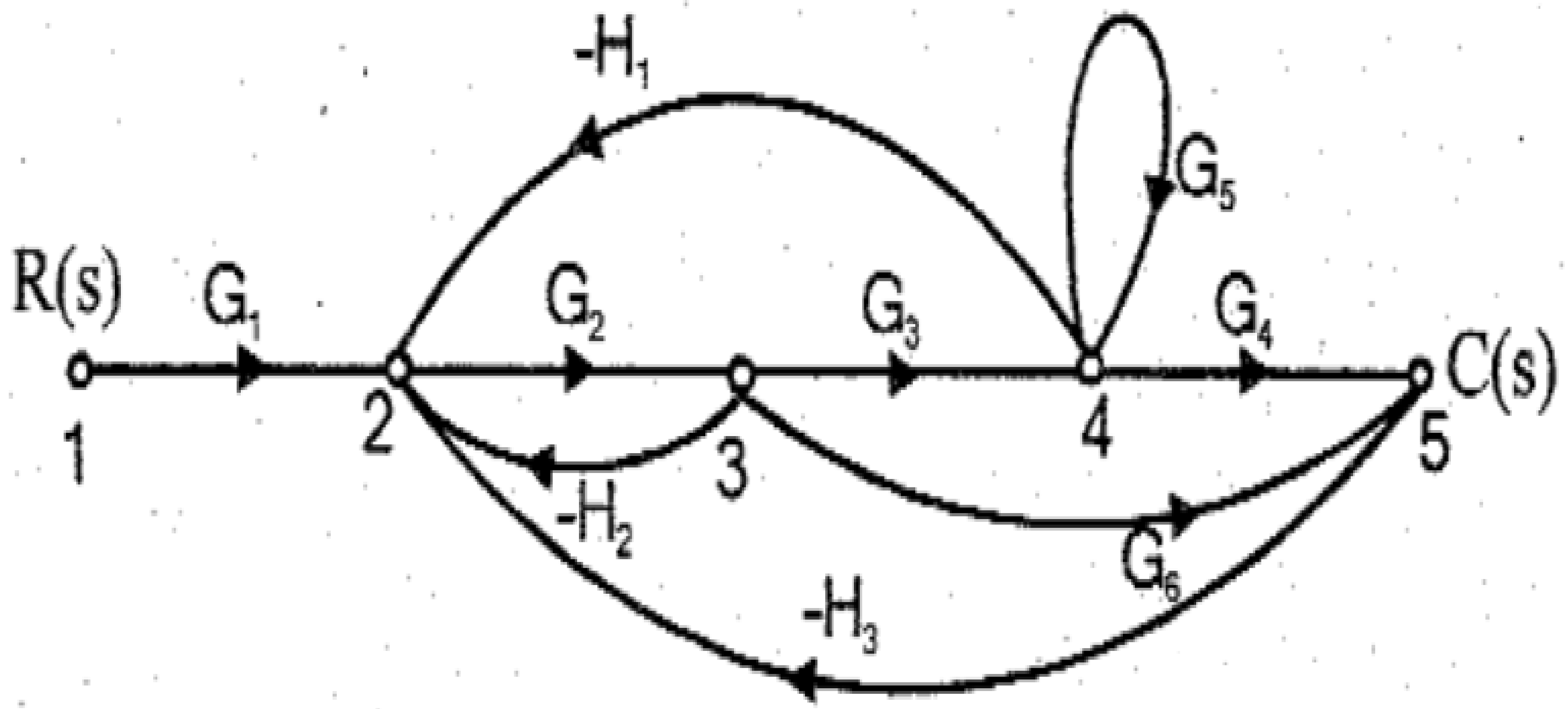
$$+ \begin{pmatrix} \text{Sum of gain products of all possible} \\ \text{combinations of two non - touching loops} \end{pmatrix}$$

$$- \begin{pmatrix} \text{Sum of gain products of all possible} \\ \text{combinations of three non - touching loops} \end{pmatrix}$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots$$

Δ_K = Δ for that part of the graph which is not touching K^{th} forward path

Find the overall gain C(s)/R(s) for the signal flow graph



Forward Path Gains

There are two forward paths. ∴ K = 2. Let the forward path gains be P_1 and P_2.

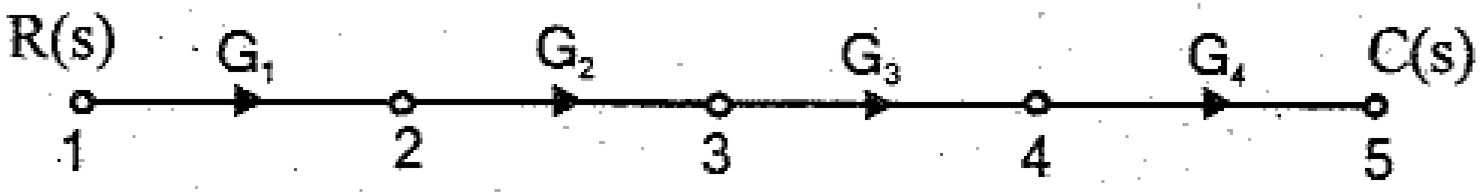


Fig 2 : Forward path-1

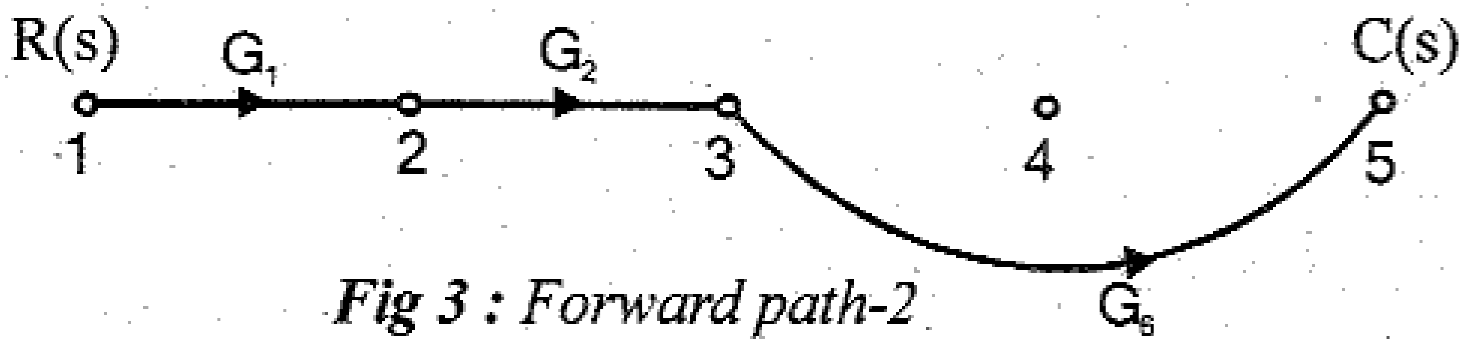


Fig 3 : Forward path-2

Gain of forward path-1, $P_1 = G_1G_2G_3G_4$

Gain of forward path-2, $P_2 = G_1G_2G_6$

Individual Loop Gain

There are five individual loops. Let the individual loop gains be p_{11}, p_{21}, p_{31}, p_{41} and p_{51}.

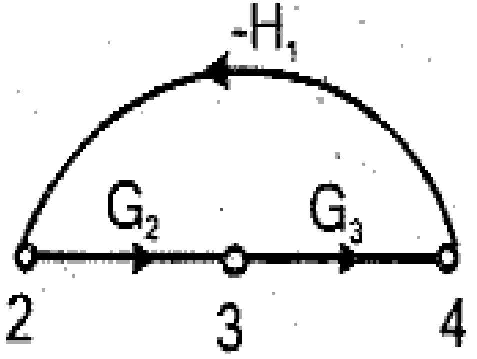


Fig 4 : loop-1

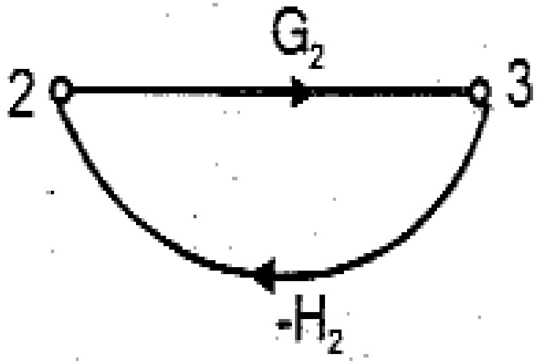


Fig 5 : loop-2

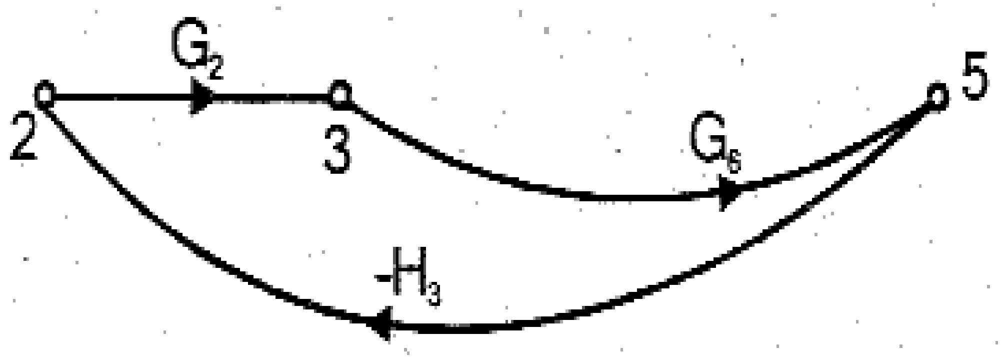


Fig 6 : loop-3

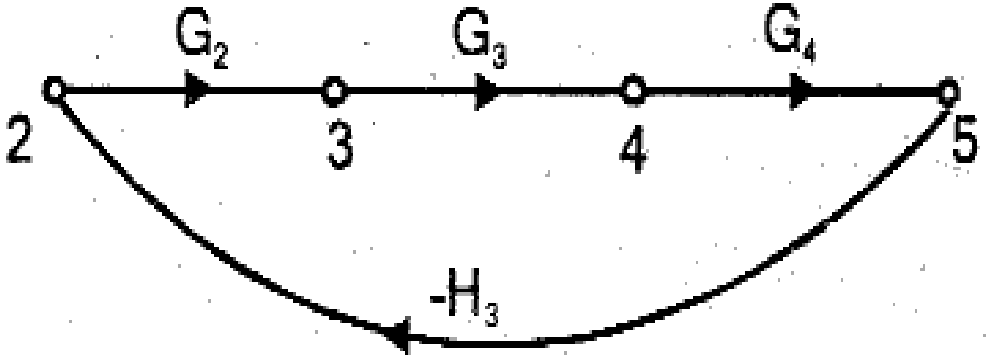


Fig 7 : loop-4

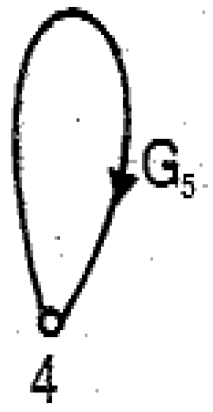


Fig 8 : loop-5

Loop gain of individual loop-1, $P_{11} = -G_2G_3H_1$

Loop gain of individual loop-2, $P_{21} = -H_2G_2$

Loop gain of individual loop-3, $P_{31} = -G_2G_6H_3$

Loop gain of individual loop-4, $P_{41} = -G_2G_3G_4H_3$

Loop gain of individual loop-5, $P_{51} = G_5$

Gain Products of Two Non-touching Loops

There are two combinations of two non-touching loops.

Let the gain products of two non-touching loops be P_{12} and P_{22}.

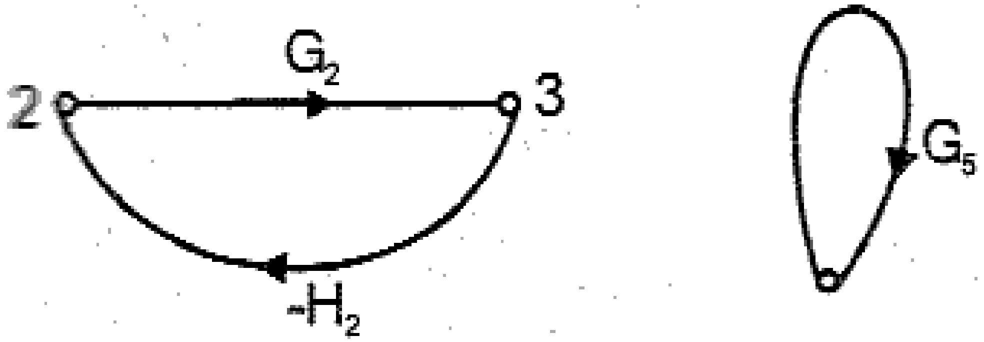


Fig 9 : *First combination of two non-touching loops*

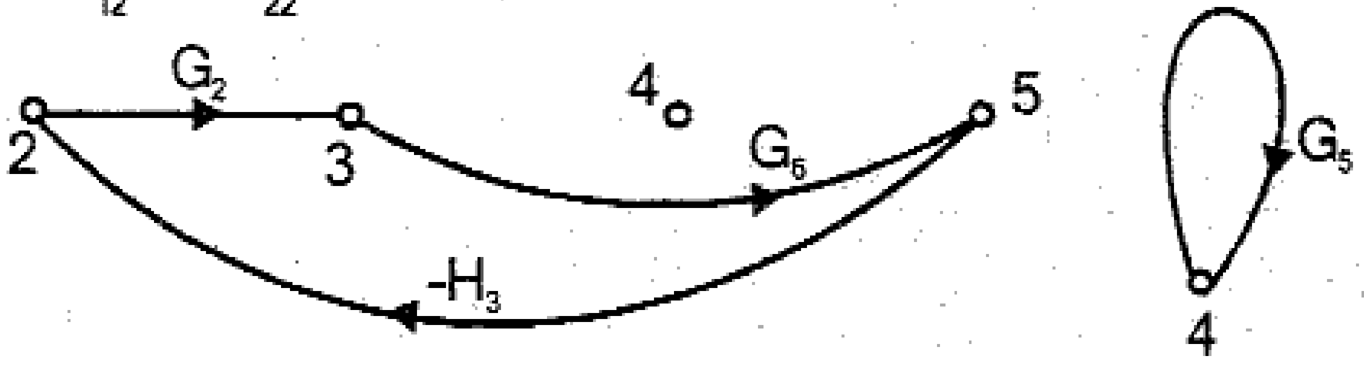


Fig 10 : *Second combination of two non-touching loops*

Gain product of first combination of two non touching loops $\Big\}$ $P_{12} = P_{21}P_{51} = (-G_2H_2)(G_5) = G_2G_5H_2$

Gain product of second combination of two non touching loops $\Big\}$ $P_{22} = P_{31}P_{51} = (-G_2G_6H_3)(G_5) = -G_2G_5G_6H_3$

Calculation of Δ and Δ_K

$$\begin{aligned}\Delta &= 1-(P_{11}+P_{21}+P_{31}+P_{41}+P_{51})+(P_{12}+P_{22})\\ &= 1-(-G_2G_3H_1-H_2G_2-G_2G_3G_4H_3+G_5-G_2G_6H_3)\\ &\quad +(-G_2H_2G_5-G_2G_5G_6H_3)\end{aligned}$$

Since there is no part of graph which is not touching forward path-1, $\Delta_1 = 1$.

The part of graph which is not touching forward path-2 is shown in fig 11.

$$\therefore\ \Delta_2 = 1 - G_5$$

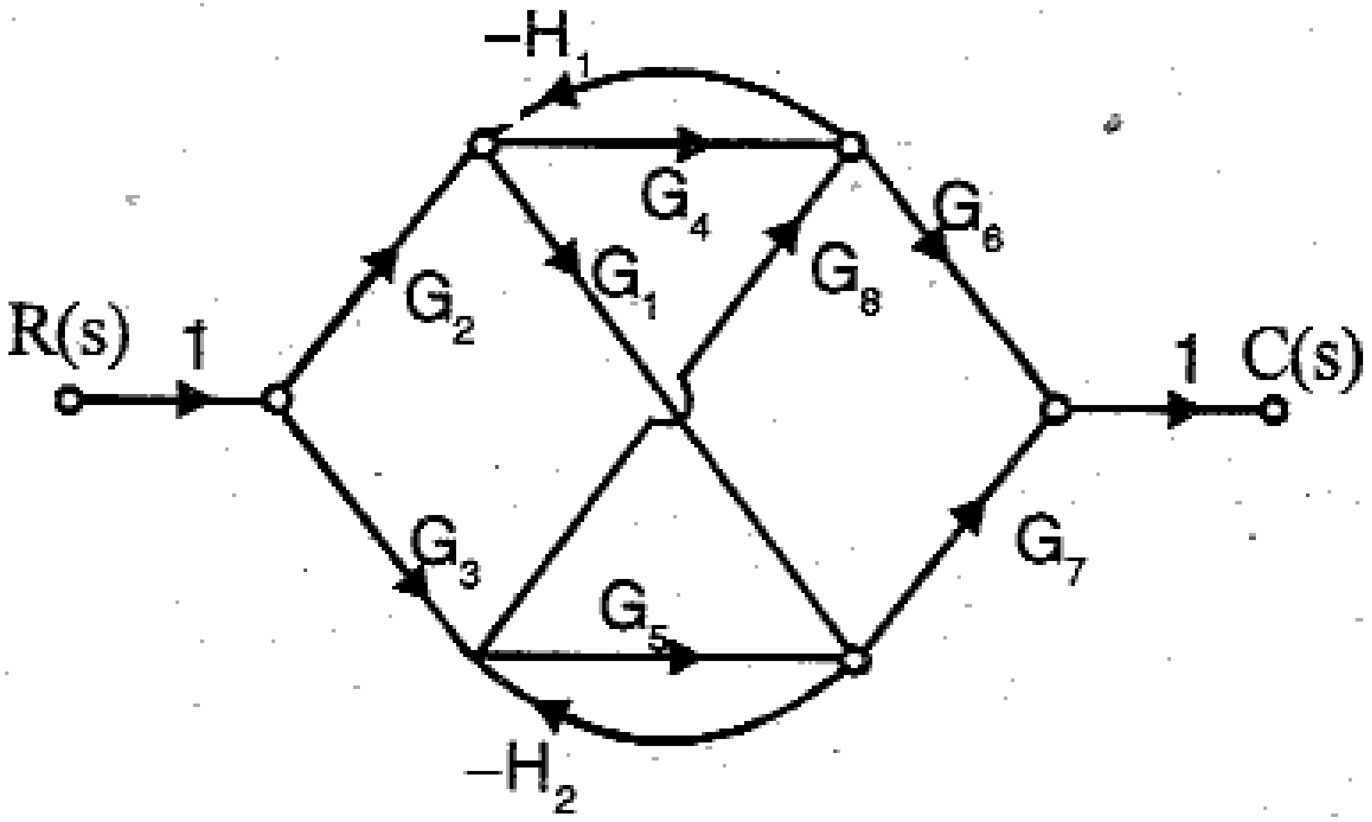
By Mason's gain formula the transfer function, T is given by,

$$T = \frac{1}{\Delta} \sum_{K} P_K \Delta_K \text{ (Number of forward path is 2 and so K = 2)}$$

$$= \frac{1}{\Delta} \left[P_1\Delta_1 + P_2\Delta_2\right] = \frac{1}{\Delta} \left[G_1G_2G_3G_4 \times 1 + G_1G_2G_6(1 - G_5)\right]$$

$$= \frac{G_1G_2G_3G_4 + G_1G_2G_6 - G_1G_2G_5G_6}{1 + G_2G_3H_1 + H_2G_2 + G_2G_3G_4H_3 - G_5 + G_2G_6H_3 - G_2H_2G_5 - G_2G_5G_6H_3}$$

Find the overall gain of the system whose signal flow graph is shown



Module II

1. Control system components: DC and AC servo motors – synchro - gyroscope - stepper motor - Tacho generator.

2. Time domain analysis of control systems:

a. Transient and steady state responses
b. Test signals
c. Order and type of systems
d. Step responses of first and second order systems.
e. Time domain specifications

TIME RESPONSE

The time response of a system is the output of a closed loop system as a function of time. It is denoted by C(t).

$$\text{Closed loop transfer function, } \frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)H(S)} = M(S)$$

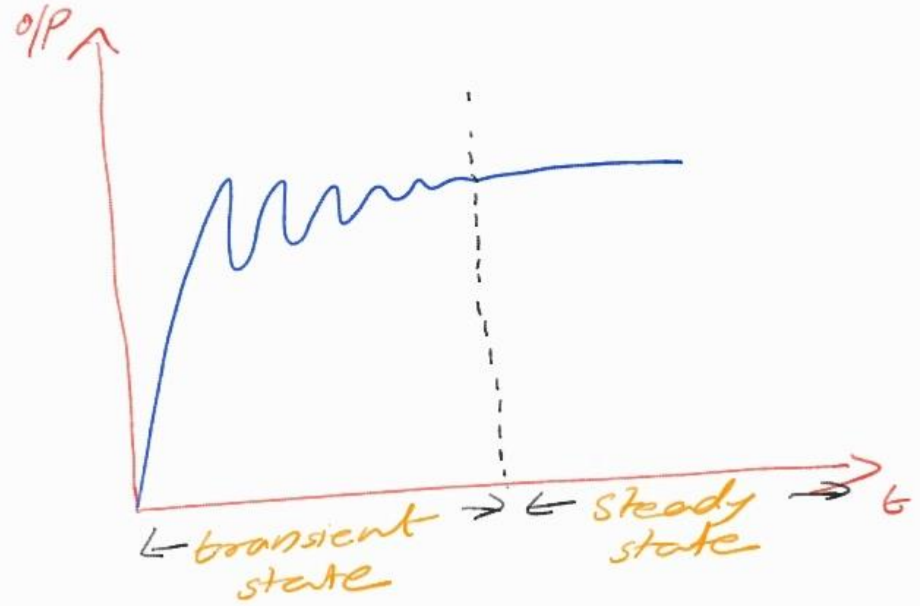
$$\text{Response in 'S' domain, } C(S) = R(S)M(S)$$

$$\text{Respomse in time domain, } C(t) = L^{-1}\{C(S)\} = L^{-1}\{R(S)M(S)\}$$

The time response of a control system consists of two parts; the transient state response and steady state response

The transient response is the response of the system when the input changes from one state to another the steady state response is the response time approaches infinity

$$C(t) = C_{tr}(t) + C_{ss}(t)$$



$$C(t) = C_{tr}(t) + C_{ss}(t)$$

TEST SIGNALS

The characteristics of actual input signals are a sudden shock, a sudden change, a constant velocity and a constant acceleration.

Test signals which resembles these characteristics are used as input signal to predict the performance of the system.

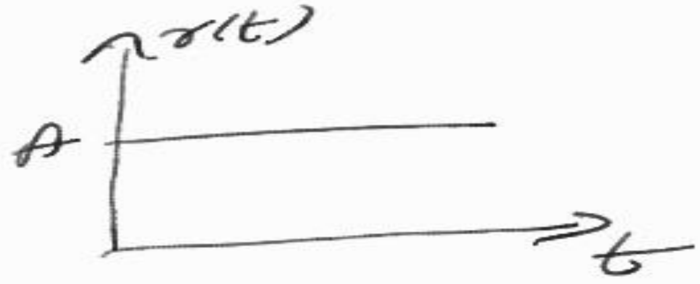
The standard test signals are step signal, unit step signal, unit ramp signal, ramp signal, unit impulse signal and sinusoidal signal

Test Signals

1. Step Signal

$r(t) = A \; ; \; t \geq 0$

$\quad\;\; = 0 \; ; \; t < 0$



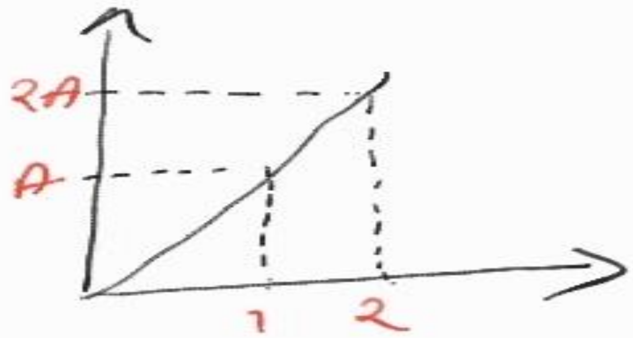
$$\mathcal{L}(r(t)) = \frac{A}{s}$$

for unity step signal '$1/s$'

2. Ramp Signal

$r(t) = At \; ; \; t \geq 0$

$\quad\;\; = 0 \; ; \; t < 0$

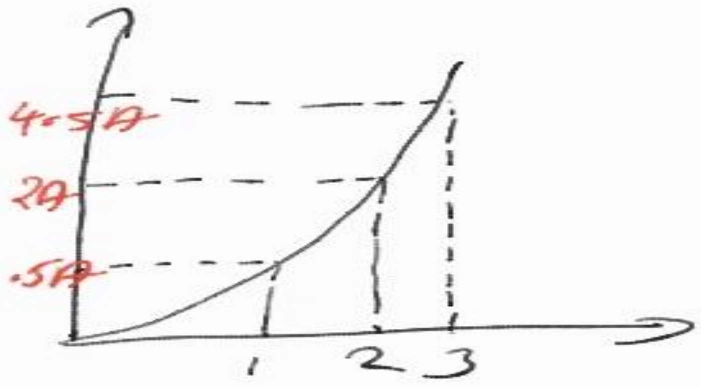


$$\mathcal{L}(r(t)) = \frac{A}{s^2}$$

for unity ramp signal '$1/s^2$'

3. Parabolic Signal

$$r(t) = \frac{At^2}{2} \; ; \; t \geq 0$$
$$= 0 \; ; \; t < 0$$



$$\boxed{L(r(t)) = \frac{A}{s^3}}$$

$$L(t^n) = \frac{n!}{s^{n+1}}$$

For unity parabolic signal
$$L(r(t)) = 1/s^3$$

4. Impulse Signal

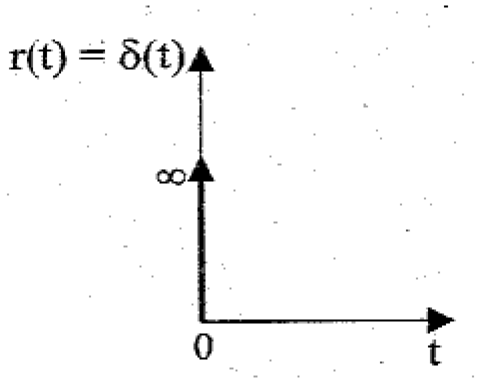
$$\delta(t) = \infty \; ; \; t = 0 \quad \& \quad \int_{-\infty}^{\infty} \delta(t)\,dt = A$$
$$= 0 \; ; \; t \neq 0$$

$$\boxed{L(\delta(t)) = 1}$$

IMPULSE SIGNAL

A signal of very large magnitude which is available for very short duration is called Impulse signal

Ideal impulse signal is a signal with infinite magnitude and zero duration but with an area of A.



Standard Test Signals

Name of the signal	Time domain equation of signal, r(t)	Laplace transform of the signal, R(s)
Step	A	$\frac{A}{s}$
Unit step	1	$\frac{1}{s}$
Ramp	At	$\frac{A}{s^2}$
Unit ramp	t	$\frac{1}{s^2}$
Parabolic	$\frac{At^2}{2}$	$\frac{A}{s^3}$
Unit parabolic	$\frac{t^2}{2}$	$\frac{1}{s^3}$
Impulse	$\delta(t)$	1

IMPULSE RESPONSE

The response of the system, with input as impulse signal is called a weighting function or impulse response of the system.

It is also given by the inverse Laplace transform of the system transfer function and denoted by m(t)

$$\text{Impulse response, } m(t) = \mathcal{L}^{-1}\{R(s)\,M(s)\} = \mathcal{L}^{-1}\{M(s)\}$$

$$\boxed{R(s) = 1, \text{ for impulse}}$$

$$\text{where, } M(s) = \frac{G(s)}{1+G(s)H(s)}$$

ORDER OF A SYSTEM

The input and output relationship of a control system can be expressed by n-th order differential equation.

The order of the system is given by the order of the differential equation governing the system.

$$a_0 \frac{d^n}{dt^n} p(t) + a_1 \frac{d^{n-1}}{dt^{n-1}} p(t) + a_2 \frac{d^{n-2}}{dt^{n-2}} p(t) + \ldots\ldots + a_{n-1} \frac{d}{dt} p(t) + a_n\, p(t) = b_0 \frac{d^m}{dt^m} q(t)$$
$$+ b_1 \frac{d^{m-1}}{dt^{m-1}} q(t) + b_2 \frac{d^{m-2}}{dt^{m-2}} q(t) + \ldots\ldots + b_{m-1} \frac{d}{dt} q(t) + b_m\, q(t)$$

If the system is governed by n-th order differential equation, then the system is called n-th order system

The order can also be determined from the transfer function of the system.

The order of the system is given by the maximum power of 'S' in the denominator polynomial

$$\text{Transfer function, } T(s) = \frac{P(s)}{Q(s)} = \frac{b_0 s^m + b_1 s^{m-1} + b_2 s^{m-2} + \dots\dots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots\dots + a_{n-1} s + a_n}$$

The order of the system is given by the maximum power of s in the denominator polynomial, Q(s).

Here, $Q(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots\dots\dots + a_{n-1} s + a_n$.

Now, **n** is the order of the system

When n = 0, the system is zero order system.

When n = 1, the system is first order system.

When n = 2, the system is second order system and so on.

TYPE NUMBER OF CONTROL SYSTEMS

The type number is specified for loop transfer function G(S)H(S).

The number of poles of the loop transfer function lying at the origin decides the type number of the system

In general if 'N' is the number of poles at the origin then the type number is 'N'

$$G(s)\,H(s) = K\frac{P(s)}{Q(s)} = K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)\,\dots\dots\dots}{s^N\,(s+p_1)\,(s+p_2)\,(s+p_3)\,\dots\dots}$$

where, $z_1, z_2, z_3, \dots\dots\dots$ are zeros of transfer function

$p_1, p_2, p_3, \dots\dots\dots$ are poles

K = Constant

N = Number of poles at the origin

If N = 0, then the system is type – 0 system

If N = 1, then the system is type – 1 system

If N = 2, then the system is type – 2 system

If N = 3, then the system is type – 3 system and so on.

TYPE NUMBER OF CONTROL SYSTEMS

$$G(s)\,H(s) = K\frac{P(s)}{Q(s)} = K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)\,\ldots\ldots\ldots}{s^N\,(s+p_1)\,(s+p_2)\,(s+p_3)\,\ldots\ldots}$$

where, z_1, z_2, z_3, ... are zeros of transfer function

p_1, p_2, p_3, ... are poles of transfer function

K = Constant

N = Number of poles at the origin

If N = 0, then the system is type – 0 system

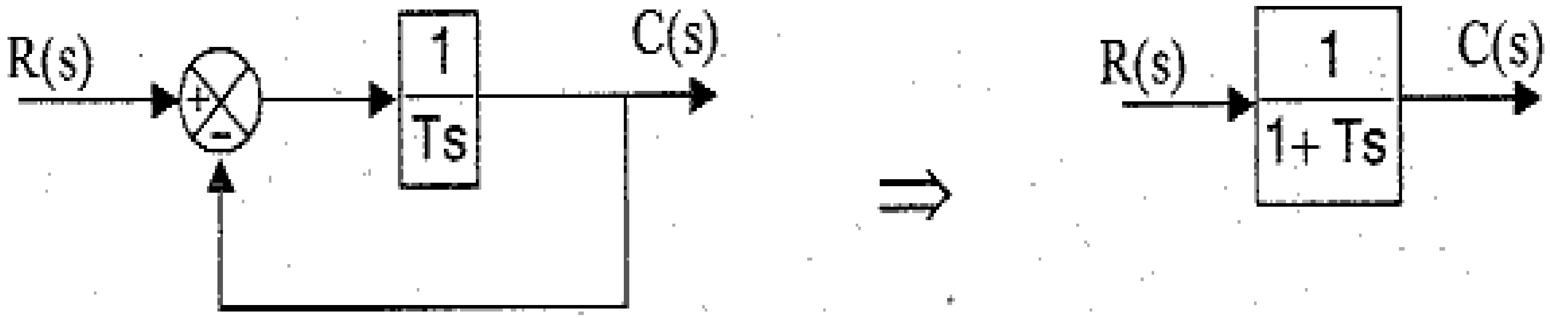
If N = 1, then the system is type – 1 system

If N = 2, then the system is type – 2 system

If N = 3, then the system is type – 3 system and so on.

Response of first order system for unit step input

The closed loop first order system with unity feedback is



$$\frac{C(s)}{R(s)} = \frac{1}{1+Ts}$$

$$\therefore \text{The response in s-domain, } C(s) = R(s)\frac{1}{(1+Ts)}$$

If the input is unit step then, $r(t) = 1$ and $R(s) = \frac{1}{s}$.

$$C(s) = R(s)\frac{1}{(1+Ts)} = \frac{1}{s}\frac{1}{(1+Ts)} = \frac{1}{sT\left(\frac{1}{T}+s\right)} = \frac{\frac{1}{T}}{s\left(s+\frac{1}{T}\right)}$$

By partial fraction expansion,

$$C(s) = \frac{\frac{1}{T}}{s\left(s+\frac{1}{T}\right)} = \frac{A}{s} + \frac{B}{\left(s+\frac{1}{T}\right)}$$

A is obtained by multiplying C(s) by s and letting s = 0.

$$A = C(s) \times s\Big|_{s=0} = \frac{\frac{1}{T}}{s\left(s+\frac{1}{T}\right)} \times s\Bigg|_{s=0} = \frac{\frac{1}{T}}{s+\frac{1}{T}}\Bigg|_{s=0} = \frac{\frac{1}{T}}{\frac{1}{T}} = 1$$

B is obtained by multiplying C(s) by (s+1/T) and letting s = −1/T.

$$B = C(s) \times \left(s+\frac{1}{T}\right)\Bigg|_{s=-\frac{1}{T}} = \frac{\frac{1}{T}}{s\left(s+\frac{1}{T}\right)} \times \left(s+\frac{1}{T}\right)\Bigg|_{s=-\frac{1}{T}} = \frac{\frac{1}{T}}{s}\Bigg|_{s=-\frac{1}{T}} = \frac{\frac{1}{T}}{\frac{-1}{T}} = -1$$

$$\therefore C(s) = \frac{1}{s} - \frac{1}{s+\frac{1}{T}}$$

The response in time domain is given by,

$$c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+\frac{1}{T}}\right\} = 1 - e^{-\frac{t}{T}}$$

When

$t = 0, \quad c(t) = 1 - e^{0} = 0$

$t = 1T, \quad c(t) = 1 - e^{-1} = 0.632$

$t = 2T, \quad c(t) = 1 - e^{-2} = 0.865$

$t = 3T, \quad c(t) = 1 - e^{-3} = 0.95$

$t = 4T, \quad c(t) = 1 - e^{-4} = 0.9817$

$t = 5T, \quad c(t) = 1 - e^{-5} = 0.993$

$t = \infty, \quad c(t) = 1 - e^{-\infty} = 1$

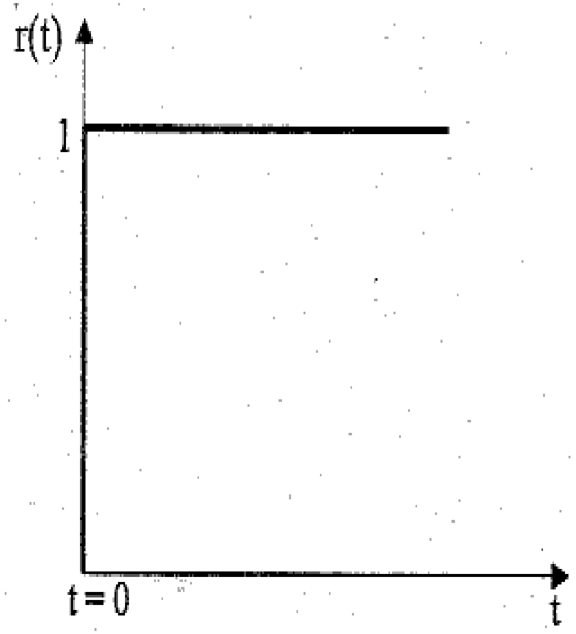


Fig 2.7a : Unit step input.

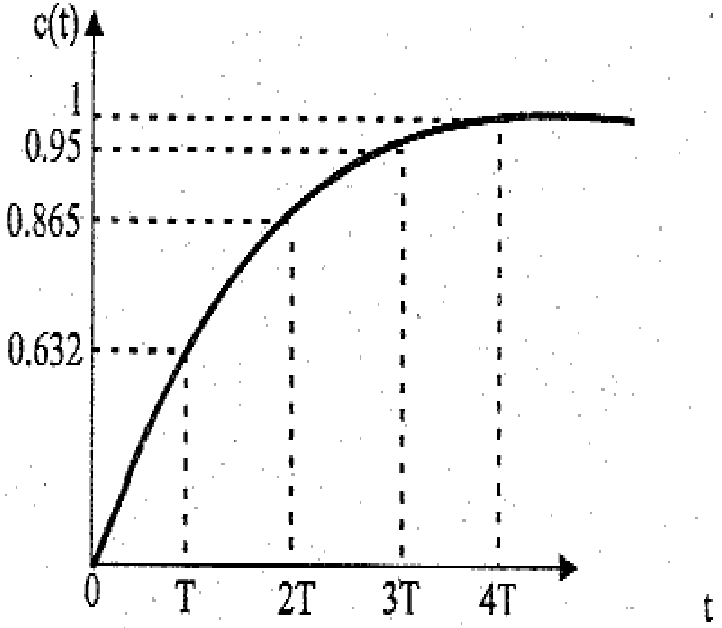
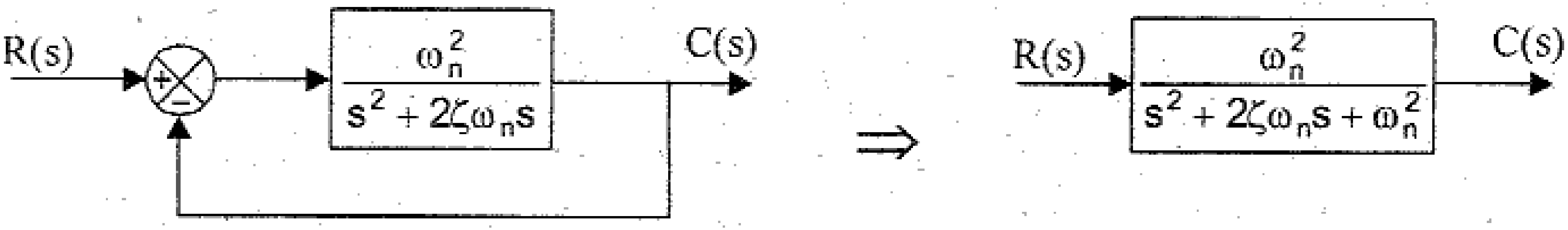


Fig 2.7b : *Response for Unit step input.*

SECOND ORDER SYSTEM

The closed loop second order system is



The standard form of closed loop transfer function of second order system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where, ω_n = Undamped natural frequency, rad/sec.

ζ = Damping ratio.

The damping ratio is defined as the ratio of the actual damping to the critical damping.

The response C(t) of second order system depends on the value of damping ratio.

Depending on the value of damping ratio, the system can be classified into four.

Case 1	:	Undamped system,	$\zeta = 0$
Case 2	:	Under damped system,	$0 < \zeta < 1$
Case 3	:	Critically damped system,	$\zeta = 1$
Case 4	:	Over damped system,	$\zeta > 1$

The characteristics equation of the second order system is,

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

The characteristics equation of thc sccond order system is,

$$s^2 + 2\zeta\omega_n s + \omega_n^{\,2} = 0$$

It is a quadratic equation and the roots of this equation is given by,

$$s_1,\ s_2 = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2} = \frac{-2\zeta\omega_n \pm \sqrt{4\omega_n^2(\zeta^2 - 1)}}{2}$$

$$= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

When $\zeta = 0$, $s_1, s_2 = \pm j\omega_n$; $\begin{cases} \text{roots are purely imaginary} \\ \text{and the system is undamped} \end{cases}$

When $\zeta = 1$, $s_1, s_2 = -\omega_n$; $\begin{cases} \text{roots are real and equal and} \\ \text{the system is critically damped} \end{cases}$

When $\zeta > 1$, $s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$; $\begin{cases} \text{roots are real and unequal and} \\ \text{the system is overdamped} \end{cases}$

When $0 < \zeta < 1$, $s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\zeta\omega_n \pm \omega_n\sqrt{(-1)\,(1-\zeta^2)}$

$$= -\zeta\omega_n \pm \omega_n\sqrt{-1}\,\sqrt{1-\zeta^2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$$= -\zeta\omega_n \pm j\omega_d \; ; \begin{cases} \text{roots are complex conjugate} \\ \text{the system is underdamped} \end{cases}$$

where, $\omega_d = \omega_n\sqrt{1-\zeta^2}$

Here ω_d is called damped frequency of oscillation of the system and its unit is rad/sec.

RESPONSE OF UNDAMPED SECOND ORDER SYSTEM FOR UNIT STEP INPUT

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For undamped system, $\zeta = 0$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + \omega_n^2}$$

the input is unit step, $r(t) = 1$ and $R(s) = \frac{1}{s}$.

The response in s-domain, $C(s) = R(s)\,\frac{\omega_n^2}{s^2 + \omega_n^2} = \frac{1}{s}\,\frac{\omega_n^2}{s^2 + \omega_n^2}$

By partial fraction expansion,

$$C(s) = \frac{\omega_n^2}{s\,(s^2 + \omega_n^2)} = \frac{A}{s} + \frac{B}{s^2 + \omega_n^2}$$

A is obtained by multiplying C(s) by s and letting s = 0.

$$A = C(s) \times s\Big|_{s=0} = \frac{\omega_n^2}{s\,(s^2+\omega_n^2)} \times s\Bigg|_{s=0} = \frac{\omega_n^2}{s^2+\omega_n^2}\Bigg|_{s=0} = \frac{\omega_n^2}{\omega_n^2} = 1$$

B is obtained by multiplying C(s) by $(s^2 + \omega_n^2)$ and letting $s^2 = -\omega_n{}^2$ or $s = j\omega_n$.

$$B = C(s) \times (s^2+\omega_n^2)\Big|_{s=j\omega} = \frac{\omega_n^2}{s\,(s^2+\omega_n^2)} \times (s^2+\omega_n^2)\Bigg|_{s=j\omega} = \frac{\omega_n^2}{s}\Bigg|_{s=j\omega_n} = \frac{\omega_n^2}{j\omega_n} = -j\omega_n = -s$$

$$\therefore\; C(s) = \frac{A}{s} + \frac{B}{s^2+\omega_n^2} = \frac{1}{s} - \frac{s}{s^2+\omega_n^2}$$

$\mathcal{L}\{1\} = \frac{1}{s}$	$\mathcal{L}\{\cos\omega t\} = \frac{s}{s^2+\omega^2}$

Time domain response, $c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+\omega_n^2}\right\} = 1 - \cos\omega_n t$

RESPONSE OF CRITICALLY DAMPED SECOND ORDER SYSTEM FOR UNIT STEP INPUT

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For critical damping $\zeta = 1$.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\omega_n)^2}$$

When input is unit step, $r(t) = 1$ and $R(s) = 1/s$.

$\therefore$ The response in s-domain,

$$C(s) = R(s)\frac{\omega_n^2}{(s+\omega_n)^2} = \frac{1}{s}\,\frac{\omega_n^2}{(s+\omega_n)^2} = \frac{\omega_n^2}{s\,(s+\omega_n)^2}$$

By partial fraction expansion, we can write,

$$C(s) = \frac{\omega_n^2}{s\,(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{(s+\omega_n)^2} + \frac{C}{s+\omega_n}$$

$$A = s \times C(s)\Big|_{s=0} = \frac{\omega_n^2}{(s+\omega_n)^2}\Bigg|_{s=0} = \frac{\omega_n^2}{\omega_n^2} = 1$$

$$B = (s+\omega_n)^2 \times C(s)\Big|_{s=-\omega_n} = \frac{\omega_n^2}{s}\Bigg|_{s=-\omega_n} = -\omega_n$$

$$C = \frac{d}{ds}\Big[(s+\omega_n)^2 \times C(s)\Big]\Bigg|_{s=-\omega_n} = \frac{d}{ds}\left(\frac{\omega_n^2}{s}\right)\Bigg|_{s=-\omega_n} = \frac{-\omega_n^2}{s^2}\Bigg|_{s=-\omega_n} = -1$$

$$\therefore C(s) = \frac{A}{s} + \frac{B}{(s+\omega_n)^2} + \frac{C}{s+\omega_n} = \frac{1}{s} - \frac{\omega_n}{(s+\omega_n)^2} - \frac{1}{s+\omega_n}$$

$\mathcal{L}\{1\} = \frac{1}{s}$
$\mathcal{L}\{te^{-at}\} = \frac{1}{(s+a)^2}$
$\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$

$$c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{\omega_n}{(s+\omega_n)^2} - \frac{1}{s+\omega_n}\right\}$$

$$c(t) = 1 - \omega_n t\, e^{-\omega_n t} - e^{-\omega_n t}$$

$$c(t) = 1 - e^{-\omega_n t}(1 + \omega_n t)$$

RESPONSE OF OVER DAMPED SECOND ORDER SYSTEM FOR UNIT STEP INPUT

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For overdamped system $\zeta > 1$

The roots of the denominator of transfer function are

$$s_a, s_b = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\left[\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}\right]$$

Let $s_1 = -s_a$ and $s_2 = -s_b$ $\quad \therefore s_1 = \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$

$$s_2 = \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s + s_1)(s + s_2)}$$

For unit step input r(t) = 1 and R(s) = 1/s.

$$\therefore C(s) = R(s)\frac{\omega_n^2}{(s+s_1)\,(s+s_2)} = \frac{\omega_n^2}{s\,(s+s_1)\,(s+s_2)}$$

By partial fraction expansion we can write,

$$C(s) = \frac{\omega_n^2}{s\,(s+s_1)\,(s+s_2)} = \frac{A}{s} + \frac{B}{s+s_1} + \frac{C}{s+s_2}$$

$$A = s \times C(s)\big|_{s=0} = s \times \frac{\omega_n^2}{s\,(s+s_1)\,(s+s_2)}\bigg|_{s=0} = \frac{\omega_n^2}{s_1 s_2}$$

$$= \frac{\omega_n^2}{\left[\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right]\left[\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right]} = \frac{\omega_n^2}{\zeta^2\omega_n^2 - \omega_n^2\left(\zeta^2-1\right)} = \frac{\omega_n^2}{\omega_n^2} = 1$$

$$B = (s+s_1) \times C(s)\Big|_{s=-s_1} = \frac{\omega_n^2}{s(s+s_2)}\bigg|_{s=-s_1} = \frac{\omega_n^2}{-s_1(-s_1+s_2)}$$

$$= \frac{-\omega_n^2}{s_1\left[-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1} + \zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right]} = \frac{-\omega_n^2}{\left[2\omega_n\sqrt{\zeta^2-1}\right]s_1} = \frac{-\omega_n}{2\sqrt{\zeta^2-1}}\ \frac{1}{s_1}$$

$$C = C(s) \times (s+s_2)\Big|_{s=-s_2} = \frac{\omega_n^2}{s\,(s+s_1)}\bigg|_{s=-s_2} = \frac{\omega_n^2}{-s_2(-s_2+s_1)}$$

$$= \frac{\omega_n^2}{-s_2\left[-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1} + \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right]} = \frac{\omega_n^2}{\left[2\omega_n\sqrt{\zeta^2-1}\right]s_2} = \frac{\omega_n}{2\sqrt{\zeta^2-1}}\ \frac{1}{s_2}$$

$$c(t) = \mathcal{L}^{-1}\left\{=\frac{1}{s} - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\,\frac{1}{s_1}\,\frac{1}{(s+s_1)} + \frac{\omega_n}{2\sqrt{\zeta^2-1}}\,\frac{1}{s_2}\,\frac{1}{(s+s_2)}\right\}$$

$$c(t) = 1 - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\,\frac{1}{s_1}\,e^{-s_1 t} + \frac{\omega_n}{2\sqrt{\zeta^2-1}}\,\frac{1}{s_2}\,e^{-s_2 t}$$

$$c(t) = 1 - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right)$$

where, $s_1 = \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}$

$s_2 = \zeta\omega_n + \omega_n\sqrt{\zeta^2-1}$

RESPONSE OF UNDERDAMPED SECOND ORDER SYSTEM FOR UNIT STEP INPUT

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$C(s) = R(s)\,\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For unit step input, $r(t) = 1$ and $R(s) = 1/s$.

$$\therefore C(s) = \frac{\omega_n^2}{s\,(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

$$C(s) = \frac{\omega_n^2}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\therefore\ A = s \times C(s)\Big|_{s=0} = s \times \frac{\omega_n^2}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)}\Bigg|_{s=0} = \frac{\omega_n^2}{\omega_n^2} = 1$$

On cross multiplication after substituting A = 1

$$\omega_n^2 = s^2 + 2\zeta\omega_n s + \omega_n^2 + (Bs + C)s$$

$$\omega_n^2 = s^2 + 2\zeta\omega_n s + \omega_n^2 + Bs^2 + Cs$$

Equating coefficients of s^2 we get, $0 = 1 + B$ $\quad\therefore B = -1$

Equating coefficient of s we get, $0 = 2\zeta\omega_n + C$ $\quad\therefore C = -2\zeta\omega_n$

$$\therefore\ C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Let us add and subtract $\zeta^2\omega_n^{\ 2}$ to the denominator of second term

$$C(s) = \frac{1}{s} - \frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2+\zeta^2\omega_n^2-\zeta^2\omega_n^2} = \frac{1}{s} - \frac{s+2\zeta\omega_n}{(s^2+2\zeta\omega_n s+\zeta^2\omega_n^2)+(\omega_n^2-\zeta^2\omega_n^2)}$$

$$= \frac{1}{s} - \frac{s+2\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)} = \frac{1}{s} - \frac{s+2\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}$$

$$\boxed{\omega_d = \omega_n\sqrt{1-\zeta^2}}$$

$$= \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} - \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}$$

multiply and divide by ω_d in the third term

$$\therefore C(s) = \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} - \frac{\zeta\omega_n}{\omega_d}\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}$$

The response in time domain is given by,

$$c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} - \frac{\zeta\omega_n}{\omega_d}\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}\right\}$$

$\mathcal{L}\{1\} = \frac{1}{s}$
$\mathcal{L}\{e^{-at}\sin\omega t\} = \frac{\omega}{(s+a)^2+\omega^2}$
$\mathcal{L}\{e^{-at}\cos\omega t\} = \frac{s+a}{(s+a)^2+\omega^2}$

$$= 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta\omega_n}{\omega_d} e^{-\zeta\omega_n t}\sin\omega_d t = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}}\sin\omega_d t\right)$$

$$\boxed{\omega_d = \omega_n\sqrt{1-\zeta^2}}$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\sqrt{1-\zeta^2}\cos\omega_d t + \zeta\sin\omega_d t\right) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\sin\omega_d t \times \zeta + \cos\omega_d t \times \sqrt{1-\zeta^2}\right)$$

Let us express c(t) in a standard form as shown below.

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\sin\omega_d t \times \cos\theta + \cos\omega_d t \times \sin\theta\right)$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \theta) \qquad .....(2.28)$$

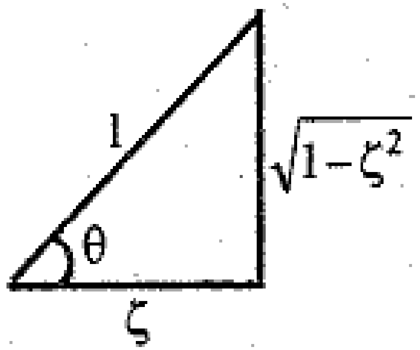
$$\text{where, } \left(\theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

Note : On constructing right angle triangle with ζ and $\sqrt{1-\zeta^2}$, we get

$\sin\theta = \sqrt{1-\zeta^2}$

$\cos\theta = \zeta$

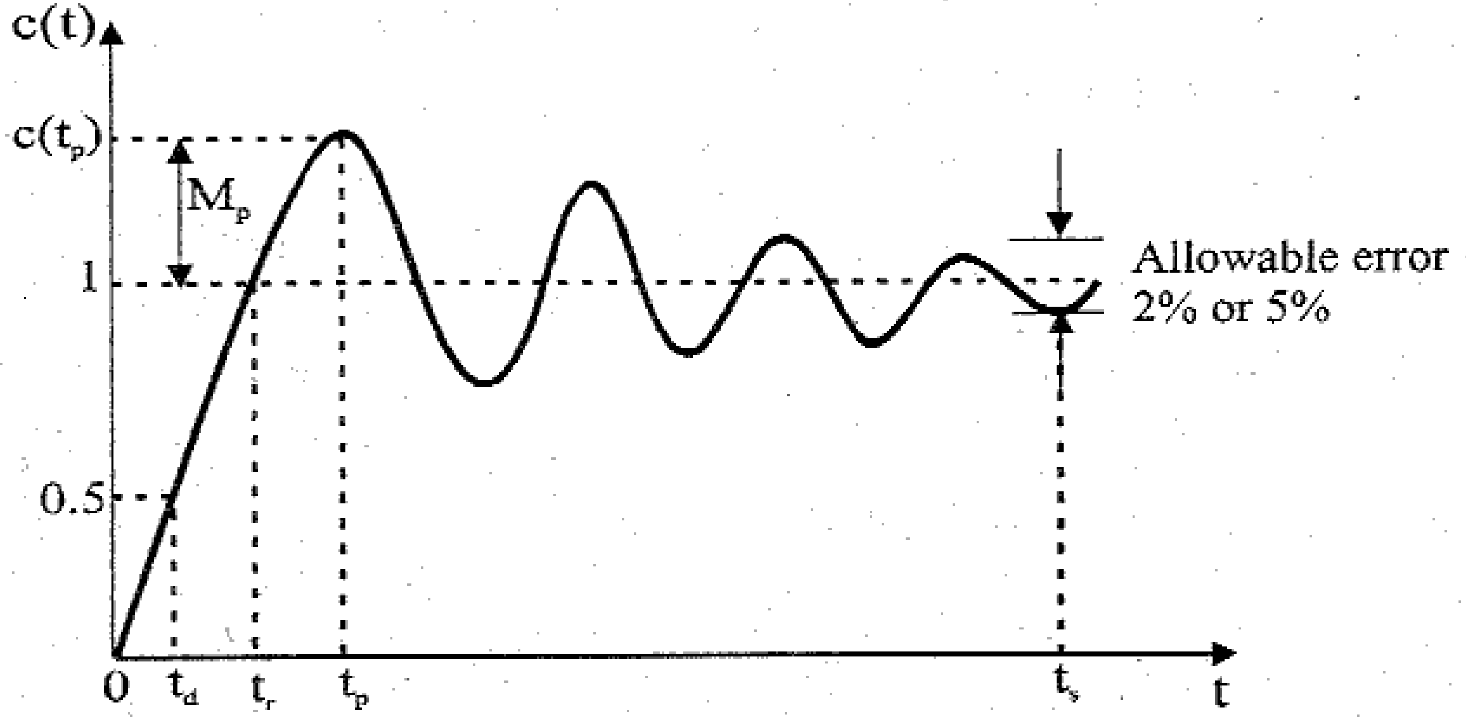
$\tan\theta = \dfrac{\sqrt{1-\zeta^2}}{\zeta}$



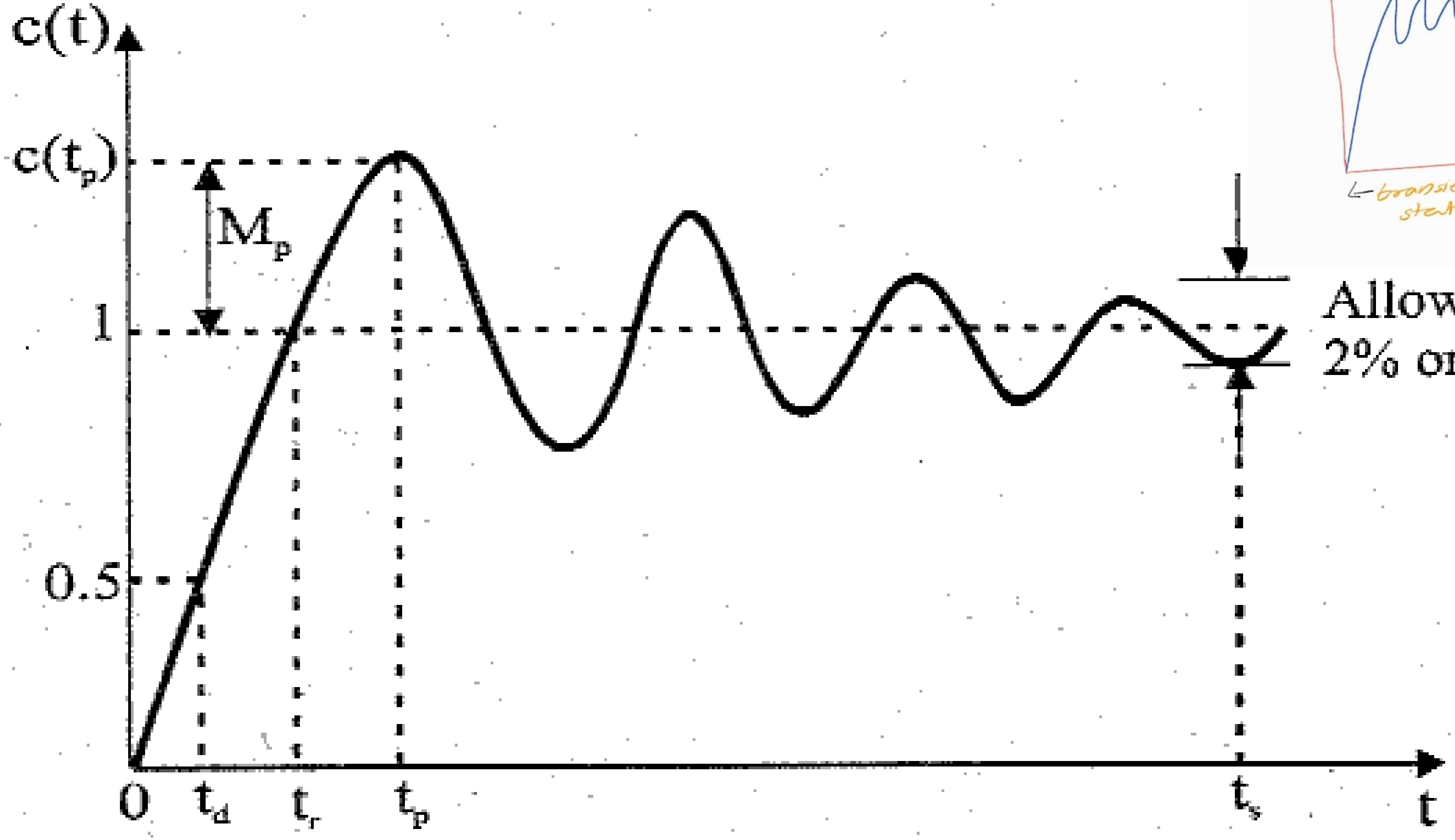
TIME DOMAIN SPECIFICATIONS

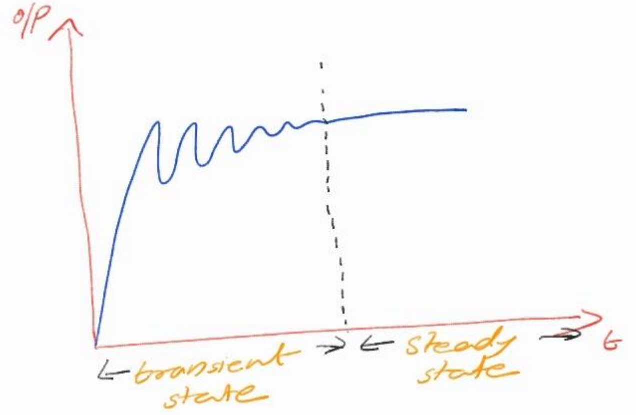
The transient response characteristics of a control system to a unit step input is specified in terms of the following time domain specifications.

1. Delay time, t_d
2. Rise time, t_r
3. Peak time, t_p
4. Maximum overshoot, M_p
5. Settling time, t_s



TIME DOMAIN SPECIFICATIONS





Delay Time (td): It is the time taken for response to reach 50 % of the final value for the very first time

Rise Time (Tr): It is the time taken for response to rise from 0 to 100 % for the very first time.

$$\text{Rise Time, } t_r = \frac{\pi - \theta}{\omega_d}$$

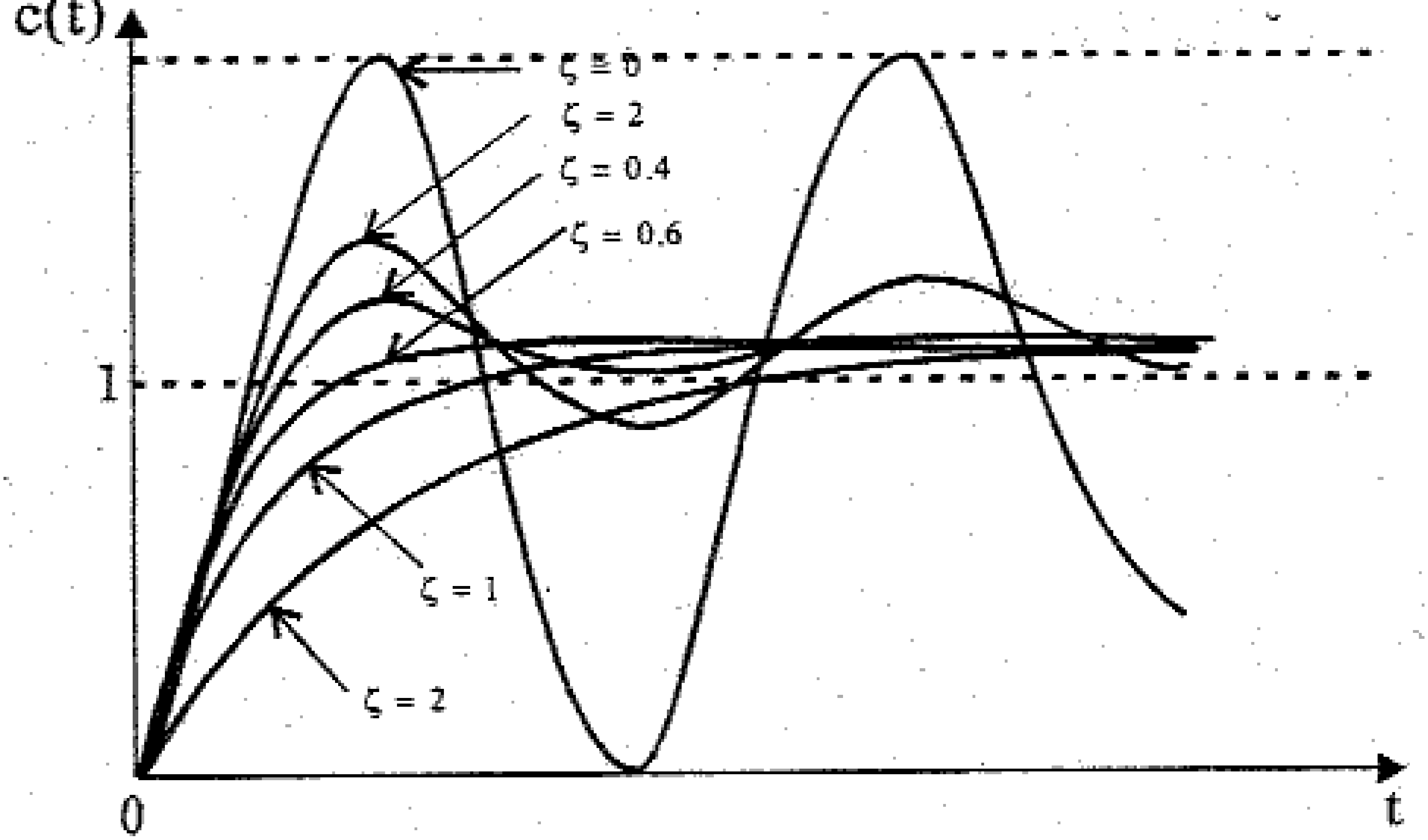
$$\omega_d = \omega_n \sqrt{1-\zeta^2}$$

θ or $\tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}$ *should be measured in radians.*

For underdamped system the rise time is calculated from 0 to 100 %

For overdamped system it is the time taken by the response to rise from 10 to 90%

For critically damped system it is the time taken for response to rise from 5 to 95 %



c(t)
ζ = 0
ζ = 2
ζ = 0.4
ζ = 0.6
1
ζ = 1
ζ = 2
0
t

Peak Time (tp): It is the time taken for the response to reach the peak value the very first time

$$\text{Peak time, } t_p = \frac{\pi}{\omega_d} \qquad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$\text{Peak time, } t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

Peak Overshoot (Mp): It is defined as the ratio of the maximum peak value to the final value were the maximum peak value is measured from final value

$$\text{Percentage Peak Overshoot, } \%M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100$$

Settling Time (t_s): It is defined as the time taken by the response to reach and stay within a specified error.

It is usually expressed as percentage of final value.

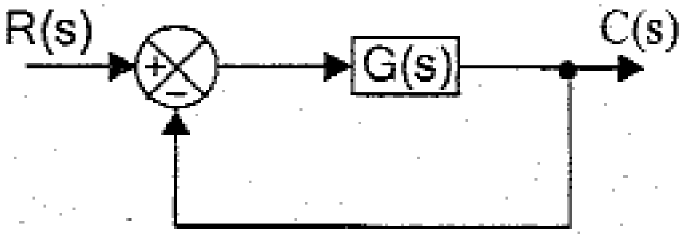
The usual tolerable error is 2% or 5% of the final value.

$$\text{Settling time } (t_s) = \frac{-\ln(\%\ \text{error})}{\zeta\omega_n} = -T.\ln(\%\ \text{error})$$

For the second order system, the time constant, $T = \dfrac{1}{\zeta\omega_n}$

Obtain the response of unity feedback system whose open loop transfer function is

$G(s) = \frac{4}{s(s+5)}$ and when the input is unit step.



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$\frac{C(s)}{R(s)} = \frac{\frac{4}{s\,(s+5)}}{1+\frac{4}{s\,(s+5)}} = \frac{\frac{4}{s\,(s+5)}}{\frac{s\,(s+5)+4}{s\,(s+5)}}$$

$$= \frac{4}{s\,(s+5)+4} = \frac{4}{s^2+5s+4} = \frac{4}{(s+4)\,(s+1)}$$

The response in s-domain, $C(s) = R(s)\ \dfrac{4}{(s+1)\,(s+4)}$

Since the input is unit step, $R(s) = \dfrac{1}{s}$; $\therefore C(s) = \dfrac{4}{s\,(s+1)\,(s+4)}$

By partial fraction expansion, we can write,

$$C(s) = \frac{4}{s\,(s+1)\,(s+4)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+4}$$

$$A = C(s) \times s\Big|_{s=0} = \frac{4}{(s+1)(s+4)}\Bigg|_{s=0} = \frac{4}{1 \times 4} = 1$$

$$B = C(s) \times (s+1)\Big|_{s=-1} = \frac{4}{s(s+4)}\Bigg|_{s=-1} = \frac{4}{-1(-1+4)} = \frac{-4}{3}$$

$$C = C(s) \times (s+4)\Big|_{s=-4} = \frac{4}{s(s+1)}\Bigg|_{s=-4} = \frac{4}{-4(-4+1)} = \frac{1}{3}$$

Response in time domain, $c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{4}{3}\,\frac{1}{s+1} + \frac{1}{3}\,\frac{1}{s+4}\right\}$

$$= 1 - \frac{4}{3}\,e^{-t} + \frac{1}{3}\,e^{-4t} = 1 - \frac{1}{3}\left[4e^{-t} - e^{-4t}\right]$$

The response of a servomechanism is $c(t) = 1 + 0.2e^{-60t} - 1.2e^{-10t}$ when subject to a unit step input. Obtain an expression for closed loop transfer function. Determine the undamped natural frequency and damping ratio.

$$c(t) = 1 + 0.2e^{-60t} - 1.2e^{-10t}$$

Take the Laplace transform

$$C(s) = \frac{1}{s} + 0.2\frac{1}{(s+60)} - 1.2\frac{1}{(s+10)}$$

$$= \frac{(s+60)(s+10) + 0.2\,s\,(s+10) - 1.2\,s\,(s+60)}{s\,(s+60)\,(s+10)}$$

$$= \frac{s^2 + 70s + 600 + 0.2\,s^2 + 2s - 1.2s^2 - 72s}{s\,(s+60)\,(s+10)}$$

$$= \frac{600}{s\,(s+60)\,(s+10)} = \frac{1}{s}\ \frac{600}{(s+60)\,(s+10)}$$

input is unit step, R(s) = 1/s.

$$C(s) = R(s)\ \frac{60}{(s+60)\,(s+10)} = R(s)\ \frac{600}{s^2+70s+600}$$

$$\frac{C(s)}{R(s)} = \frac{600}{s^2+70s+600}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{600}{s^2+70s+600}$$

On comparing we get,

$$\omega_n^2 = 600$$

$$\therefore\ \omega_n = \sqrt{600} = 24.49 \text{ rad / sec}$$

$$2\zeta\omega_n = 70$$

$$\therefore\ \zeta = \frac{70}{2\omega_n} = \frac{70}{2 \times 24.49} = 1.43$$

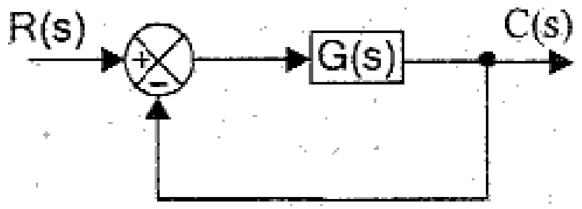
RESULT

The closed loop transfer function of the system, $\frac{C(s)}{R(s)} = \frac{600}{s^2 + 70s + 600}$

Natural frequency of oscillation, $\omega_n = 24.49$ rad/sec

Damping ratio, $\zeta = 1.43$

The unity feedback system is characterized by an open loop transfer function G(s) =K/s (s +10). Determine the gain so that the system will have a damping ratio of 0.5 for this value of K. Determine peak overshoot and time at peak overshoot for a unit step input.



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$G(s) = K/s\,(s+10)$$

$$\frac{C(s)}{R(s)} = \frac{\frac{K}{s(s+10)}}{1+\frac{K}{s(s+10)}}$$

$$= \frac{K}{s\,(s+10)+K} = \frac{K}{s^2+10s+K}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{K}{s^2 + 10s + K}$$

On comparing we get,

$\omega_n^2 = K$	$2\zeta\omega_n = 10$	$K = 100$
$\therefore \omega_n = \sqrt{K}$	Put $\zeta = 0.5$ and $\omega_n = \sqrt{K}$	$\omega_n = 10$ rad / sec
	$\therefore 2 \times 0.5 \times \sqrt{K} = 10$	
	$\sqrt{K} = 10$	

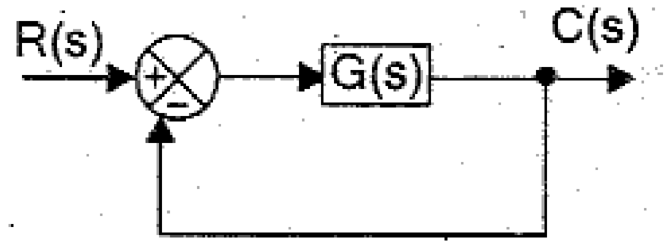
The value of gain, K =100.

Percentage peak overshoot, $\%M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100$

$$= e^{-0.5\pi/\sqrt{1-0.5^2}} \times 100 = 0.163 \times 100 = 16.3\%$$

Peak time, $t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{10\sqrt{1-0.5^2}} = 0.363$ sec

A unity feedback control system has an open loop transfer function, $G(s) = 10/s(s+2)$. Find the rise time, percentage overshoot, peak time and settling time for a step input of 12 units.



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$G(s) = 10/s\,(s+2)$$

$$\frac{C(s)}{R(s)} = \frac{\dfrac{10}{s\,(s+2)}}{1+\dfrac{10}{s\,(s+2)}} = \frac{10}{s\,(s+2)+10}$$

$$= \frac{10}{s^2+2s+10} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$\omega_n^2 = 10$$

$$\omega_n = \sqrt{10} = 3.162 \text{ rad / sec}$$

$$2\zeta\omega_n = 2$$

$$\therefore \zeta = \frac{2}{2\omega_n} = \frac{1}{3.162} = 0.316$$

$$\theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} = \tan^{-1}\frac{\sqrt{1-0.316^2}}{0.316} = 1.249 \text{ rad}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 3.162\sqrt{1-0.316^2} = 3 \text{ rad / sec}$$

$$\text{Rise time, } t_r = \frac{\pi-\theta}{\omega_d} = \frac{\pi-1.249}{3} = 0.63 \text{ sec}$$

$$\text{Percentage overshoot, } \%M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100 = e^{\frac{-0.316\pi}{\sqrt{1-0.316^2}}} \times 100$$

$$= 0.3512 \times 100 = 35.12\%$$

$$\text{Peak overshoot} = \frac{35.12}{100} \times 12 \text{ units} = 4.2144 \text{ units}$$

$$\text{Peak time,} \quad t_p = \frac{\pi}{\omega_d} = \frac{\pi}{3} = 1.047 \text{ sec}$$

$$\text{Time constant, } T = \frac{1}{\zeta\omega_n} = \frac{1}{0.316 \times 3.162} = 1 \text{ sec}$$

$\therefore$ For 5% error, Settling time, $t_s = 3T = 3$ sec

For 2% error, Settling time, $t_s = 4T = 4$ sec

A closed loop servo is represented by the differential equation $\frac{d^2c}{dt^2} + 8\frac{dc}{dt} = 64\ e$

Where c is the displacement of the output shaft, r is the displacement of the input shaft and $e = r - c$.

Determine undamped natural frequency, damping ratio and percentage maximum overshoot for unit step input.

$$\frac{d^2c}{dt^2} + 8\frac{dc}{dt} = 64\ e$$

$$e = r - c$$

$$\frac{d^2c}{dt^2} + 8\frac{dc}{dt} = 64(r - c)$$

Take Laplace Transform

$$s^2\,C(s) + 8s\,C(s) = 64\,[R(s) - C(s)]$$

$$s^2\,C(s) + 8s\,C(s) + 64\,C(s) = 64\,R(s)$$

$$(s^2 + 8s + 64)\,C(s) = 64\,R(s)$$

$$\frac{C(s)}{R(s)} = \frac{64}{s^2 + 8s + 64} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega_n^2 = 64$$

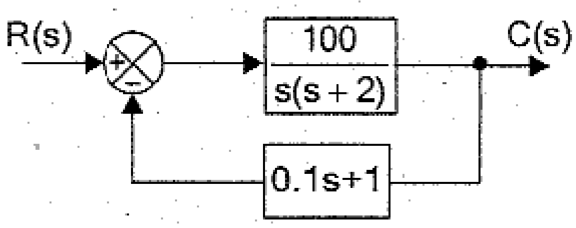
$$\omega_n = 8 \text{ rad / sec}$$

$$2\zeta\omega_n = 8$$

$$\zeta = \frac{8}{2\omega_n} = \frac{8}{2 \times 8} = 0.5$$

$$\%M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100 = e^{\frac{-0.5\pi}{\sqrt{1-0.5^2}}} \times 100 = 16.3\%$$

A positional control system with velocity feedback is shown in fig
What is the response of the system for unit step input.



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)\,H(s)}$$

$$\frac{C(s)}{R(s)} = \frac{\dfrac{100}{s\,(s+2)}}{1+\left(\dfrac{100}{s\,(s+2)}\right)(0.1s+1)} = \frac{\dfrac{100}{s\,(s+2)}}{\dfrac{s(s+2)+100\,(0.1s+1)}{s\,(s+2)}}$$

$$= \frac{100}{s^2+2s+10s+100} = \frac{100}{s^2+12s+100}$$

The roots of the characteristic polynomial are,

$$s_1, s_2 = \frac{-12 \pm \sqrt{144-400}}{2} = \frac{-12 \pm j16}{2} = -6 \pm j8$$

$$C(s) = R(s)\ \frac{100}{s^2+12s+100}$$

$$R(s) = \frac{1}{s}$$

$$C(s) = \frac{1}{s}\ \frac{100}{s^2+12s+100} = \frac{100}{s(s^2+12s+100)}$$

By partial fraction expansion we can write,

$$C(s) = \frac{100}{s\,(s^2+12s+100)} = \frac{A}{s} + \frac{Bs+C}{s^2+12s+100}$$

The residue A is obtained by multiplying C(s) by s and letting s = 0.

$$A = C(s) \times s\big|_{s=0} = \left.\frac{100}{s^2+12s+100}\right|_{s=0} = \frac{100}{100} = 1$$

The residue B and C are evaluated by cross multiplying the following equation

and equating the coefficients of like power of s.

$$100 = A(s^2+12s+100) + (Bs+C)\,s$$
$$100 = As^2 + 12As + 100A + Bs^2 + Cs$$

On equating the coefficients of s^2 we get, $0 = A + B$ $\therefore B = -A = -1$

On equating coefficients of s we get, $0 = 12A + C$ $\therefore C = -12A = -12$

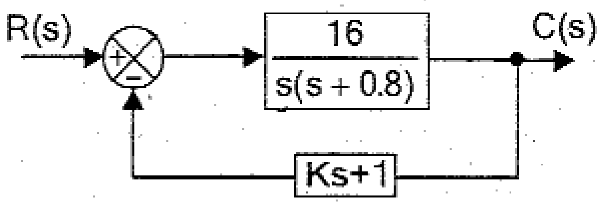
$$C(s) = \frac{1}{s} + \frac{-s-12}{s^2+12s+100} = \frac{1}{s} - \frac{s+12}{s^2+12s+36+64}$$

$$= \frac{1}{s} - \frac{s+6+6}{(s+6)^2+8^2} = \frac{1}{s} - \frac{s+6}{(s+6)^2+8^2} - \frac{6}{(s+6)^2+8^2}$$

$$c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s+6}{(s+6)^2+8^2} - \frac{6}{8}\,\frac{8}{(s+6)^2+8^2}\right\}$$

$$= 1 - e^{-6t}\cos 8t - \frac{6}{8}\,e^{-6t}\sin 8t = 1 - e^{-6t}\left[\frac{6}{8}\sin 8t + \cos 8t\right]$$

A positional control system with velocity feedback is shown in fig
What is the response c(t) to the unit step input. Given that $\zeta = 0.5$.
Also calculate rise time, peak time, maximum overshoot and settling time.



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)\,H(s)}$$

G(s) = 16/s(s + 0.8) and H(s) = Ks +1

$$\frac{C(s)}{R(s)} = \frac{\dfrac{16}{s\,(s+0.8)}}{1 + \dfrac{16}{s\,(s+0.8)}\,(Ks+1)} = \frac{16}{s\,(s+0.8) + 16(Ks+1)}$$

$$= \frac{16}{s^2 + 0.8s + 16Ks + 16} = \frac{16}{s^2 + (0.8 + 16K)\,s + 16}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{16}{s^2 + (0.8 + 16K)\,s + 16}$$

$\omega_n^2 = 16$

$\omega_n = 4 \text{ rad / sec}$

$0.8 + 16K = 2\zeta\omega_n$

$$\therefore\ K = \frac{2\zeta\omega_n - 0.8}{16} = \frac{2 \times 0.5 \times 4 - 0.8}{16} = 0.2$$

$$\frac{C(s)}{R(s)} = \frac{16}{s^2 + (0.8 + 16 \times 0.2)\,s + 16} = \frac{16}{s^2 + 4s + 16}$$

$$C(s) = R(s)\ \frac{16}{s^2 + 4s + 16} \qquad\qquad R(s) = 1/s.$$

$$C(s) = \frac{1}{s}\ \frac{16}{s^2 + 4s + 16} = \frac{16}{s\ (s^2 + 4s + 16)}$$

$$C(s) = \frac{16}{s\ (s^2 + 4s + 16)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4s + 16}$$

The residue A is obtained by multiplying C(s) by s and letting s = 0.

$$A = C(s) \times s\big|_{s=0} = \frac{16}{s^2 + 4s + 16}\Bigg|_{s=0} = \frac{16}{16} = 1$$

On cross multiplication we get, $16 = A\,(s^2 + 4s + 16) + (Bs + C)\,s$

$$16 = As^2 + 4As + 16A + Bs^2 + Cs$$

On equating the coefficients of s^2 we get, $0 = A + B \quad \therefore\ B = -A = -1$

On equating the coefficients of s we get, $0 = 4A + C \ \therefore\ C = -4A = -4$

$$C(s) = \frac{1}{s} + \frac{-s - 4}{s^2 + 4s + 16} = \frac{1}{s} - \frac{s + 4}{s^2 + 4s + 4 + 12}$$

$$= \frac{1}{s} - \frac{s + 2 + 2}{(s + 2)^2 + 12}$$

$$= \frac{1}{s} - \frac{s + 2}{(s + 2)^2 + 12} - \frac{2}{\sqrt{12}}\ \frac{\sqrt{12}}{(s + 2)^2 + 12}$$

$$c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s+2}{(s+2)^2+12} - \frac{2}{\sqrt{12}}\ \frac{\sqrt{12}}{(s+2)^2+12}\right\}$$

$$= 1 - e^{-2t}\cos\sqrt{12}\,t - \frac{2}{2\sqrt{3}}\ e^{-2t}\sin\sqrt{12}\,t$$

$$= 1 - e^{-2t}\left[\frac{1}{\sqrt{3}}\sin(\sqrt{12}\,t) + \cos(\sqrt{12}\,t)\right]$$

$$\left.\begin{array}{l}\text{Damped frequency}\\ \text{of oscillation}\end{array}\right\}\omega_d = \omega_n\sqrt{1-\zeta^2} = 4\sqrt{1-0.5^2} = 3.464 \text{ rad / sec}$$

$$\text{Rise time, } t_r = \frac{\pi - \theta}{\omega_d} = \frac{\pi - 1047}{3.464} = 0.6046 \text{ sec}$$

$$\text{Peak time, } t_p = \frac{\pi}{\omega_d} = \frac{\pi}{3.464} = 0.907 \text{ sec}$$

$$\left.\begin{array}{l}\text{\% Maximum}\\ \text{overshoot}\end{array}\right\}\%M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100 = e^{\frac{-0.5\times\pi}{\sqrt{1-0.5^2}}} \times 100 = 0.163 \times 100 = 16.3\%$$

$$\text{Time constant, } T = \frac{1}{\zeta\omega_n} = \frac{1}{0.5 \times 4} = 0.5 \text{ sec}$$

For 5% error, Settling time, $t_s = 3T = 3 \times 0.5 = 1.5$ sec

For 2% error, Settling time, $t_s = 4T = 4 \times 0.5 = 2$ sec

Module III

Error analysis:

steady state error analysis

static error coefficient of type 0,1, 2 systems

Dynamic error coefficients

Concept of stability:

Time response for various pole locations

stability of feedback system

Routh's stability criterion

ORDER OF A SYSTEM

The input and output relationship of a control system can be expressed by n-th order differential equation.

The order of the system is given by the order of the differential equation governing the system.

$$a_0 \frac{d^n}{dt^n} p(t) + a_1 \frac{d^{n-1}}{dt^{n-1}} p(t) + a_2 \frac{d^{n-2}}{dt^{n-2}} p(t) + \ldots\ldots + a_{n-1} \frac{d}{dt} p(t) + a_n\, p(t) = b_0 \frac{d^m}{dt^m} q(t)$$
$$+ b_1 \frac{d^{m-1}}{dt^{m-1}} q(t) + b_2 \frac{d^{m-2}}{dt^{m-2}} q(t) + \ldots\ldots + b_{m-1} \frac{d}{dt} q(t) + b_m\, q(t)$$

If the system is governed by n-th order differential equation, then the system is called n-th order system

The order can also be determined from the transfer function of the system.

The order of the system is given by the maximum power of 'S' in the denominator polynomial

$$\text{Transfer function, } T(s) = \frac{P(s)}{Q(s)} = \frac{b_0 s^m + b_1 s^{m-1} + b_2 s^{m-2} + \dots\dots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots\dots + a_{n-1} s + a_n}$$

The order of the system is given by the maximum power of s in the denominator polynomial, Q(s).

Here, $Q(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots\dots\dots\dots + a_{n-1} s + a_n$.

Now, **n** is the order of the system

When n = 0, the system is zero order system.

When n = 1, the system is first order system.

When n = 2, the system is second order system and so on.

TYPE NUMBER OF CONTROL SYSTEMS

The type number is specified for loop transfer function G(S)H(S).

The number of poles of the loop transfer function lying at the origin decides the type number of the system

In general if 'N' is the number of poles at the origin then the type number is 'N'

$$G(s)\,H(s) = K\frac{P(s)}{Q(s)} = K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)\,\ldots\ldots\ldots\ldots}{s^N\,(s+p_1)\,(s+p_2)\,(s+p_3)\,\ldots\ldots\ldots}$$

where, $z_1, z_2, z_3, \ldots\ldots\ldots\ldots$ are zeros of transfer function

$p_1, p_2, p_3, \ldots\ldots\ldots\ldots$ are poles of transfer function

K = Constant

N = Number of poles at the origin

TYPE NUMBER OF CONTROL SYSTEMS

$$G(s)\,H(s) = K\frac{P(s)}{Q(s)} = K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)\,\ldots\ldots\ldots\ldots}{s^N\,(s+p_1)\,(s+p_2)\,(s+p_3)\,\ldots\ldots}$$

where, z_1, z_2, z_3, ... are zeros of transfer function

p_1, p_2, p_3, ... are poles of transfer function

K = Constant

N = Number of poles at the origin

If N = 0, then the system is type – 0 system

If N = 1, then the system is type – 1 system

If N = 2, then the system is type – 2 system

If N = 3, then the system is type – 3 system and so on.

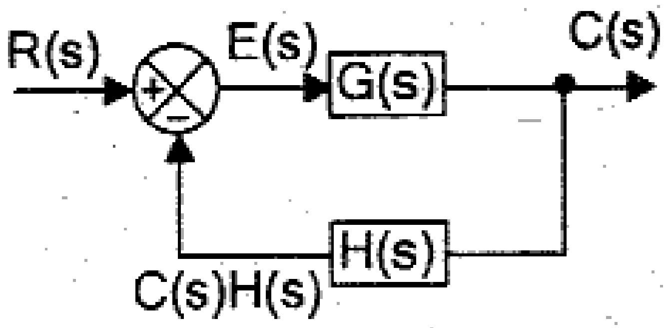
STEADY STATE ERROR

The steady state error is the value of error signal e(t), when ‘t’ tends to infinity.

Steady state error is a measure of system accuracy.

These errors arise from the nature of in inputs, type of system and from non linearity of system components.

The steady-state performance of a stable control system is generally judged by its steady state error to step, ramp and parabolic inputs



Consider a closed loop system

R(s) = Input signal

E(s) = Error signal

C(s) H(s) = Feedback signal

C(s) = Output signal or response

$$E(s) = R(s) - C(s)\,H(s)$$

$$C(s) = E(s)\,G(s)$$

$$E(s) = R(s) - [E(s)\,G(s)]\,H(s)$$

$$E(s) + E(s)\,G(s)\,H(s) = R(s)$$

$$E(s)\,[1 + G(s)\,H(s)] = R(s)$$

$$E(s) = \frac{R(s)}{1 + G(s)\,H(s)}$$

e(t) = error signal in time domain.

$$e(t) = \mathcal{L}^{-1}\{E(s)\} = \mathcal{L}^{-1}\left\{\frac{R(s)}{1 + G(s)\,H(s)}\right\}$$

The steady state error is the value of error signal e(t), when ‘t’ tends to infinity.

steady state error. $e_{ss} = \underset{t \to \infty}{Lt}\ e(t)$

The final value theorem of Laplace transform states that,

If, $F(s) = \mathcal{L}\{f(t)\}$ **then,** $\underset{t\to\infty}{Lt}\ f(t) = \underset{s\to 0}{Lt}\ s\,F(s)$

$$e_{ss} = \underset{t\to\infty}{Lt}\ e(t) = \underset{s\to 0}{Lt}\ sE(s) = \underset{s\to 0}{Lt}\ \frac{sR(s)}{1+G(s)\,H(s)}$$

STATIC ERROR CONSTANTS

Positional error constant, $K_p = \underset{s \to 0}{Lt} \; G(s) \, H(s)$

Velocity error constant, $K_v = \underset{s \to 0}{Lt} \; s \, G(s) \, H(s)$

Acceleration error constant, $K_a = \underset{s \to 0}{Lt} \; s^2 G(s) \, H(s)$

STEADY STATE ERROR WHEN THE INPUT IS UNIT STEP SIGNAL

$$e_{ss} = \underset{s \to 0}{Lt} \frac{sR(s)}{1+G(s)\,H(s)} \qquad R(s) = 1/s$$

$$e_{ss} = \underset{s \to 0}{Lt} \frac{s\frac{1}{s}}{1+G(s)\,H(s)} = \underset{s \to 0}{Lt} \frac{1}{1+G(s)\,H(s)}$$

$$= \frac{1}{1+\underset{s \to 0}{Lt}\, G(s)\,H(s)} = \frac{1}{1+K_p}$$

where, $K_p = \underset{s \to 0}{Lt}\, G(s)\,H(s)$

The constant K_p is called ***positional error constant.***

Type-0 system

$$K_p = \underset{s \to 0}{Lt}\, G(s)\, H(s) = \underset{s \to 0}{Lt}\, K \frac{(s+z_1)\,(s+z_2)\,(s+z_3)\ldots\ldots}{(s+p_1)\,(s+p_2)\,(s+p_3)\ldots\ldots} = K \frac{z_1.z_2.z_3\ldots\ldots}{p_1.p_2.p_3\ldots\ldots} = \text{constant}$$

$$e_{ss} = \frac{1}{1+K_p} = \text{constant}$$

in type-0 systems when the input is unit step there will be a constant steady state error.

Type-1 system

$$K_p = \underset{s \to 0}{Lt}\, G(s)\, H(s) = \underset{s \to 0}{Lt}\, K \frac{(s+z_1)\,(s+z_2)\,(s+z_3)\ldots\ldots}{s\,(s+p_1)\,(s+p_2)\,(s+p_3)\ldots\ldots} = \infty$$

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0$$

In systems with type number 1 and above, for unit step input the value of K_p is infinity and so the steady state error is zero.

STEADY STATE ERROR WHEN THE INPUT IS UNIT RAMP SIGNAL

$$e_{ss} = \underset{s \to 0}{Lt} \frac{sR(s)}{1+G(s)\,H(s)}$$ the input is unit ramp, $R(s) = \frac{1}{s^2}$

$$e_{ss} = \underset{s \to 0}{Lt} \frac{s\frac{1}{s^2}}{1+G(s)\,H(s)} = \underset{s \to 0}{Lt} \frac{1}{s+sG(s)\,H(s)} = \frac{1}{\underset{s \to 0}{Lt}\, sG(s)\,H(s)} = \frac{1}{K_v}$$

where, $K_v = \underset{s \to 0}{Lt}\, s\,G(s)\,H(s)$ The constant K_v is called ***velocity error constant.***

Type-0 system

$$K_v = \underset{s \to 0}{Lt}\, sG(s)\,H(s) = \underset{s \to 0}{Lt}\, sK\frac{(s+z_1)\,(s+z_2)\,(s+z_3)\ldots\ldots}{(s+p_1)\,(s+p_2)\,(s+p_3)\ldots\ldots} = 0$$

$$\therefore e_{ss} = 1/K_v = 1/0 = \infty$$

Hence in type-0 systems when the input is unit ramp, the steady state error is infinity.

Type-1 system

$$K_v = \underset{s\to 0}{Lt}\, sG(s)\,H(s) = \underset{s\to 0}{Lt}\; sK\frac{(s+z_1)\,(s+z_2)\,(s+z_3)\ldots\ldots}{s\,(s+p_1)\,(s+p_2)\,(s+p_3)\ldots\ldots} = K\frac{z_1.z_2.z_3\ldots\ldots}{p_1.p_2.p_3\ldots\ldots} = \text{constant}$$

$$\therefore e_{ss} = 1/K_v = \text{constant}$$

- Hence in type-1 systems when the input is unit ramp there will be a constant steady state error.

Type-2 system

$$K_v = \underset{s\to 0}{Lt}\, sG(s)\,H(s) = \underset{s\to 0}{Lt}\; sK\frac{(s+z_1)\,(s+z_2)\,(s+z_3)\ldots\ldots}{s^2\,(s+p_1)\,(s+p_2)\,(s+p_3)\ldots\ldots} = \infty$$

$$\therefore e_{ss} = 1/K_v = 1/\infty = 0$$

In systems with type number 2 and above, for unit ramp input, the value of K_v is infinity so the steady state error is zero.

STEADY STATE ERROR WHEN THE INPUT IS UNIT PARABOLIC SIGNAL

$$e_{ss} = \underset{s\to 0}{Lt} \frac{sR(s)}{1+G(s)\,H(s)} \qquad R(s) = \frac{1}{s^3}$$

$$e_{ss} = \underset{s\to 0}{Lt} \frac{s\frac{1}{s^3}}{1+G(s)\,H(s)} = \underset{s\to 0}{Lt} \frac{1}{s^2+s^2\,G(s)\,H(s)} = \frac{1}{\underset{s\to 0}{Lt}\; s^2G(s)\,H(s)} = \frac{1}{K_a}$$

$K_a = \underset{s\to 0}{Lt}\; s^2G(s)\,H(s)$ The constant K_a is called ***acceleration error constant.***

Type-0 system

$$K_a = \underset{s\to 0}{Lt}\, s^2G(s)\,H(s) = \underset{s\to 0}{Lt}\; s^2K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)\ldots\ldots}{(s+p_1)\,(s+p_2)\,(s+p_3)\ldots\ldots} = 0$$

$$\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$$

Hence in type-0 systems for unit parabolic input, the steady state error is infinity.

Type-1 system

$$K_a = \underset{s\to 0}{Lt}\, s^2G(s)\,H(s) = \underset{s\to 0}{Lt}\; s^2K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{s\,(s+p_1)\,(s+p_2)\,(s+p_3)......} = 0$$

$$\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$$

Hence in type-1 systems for unit parabolic input, the steady state error is infinity.

Type-2 system

$$K_a = \underset{s\to 0}{Lt}\, s^2G(s)\,H(s) = \underset{s\to 0}{Lt}\; s^2K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{s^2(s+p_1)\,(s+p_2)\,(s+p_3)......} = K\frac{z_1.z_2.z_3......}{p_1.p_2.p_3.....} = \text{constant}$$

$$\therefore e_{ss} = \frac{1}{K_a} = \text{constant}$$

Hence in type-2 system when the input is unit parabolic signal there will be a constant steady state error.

Type-3 system

$$K_a = \underset{s \to 0}{Lt}\, s^2 G(s)\, H(s) = \underset{s \to 0}{Lt}\, s^2 K \frac{(s+z_1)(s+z_2)(s+z_3)......}{s^3 (s+p_1)(s+p_2)(s+p_3)......} = \infty$$

$$\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{\infty} = 0$$

In systems with type number 3 and above for unit parabolic input the value of K_a is infinity and so the steady state error is zero.

Static Error Constant for Various Type Number of Systems

Error Constant	Type number of system			
	0	1	2	3
K_p	constant	∞	∞	∞
K_v	0	constant	∞	∞
K_a	0	0	constant	∞

Steady State Error for Various Types of Inputs

Input Signal	Type number of system			
	0	1	2	3
Unit Step	$\frac{1}{1+K_p}$	0	0	0
Unit Ramp	∞	$\frac{1}{K_v}$	0	0
Unit Parabolic	∞	∞	$\frac{1}{K_a}$	0

For a unity feedback control system the open loop transfer function, $G(s) = \frac{10(s+2)}{s^2(s+1)}$. Find

a) the position, velocity and acceleration error constants,

b) the steady state error when the input is R(s), where $R(s) = \frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}$

---

static error constants

$$H(s)=1$$

Position error constant, $K_p = \underset{s \to 0}{Lt}\, G(s)H(s) = \underset{s \to 0}{Lt}\, G(s) = \underset{s \to 0}{Lt}\, \frac{10(s+2)}{s^2(s+1)} = \infty$

Velocity error constant, $K_v = \underset{s \to 0}{Lt}\, s\, G(s)H(s) = \underset{s \to 0}{Lt}\, s\, G(s) = \underset{s \to 0}{Lt}\, s \frac{10(s+2)}{s^2(s+1)} = \infty$

Acceleration error constant, $K_a = \underset{s \to 0}{Lt}\, s^2 G(s)H(s) = \underset{s \to 0}{Lt}\, s^2 G(s)$

$$= \underset{s \to 0}{Lt}\, s^2 \frac{10(s+2)}{s^2(s+1)} = \frac{10 \times 2}{1} = 20$$

$$E(s) = \frac{R(s)}{1 + G(s)H(s)}$$

$$R(s) = \frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}$$

$$G(s) = \frac{10(s+2)}{s^2(s+1)} \quad ; \quad H(s) = 1$$

$$E(s) = \frac{\frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}}{1 + \frac{10(s+2)}{s^2(s+1)}} = \frac{\frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}}{\frac{s^2(s+1) + 10(s+2)}{s^2(s+1)}}$$

$$= \frac{3}{s}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] - \frac{2}{s^2}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] + \frac{1}{3s^3}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right]$$

$$e_{ss} = \underset{t\to\infty}{Lt}\, e(t) = \underset{s\to 0}{Lt}\, s\, E(s)$$

$$e_{ss} = \underset{s\to 0}{Lt}\; s\left\{\frac{3}{s}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] - \frac{2}{s^2}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] + \frac{1}{3s^3}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right]\right\}$$

$$= \underset{s\to 0}{Lt}\left\{\frac{3s^2(s+1)}{s^2(s+1)+10(s+2)} - \frac{2s(s+1)}{s^2(s+1)+10(s+2)} + \frac{(s+1)}{3s^2(s+1)+30\,(s+2)}\right\} = 0 - 0 + \frac{1}{60}$$

$$= \frac{1}{60}$$

For servomechanisms with open loop transfer function given below explain what type of input signal give rise to a constant steady state error and calculate their values.

a) $G(s) = \dfrac{20(s+2)}{s(s+1)(s+3)}$; b) $G(s) = \dfrac{10}{(s+2)(s+3)}$; c) $G(s) = \dfrac{10}{s^2(s+1)(s+2)}$

a) $G(s) = \dfrac{20(s+2)}{s(s+1)(s+3)}$

Let us assume unity feedback system, $\therefore H(s)=1$

type-1 system.

the velocity (ramp) input will give a constant steady state error.

The steady state error with unit velocity input, $e_{ss} = \dfrac{1}{K_v}$

Velocity error constant, $K_v = \underset{s\to 0}{Lt}\ s\,G(s)\,H(s) = \underset{s\to 0}{Lt}\ s\,G(s)$

$$= \underset{s\to 0}{Lt}\ s\ \frac{20(s+2)}{s(s+1)(s+3)} = \frac{20\times 2}{1\times 3} = \frac{40}{3}$$

Steady state error, $e_{SS} = \dfrac{1}{K_v} = \dfrac{3}{40} = 0.075$

b) $G(s) = \frac{10}{(s+2)(s+3)}$

$H(s)=1.$

it is a type-0 system

the step input will give a constant steady state error.

$$e_{ss} = \frac{1}{1+K_p}$$

$$K_p = \underset{s\to 0}{Lt}\ G(s)H(s) = \underset{s\to 0}{Lt}\ G(s) = \underset{s\to 0}{Lt}\ \frac{10}{(s+2)(s+3)} = \frac{10}{2\times 3} = \frac{5}{3}$$

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\frac{5}{3}} = \frac{3}{3+5} = \frac{3}{8} = 0.375$$

c) $G(s) = \dfrac{10}{s^2(s+1)(s+2)}$

$H(s)=1.$

it is a type-2 system.

acceleration (parabolic) input will give a constant steady state error.

$$e_{ss} = \frac{1}{K_a}$$

$$K_a = \underset{s\to 0}{Lt}\ s^2\, G(s)H(s) = \underset{s\to 0}{Lt}\ s^2\, G(s) = \underset{s\to 0}{Lt}\ s^2 \frac{10}{s^2(s+1)(s+2)} = \frac{10}{1\times 2} = 5$$

$$e_{ss} = \frac{1}{K_a} = \frac{1}{5} = 0.2$$

The open loop transfer function of a servo system with unity feedback is G(s) = 10/s(0.1s+1). Evaluate the static error constants of the system. Obtain the steady state error of the system, when subjected to an input given by the polynominal, $r(t) = a_0 + a_1 t + \frac{a_2}{2} t^2$

$$H(s) = 1$$

$$G(s)\,H(s) = G(s)$$

Position error constant, $K_p = \underset{s \to 0}{Lt}\ G(s) = \underset{s \to 0}{Lt} \frac{10}{s(0.1s+1)} = \infty$

Velocity error constant, $K_v = \underset{s \to 0}{Lt}\ sG(s) = \underset{s \to 0}{Lt}\ s \frac{10}{s(0.1s+1)} = 10$

Acceleration error constant, $K_a = \underset{s \to 0}{Lt}\ s^2 G(s) = \underset{s \to 0}{Lt}\ s^2 \frac{10}{s(0.1s+1)} = 0$

$$E(s) = \frac{R(s)}{1+G(s)H(s)}$$

$$r(t) = a_o + a_1 t + \frac{a_2}{2} t^2$$

$$G(s) = \frac{10}{s(0.1s+1)}$$

$$R(s) = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{a_2}{2}\frac{2!}{s^3} = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{a_2}{s^3}$$

$$E(s) = \frac{R(s)}{1+G(s)H(s)} = \frac{\frac{a_o}{s} + \frac{a_1}{s^2} + \frac{a_2}{s^3}}{1 + \frac{10}{s\,(0.1s+1)}}$$

$$= \frac{\dfrac{a_0}{s} + \dfrac{a_1}{s^2} + \dfrac{a_2}{s^3}}{\dfrac{s\,(0.1s+1)+10}{s\,(0.1s+1)}}$$

$$= \frac{a_0}{s}\left[\frac{s\,(0.1s+1)}{s\,(0.1s+1)+10}\right] + \frac{a_1}{s^2}\left[\frac{s\,(0.1s+1)}{s\,(0.1s+1)+10}\right] + \frac{a_2}{s^3}\left[\frac{s\,(0.1s+1)}{s\,(0.1s+1)+10}\right]$$

$$e_{ss} = \underset{t\to\infty}{Lt}\; e(t) = \underset{s\to 0}{Lt}\; s\,E(s)$$

$$e_{ss} = \underset{s\to 0}{Lt}\, s\left\{\frac{a_o}{s}\left[\frac{s(0.1s+1)}{s(0.1s+1)+10}\right] + \frac{a_1}{s^2}\left[\frac{s(0.1s+1)}{s(0.1s+1)+10}\right] + \frac{a_2}{s^3}\left[\frac{s(0.1s+1)}{s(0.1s+1)+10}\right]\right\}$$

$$= \underset{s\to 0}{Lt}\left\{\frac{a_o s(0.1s+1)}{s(0.1s+1)+10} + \frac{a_1(0.1s+1)}{s(0.1s+1)+10} + \frac{a_2(0.1s+1)}{s\left[s(0.1s+1)+10\right]}\right\} = 0 + \frac{a_1}{10} + \infty \ = \infty$$

RESULT

(a) Position error constant, $K_p = \infty$

(b) Velocity error constant, $K_v = 10$

(c) Acceleration error constant, $K_a = 0$

(d) When input, $r(t) = a_o + a_1 t + \frac{a_2 t^2}{2}$, Steady state error, $e_{ss} = \infty$

Consider a unity feedback system with a closed loop transfer function $\frac{C(s)}{R(s)} = \frac{Ks+b}{s^2+as+b}$

Determine open loop transfer function G(s). Show that steady state error with unit ramp input is given by

$$\frac{(a-K)}{b}$$

$$H(s)=1$$

$$M(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{G(s)}{1+G(s)}$$

$$M(s) = \frac{G(s)}{1+G(s)}$$

$$G(s) = M(s)[1+G(s)] = M(s) + M(s)\,G(s)$$

$$G(s) - M(s)\,G(s) = M(s)$$

$$G(s)[1 - M(s)] = M(s)$$

$$G(s) = \frac{M(s)}{1 - M(s)}$$

$$M(s) = \frac{Ks + b}{s^2 + as + b}$$

$$= \frac{\dfrac{Ks + b}{s^2 + as + b}}{1 - \dfrac{Ks + b}{s^2 + as + b}} = \frac{Ks + b}{\left(s^2 + as + b\right) - (Ks + b)}$$

$$= \frac{Ks + b}{s^2 + as + b - Ks - b} = \frac{Ks + b}{s^2 + (a - k)s} = \frac{Ks + b}{s\left[s + (a.K)\right]}$$

$$K_v = \underset{s\to 0}{Lt}\, s\, G(s)H(s) = \underset{s\to 0}{Lt}\, s\, G(s) = \underset{s\to 0}{Lt}\, s\frac{Ks+b}{s[s+(a-K)]} = \frac{b}{a-K}$$

$$e_{ss} = \frac{1}{K_v} = \frac{a-K}{b}$$

RESULT

$$G(s) = \frac{Ks+b}{s\,[s+(a-K)]}$$

$$e_{ss} = \frac{a-K}{b}$$

GENERALIZED ERROR COEFFICIENT

The error signal in s-domain

$$E(s) = \frac{R(s)}{1+G(s)\,H(s)}$$

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)\,H(s)} \quad = F(s)$$

The equation F(s) can be expressed as a power series of s

$$= C_0 + C_1 s + \frac{C_2}{2!}s^2 + \frac{C_3}{3!}s^3 + \ldots..$$

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)\,H(s)} = C_0 + C_1 s + \frac{C_2}{2!}s^2 + \frac{C_3}{3!}s^3 + \ldots..$$

$$E(s) = C_0 R(s) + C_1 s R(s) + \frac{C_2}{2!} s^2 R(s) + \frac{C_3}{3!} s^3 R(s) + ...$$

inverse Laplace transform

$$e(t) = C_0 r(t) + C_1 \dot{r}(t) + \frac{C_2}{2!} \ddot{r}(t) + \frac{C_3}{3!} \dddot{r}(t) + ... + \frac{C_n}{n!} \overset{n.}{r}(t).....$$

The equation is the general equation for error signal, e(t).

The coefficients C_0, C_1, C_2 ,......C_n are called the generalized error coefficients or dynamic error coefficients.

$$F(s) = \frac{1}{1 + G(s) H(s)}$$

$$\boxed{C_0 = \underset{s \to 0}{Lt} F(s)}$$

$$\boxed{C_1 = \underset{s \to 0}{Lt} \frac{d}{ds} F(s)}$$

$$\boxed{C_2 = \underset{s \to 0}{Lt} \frac{d^2}{ds^2} F(s)}$$

$$\boxed{C_n = \underset{s \to 0}{Lt} \frac{d^n}{ds^n} F(s)}$$

CORRELATION BETWEEN STATIC AND DYNAMIC ERROR COEFFICIENTS

$$C_0 = \frac{1}{1+K_p}$$

$$C_1 = \frac{1}{K_v}$$

$$C_2 = \frac{1}{K_a}$$

Proof

$$C_0 = \underset{s \to 0}{Lt}\, F(s) = \underset{s \to 0}{Lt} \frac{1}{1+G(s)\,H(s)} = \frac{1`}{1+\underset{s \to 0}{Lt}\, G(s)\,H(s)} = \frac{1}{1+K_p}$$

An open loop TF $G(s) = \frac{9}{s+1}$ of unity feed back s/m. Evaluate the 1st three co-eff error series & steady state error for i/p $r(t) = \frac{3t^2}{2}$

$G(s) = \frac{9}{s+1}$ $\quad H(s) = 1$

$r(t) = \frac{3t^2}{2}$

find C_0, C_1, C_2, e_{ss}

$C_0 = \lim_{s \to 0} F(s)$

$F(s) = \frac{1}{1+G(s)H(s)}$

$= \frac{1}{1+\frac{9}{s+1} \cdot 1} = \frac{s+1}{s+1+9}$

$= \frac{s+1}{s+10}$

$$C_0 = \lim_{s \to 0} \frac{s+1}{s+10} = \frac{1}{10} = .1$$

$$C_1 = \lim_{s \to 0} \frac{d}{ds} F(s)$$

$$= \lim_{s \to 0} \frac{d}{ds} \left[\frac{s+1}{s+10} \right]$$

$\frac{u}{v}$ format

$$\left[\frac{u}{v}\right]' = \frac{u'v - uv'}{v^2}$$

$$C_1 = \lim_{s \to 0} \frac{1 \cdot (s+10) - (s+1) \cdot 1}{(s+10)^2}$$

$$= \lim_{s \to 0} \frac{9}{(s+10)^2} = \frac{9}{100}$$

$$= .009$$

$$C_2 = \lim_{s \to 0} \frac{d^2 F(s)}{ds^2}$$

$$= \lim_{s \to 0} \frac{d}{ds}\left[\frac{9}{(s+10)^2}\right]$$

$$= \lim_{s \to 0} 9 \times -2 \cdot \frac{1}{(s+10)^3} (1+0)$$

$$= \lim_{s \to 0} \frac{-18}{(s+10)^3}$$

$$= \frac{-18}{10^3} = \frac{-18}{1000}$$

$$= \underline{\underline{-.018}}$$

$$e(t) = r(t)C_0 + r'(t)C_1 + \frac{r''(t)C_2}{2!}$$

$$r(t) = \frac{3t^2}{2}$$

$$r'(t) = 3t$$

$$r''(t) = 3$$

$$e(t) = \frac{3t^2}{2}(0.1) + 3t(0.09) + \frac{3(-0.018)}{2}$$

$$= 0.15t^2 + 0.27t - 0.027$$

$$e_{ss} = \lim_{t \to \infty} e(t)$$

$$= \lim_{t \to \infty} \left[0.15t^2 + 0.27t - 0.027\right]$$

$$= \infty$$

Open loop tf of a unity feedback s/m is given by $G(s) = \frac{20}{s(s+2)}$. The i/p fn is $r(t) = 2+3t+t^2$ Determine the generalized error coefficients & steady state error

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$$G(s) = \frac{20}{s(s+2)} \qquad H(s) = 1$$

$$r(t) = 2+3t+t^2$$

C_0, C_1, C_2, e_{ss}

$$F(s) = \frac{1}{1+G(s)\cdot H(s)}$$

$$= \frac{1}{1+G(s)}$$

$$= \frac{1}{1+\left[\frac{20}{s(s+2)}\right]} = \frac{s^2+2s}{s^2+2s+20}$$

$$C_0 = \lim_{s \to 0} F(s)$$

$$= \lim_{s \to 0} \frac{s^2+2s}{s^2+2s+20} = 0$$

$$C_1 = \lim_{s \to 0} \frac{d}{ds} F(s)$$

$$\frac{u'v - uv'}{v^2} \qquad u = s^2+2s$$

$$v = s^2+2s+20$$

$$\frac{d}{ds}F(s) = \frac{(2s+2)(s^2+2s+20) - (s^2+2s)(2s+2)}{(s^2+2s+20)^2}$$

$$C_1 = \frac{2 \times 20}{20^2} = \frac{40}{400} = 0.1$$

$$C_2 = \frac{d^2F(s)}{ds^2} = \frac{d}{ds}\left[\frac{d}{ds}F(s)\right]$$

$$\frac{d}{ds}F(s) = \frac{(s^2+2s+20)(2s+2) - (s^2+2s)(2s+2)}{(s^2+2s+20)^2}$$

$$= \frac{2s^3 + 2s^2 + 4s^2 + 4s + 40s + 40 - [2s^3 + 2s^2 + 4s^2 + 4s]}{(s^2+2s+20)^2}$$

$$= \frac{40s+40}{(s^2+2s+20)^2}$$

$$\frac{d^2F(s)}{d^2s} = \frac{40\cdot(s^2+2s+20)^2 - (40s+40)(2(s^2+2s+20)(2s+2))}{(s^2+2s+20)^4}$$

$$= 40\,\frac{\left[(s^2+2s+20)^2 - (s+1)(s^2+2s+20)(4s+4)\right]}{(s^2+2s+20)^4}$$

$$C_2 = \lim_{s \to 0} \frac{d^2 F(s)}{ds^2}$$

$$= 40 \lim_{s \to 0} \frac{\left[(s^2+2s+20)^2 - (s+1)(s^2+2s+20)(4s+4) \right]}{(s^2+2s+20)^4}$$

$$= 40 \cdot \frac{20^2 - 1 \times 20 \times 4}{20^4}$$

$$= 40 \left(\frac{400 - 80}{20^4} \right)$$

$$= 0.08$$

$$e(t) = r(t)C_0 + r'(t)C_1 + \frac{r''(t)C_2}{2!}$$

$$r(t) = 2 + 3t + t^2$$

$$r'(t) = 3 + 2t$$

$$r''(t) = 2$$

$$r'''(t) = 0$$

$$e_{ss} = \lim_{t \to \infty} e(t)$$

$$= \lim_{t \to \infty} \left[(2+3t+t^2)0 + (3+2t)(0.1) + \frac{2}{2!}(0.08) \right]$$

$$= \lim_{t \to \infty} 0.3 + 0.2t + 0.08$$

$$= \infty$$

$$e_{ss} = \infty$$

STABILITY

The term stability refers to the stable working condition of a control system. Every working system is designed to be stable. In a stable system the response or output is predictable, finite and stable for a given input.

The different definition of the stability are the following

1. A system is stable, if its output is bounded (finite) for any bounded (finite) input.

2. A system is asymptotically stable, if in the absence of the input, the output tends towards zero irrespective of initial conditions.

3. A system is stable if for a bounded disturbing input signal the output vanishes ultimately as‘t’ approaches infinity.

4. A system is unstable if for a bounded disturbing input signal the output is of finite amplitude or oscillatory.

5. For a bounded input signal, if the output has constant amplitude of oscillation then the system may be stable or unstable under some limited constraints. Such a system is called **limitedly stable**.

6. If a system output is stable for all variations of its parameters, then the system is called **absolutely stable system.**

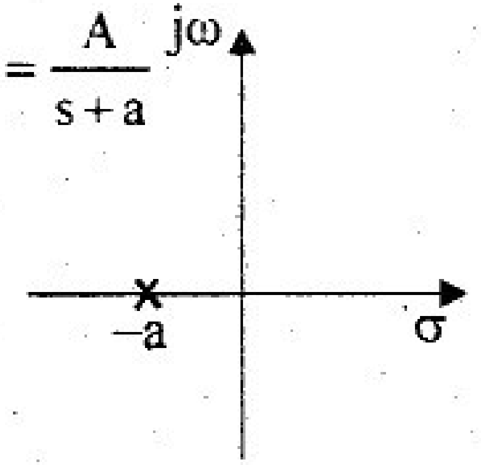
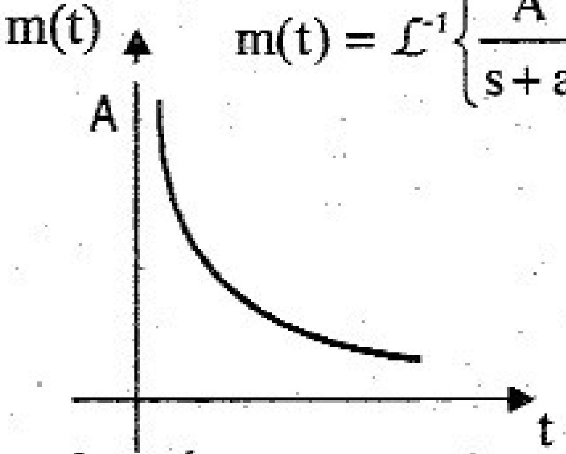
7. If a system output is stable for a limited range of variations of its parameters, then the system is called **conditionally stable system**

LOCATION OF POLES ON s-PLANE FOR STABILITY

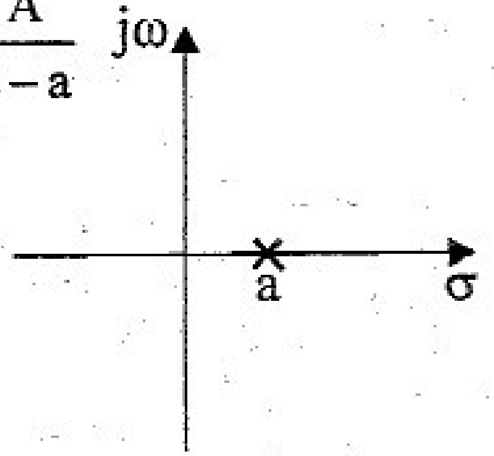
$$M(s) = \frac{C(s)}{R(s)} \qquad C(s) = M(s)\,R(s) \qquad R(s) = \mathcal{L}[\delta(t)] = 1$$

$$J\omega = S$$

IMPULSE RESPONSE OF A SYSTEM $\quad c(t) = \mathcal{L}^{-1}\{C(s)\}$

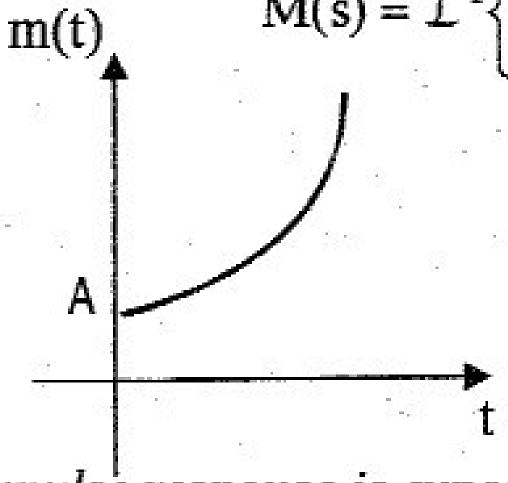
Transfer function, M(s) and location of roots on s-plane	Impulse response, m(t)
$M(s) = \frac{A}{s+a}$  *Root on negative real axis*	$m(t) = \mathcal{L}^{-1}\left\{\frac{A}{s+a}\right\} = Ae^{-at}$  *Impulse response is exponentially decaying. Stable system.*

$$M(s) = \frac{A}{s-a}$$



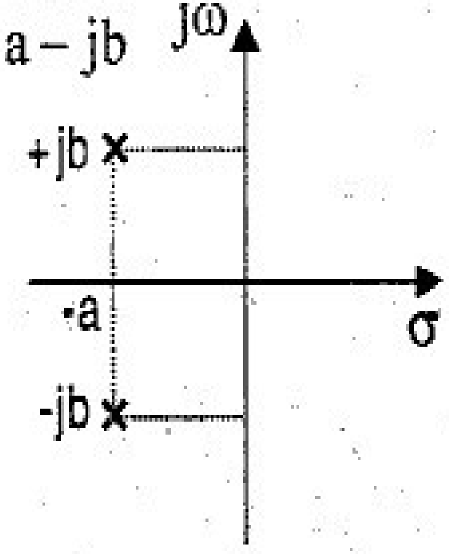
Root on positive real axis

$$M(s) = \mathcal{L}^{-1}\left\{\frac{A}{s-a}\right\} = Ae^{+at}$$



Impulse response is exponentially increasing. Unstable system.

$$M(s) = \frac{A}{s+a+jb} + \frac{A^*}{s+a-jb}$$

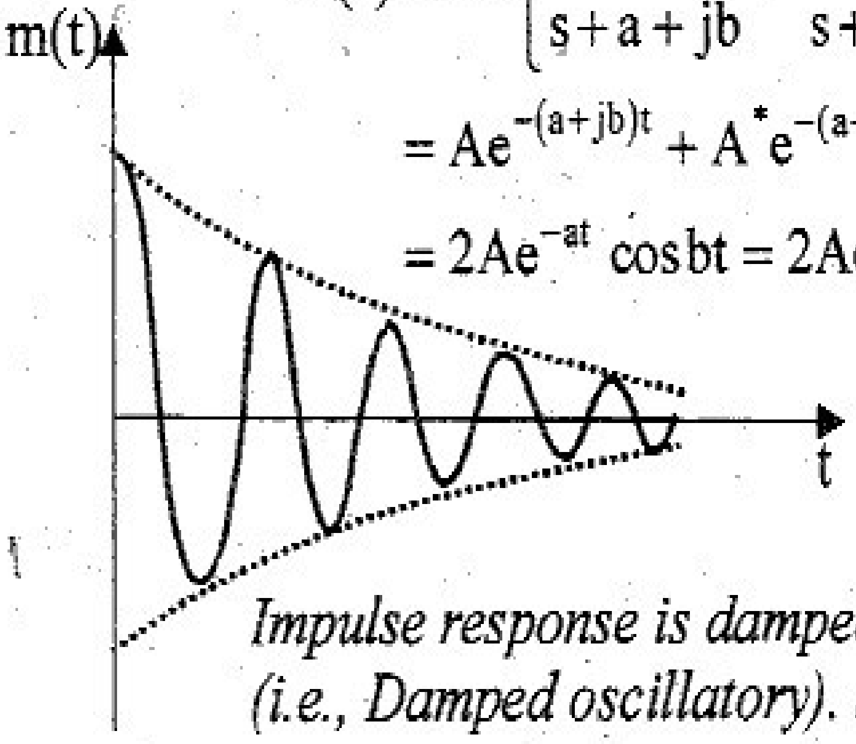


Complex conjugate roots on left half of s-plane

$$m(t) = \mathcal{L}^{-1}\left\{\frac{A}{s+a+jb} + \frac{A^*}{s+a-jb}\right\}$$

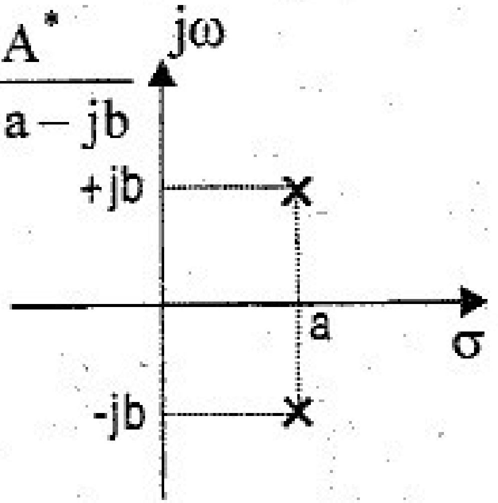
$$= Ae^{-(a+jb)t} + A^*e^{-(a-jb)t}$$

$$= 2Ae^{-at}\cos bt = 2Ae^{-at}\sin(bt + 90^\circ)$$



Impulse response is damped sinusoidal (i.e., Damped oscillatory). Stable system

$$M(s) = \frac{A}{s-a+jb} + \frac{A^*}{s-a-jb}$$

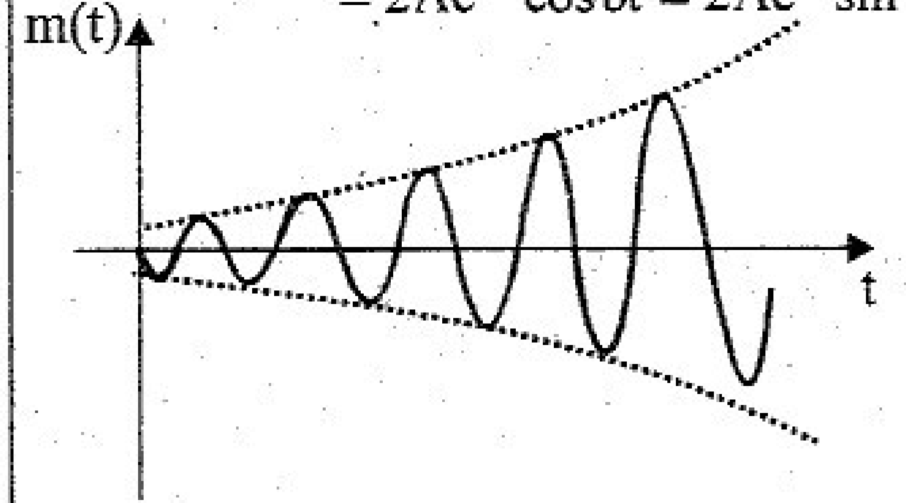


Complex conjugate roots on right half of s-plane

$$m(t) = \mathcal{L}^{-1}\left\{\frac{A}{s-a+jb} + \frac{A^*}{s-a-jb}\right\}$$

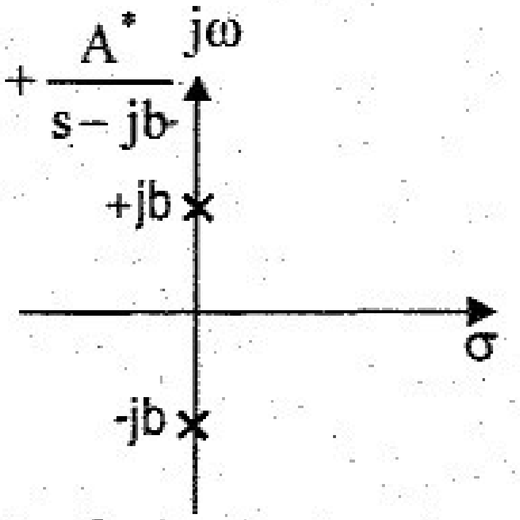
$$= Ae^{-(-a+jb)t} + A^*e^{-(-a-jb)t}$$

$$= 2Ae^{at}\cos bt = 2Ae^{at}\sin(bt+90^\circ)$$



Impulse response is exponentially increasing sinusoidal (i.e., Amplitude of oscillations exponentially increases with time). Unstable system.

$$M(s) = \frac{A}{s + jb} + \frac{A^*}{s - jb}$$

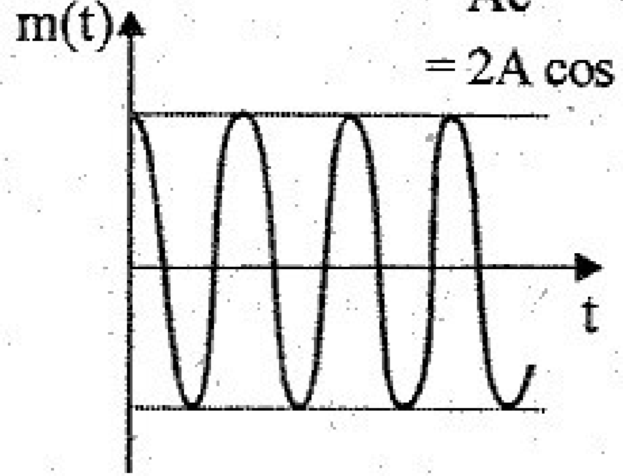


Single pair of roots on imaginary axis

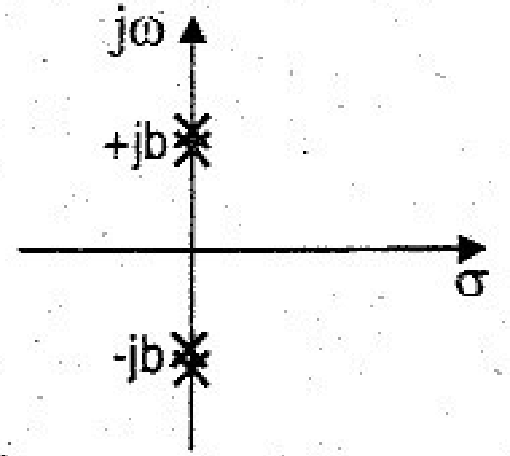
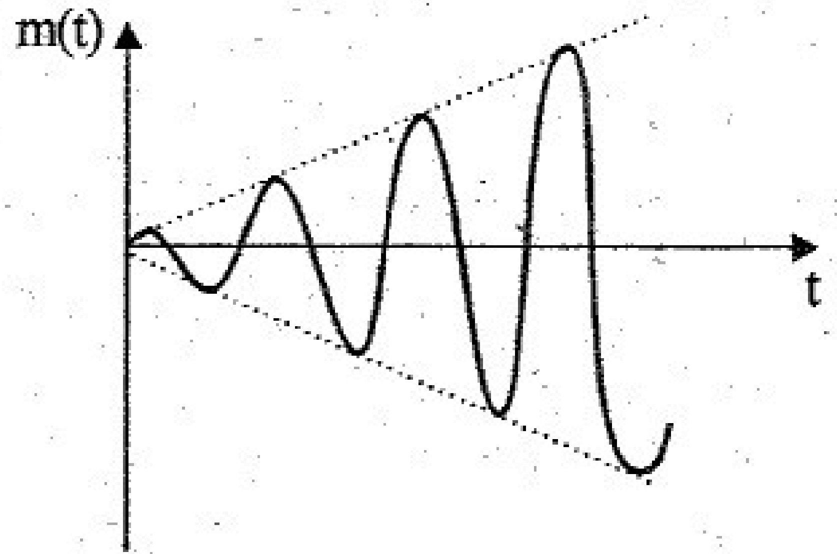
$$m(t) = \mathcal{L}^{-1}\left\{\frac{A}{s + jb} + \frac{A^*}{s - jb}\right\}$$

$$= Ae^{-jbt} + A^* e^{+jbt}$$

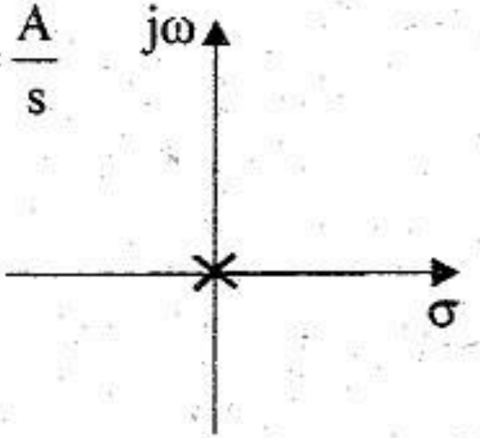
$$= 2A \cos bt = 2A \sin(bt + 90°)$$



Impulse response is oscillatory
Marginally stable

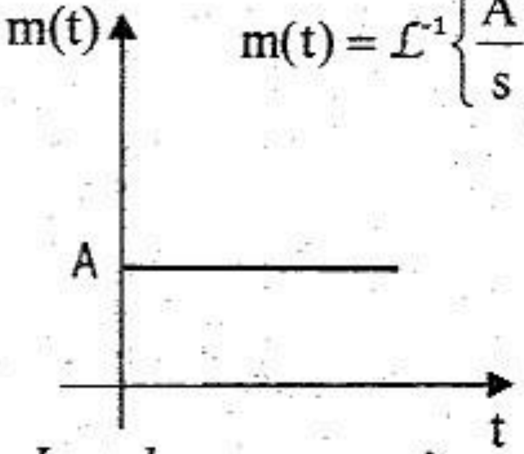
$M(s) = \frac{A}{(s+jb)^2} + \frac{A^*}{(s-jb)^2}$  *Double pair of roots on imaginary axis*	$m(t) = \mathcal{L}^{-1}\left\{\frac{A}{(s+jb)^2} + \frac{A^*}{(s-jb)^2}\right\}$ $= At\,e^{-jbt} + A^*\,t\,e^{+jbt}$ $= 2At\cos bt = 2At\sin(bt + 90°)$  *Impulse response is linearly increasing sinusoidal (i.e., amplitude of oscillations linearly increases with time). Unstable system.*

$$M(s) = \frac{A}{s}$$

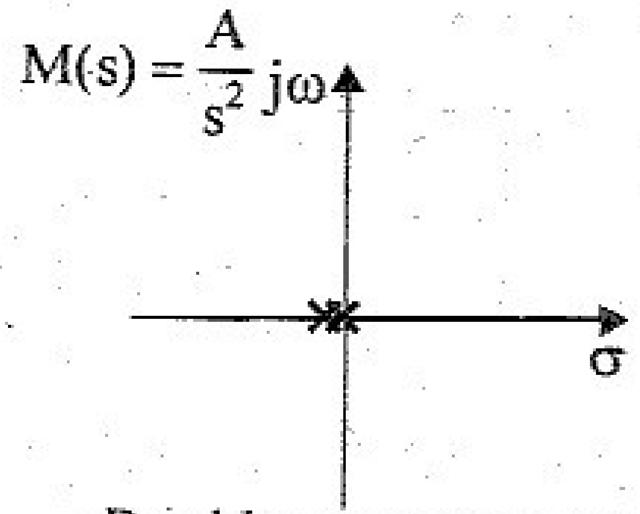


Single root at origin

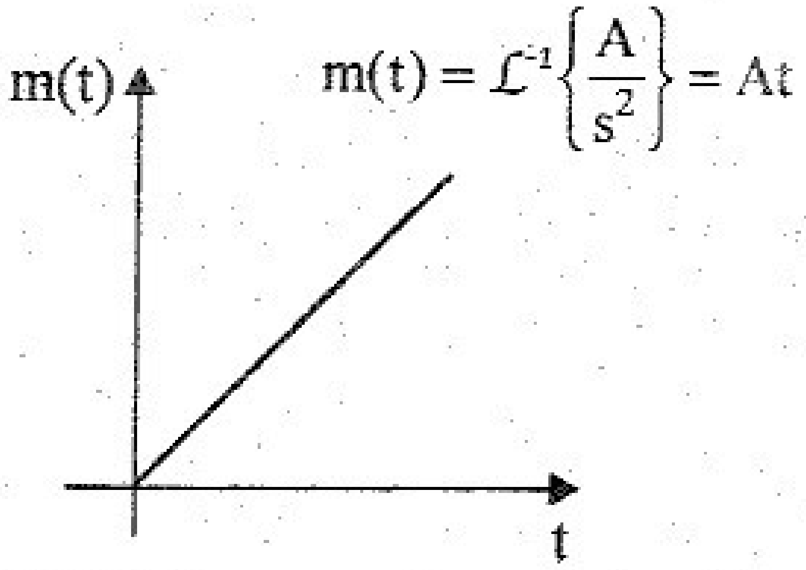
$$m(t) = \mathcal{L}^{-1}\left\{\frac{A}{s}\right\} = A$$



Impulse response is constant.
Marginally stable system.



Double root at origin



Impulse reponse linearly increases with time. Unstable system

The stability of the system depending on the location of roots of characteristic equation

1. If all the roots of characteristic equation has negative real parts, then the system is stable.
2. If any root of the characteristic equation has a positive real part or if there is a repeated root on the imaginary axis then the system is unstable.
3. If the condition (1) is satisfied except for the presence of one or more non repeated roots on imaginary axis, then the system is limitedly or marginally stable.

Methods of determining stability

1. Routh-Hurwitz criterion (RH criterion)
2. Bode plot
3. Nyquist criterion

Routh-Hurwitz criterion (RH criterion)

Necessary and Sufficient condition

The necessary and sufficient condition for stability is that all of the elements in the first column of the Routh array be positive. If this condition is not met, the system is unstable and the number of sign changes in the elements of the first column of the Routh array corresponds to the number of roots of the C.E. in the right half of S plane.

RH criterion is algebraic method for determining the location of poles of a characteristic equation with respect to left half and right half of ‘S’ plane without actually solving the equation

CONSTRUCTION OF ROUTH ARRAY

Let the characteristic polynomial be

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + a_3 s^{n-3} + \ldots a_{n-1} s^1 + a_n s^0$$

The coefficients of the polynomial are arranged in two rows

$$\begin{array}{lllll} s^n & : a_0 & a_2 & a_4 & a_6 \ldots\ldots \\ s^{n-1} & : a_1 & a_3 & a_5 & a_7 \ldots\ldots \end{array}$$

When n is even, the s^n row is formed by coefficients of even order terms

s^{n-1} row is formed by coefficients of odd order terms

When n is odd, the s^n row is formed by coefficients of odd order terms

s^{n-1} row is formed by coefficients of even order terms

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s + a_n = 0, \text{ where } a_0 > 0,$$

s^n	:	a_0	a_2	a_4	a_6	a_8	……
s^{n-1}	:	a_1	a_3	a_5	a_7	a_9	……
s^{n-2}	:	b_0	b_1	b_2	b_3	b_4	……
s^{n-3}	:	c_0	c_1	c_2	c_3	c_4	……
s^1	:	g_0					
s_0	:	h_0					

The other rows of routh array upto s^0 row can be formed by the following procedure. Each row of Routh array is constructed by using the elements of previous two rows. Consider two consecutive rows of Routh array as shown below.

$$s^{n-x} : x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \ldots\ldots$$

$$s^{n-x-1} : y_0 \quad y_1 \quad y_2 \quad y_3 \quad y_4 \quad y_5 \ldots\ldots$$

next row be

$$s^{n-x-2} : z_0 \quad z_1 \quad z_2 \quad z_3 \quad z_4 \ldots\ldots$$

$$z_0 = \frac{y_0 x_1 - y_1 x_0}{y_0} \qquad z_1 = \frac{y_0 x_2 - y_2 x_0}{y_0} \qquad z_2 = \frac{y_0 x_3 - y_3 x_0}{y_0} \qquad z_3 = \frac{y_0 x_4 - y_4 x_0}{y_0}$$

and so on.

Case-I : Normal Routh array (Non-zero elements in the first column of routh array).

Case-II : A row of all zeros.

Case-III : First element of a row is zero but some or other elements are not zero.

Case-I : Normal routh array

The routh array can be constructed as explained above

1. If there is no sign change in the first column of Routh array then all the roots are lying on left half of s-plane and the system is stable.

2. If there is sign change in the first column of routh array, then the system is unstable and the number of roots lying on the right half of s-plane is equal to number of sign changes. The remaining roots are lying on the left half of s-plane.

Case-II : A row of all zeros

METHOD-1

1. Determine the auxiliary polynomial, A(s)
2. Differentiate the auxiliary polynomial with respect to s, to get d A(s)/ds
3. The row of zeros is replaced with coefficients of dA(s)/ds.
4. Continue the construction of the array in the usual manner (as that of case-I)

METHOD-2

1. Determine the auxiliary polynomial, A(s).
2. Divide the characteristic equation by auxiliary polynomial.
3. Construct Routh array using the coefficients of quotient polynomial.

Case-III : First element of a row is zero

let $0 \rightarrow \in$ and complete the construction of array in the usual way (as that of case-I)

Finally let $\in \rightarrow 0$ and determine the values of the elements of the array which are functions of $\in$

Construct Routh array and determine the stability of the system whose characterisitc equation is

$s^6+2s^5+8s^4+12s^3+20s^2+16s+16=0.$

Also determine the number of roots lying on right half of s-plane, left half of s-plane and on imaginary axis

---

$s^6+2s^5+8s^4+12s^3+20s^2+16s+16=0$

s^6	:	1	8	20	16	.... Row-1
s^5	:	2	12	16		.... Row-2

The elements of s^5 row can be divided by 2 to simplify the calculations.

s^5	:	1	6	8	....Row-2

s^6	:	1	8	20	16	.... Row-1
s^5	:	1	6	8		....Row-2
s^4	:	1	6	8		....Row-4
s^3	:	0	0			....Row-4
s^3	:	1	3			....Row-4
s^2	:	3	8			....Row-5
s^1	:	0.33				....Row-6
s^0	:	8				....Row-7

Column-1

$$s^4 : \frac{1\times 8-6\times 1}{1} \quad \frac{1\times 20-8\times 1}{1} \quad \frac{1\times 16-0\times 1}{1}$$

s^4 : 2 12 16

divide by 2

s^4 : 1 6 8

$$s^3: \frac{1\times 6-6\times 1}{1} \quad \frac{1\times 8-8\times 1}{1}$$

s^3: 0 0

The auxiliary polynomial is

$$s^4+6s^2+8=0$$

$$\frac{dA}{ds}=4s^3+12s$$

s^6	:	1	8	20	16	.... Row-1
s^5	:	1	6	8		....Row-2
s^4	:	1	6	8		....Row-4
s^3	:	0	0			....Row-4
s^3	:	1	3			....Row-4
s^2	:	3	8			....Row-5
s^1	:	0.33				....Row-6
s^0	:	8				....Row-7

Column-1

The coefficients of dA/dS are used to form S^3 row

s^3 : 4 12

divide by 4

s^3 : 1 3

$$s^2 : \frac{1\times 6-3\times 1}{1} \quad \frac{1\times 8-0\times 1}{1}$$

s^2 : 3 8

$$s^1 : \frac{3\times 3-8\times 1}{3}$$

s^1 : 0.33

$$s^0 : \frac{0.33\times 8-0\times 3}{0.33}$$

s^0 : 8

There is no sign change in the first column. The row with all zeros indicate the possibility of roots on imaginary axis. Hence the system is limitedly or marginally stable.

The auxiliary polynomial is,

$$s^4 + 6s^2 + 8 = 0$$

Let, $s^2 = x$

$$\therefore x^2 + 6x + 8 = 0$$

The roots of quadratic are, $x = \dfrac{-6 \pm \sqrt{6^2 - 4 \times 8}}{2}$

$$= -3 \pm 1 = -2 \text{ or } -4$$

The roots of auxiliary polynomial is,

$$s = \pm\sqrt{x} = \pm\sqrt{-2} \text{ and } \pm\sqrt{-4}$$

$$= +j\sqrt{2}, -j\sqrt{2}, +j2 \text{ and } -j2$$

The roots of auxiliary polynomial are also roots of characteristic equation.

Hence roots are lying on imaginary axis and the remaining two roots are lying on the left half of S-plane

RESULT

1. The system is limitedly or marginally stable.

2. Four roots are lying on imaginary axis and remaining two roots are lying on left half of s-plane.

Construct Routh array and determine the stability of the system represented by the characteristic equation, $s^5+s^4+2s^3+2s^2+3s+5=0$. Comment on the location of the roots of characteristic equation

---

$s^5+s^4+2s^3+2s^2+3s+5=0.$

s^5	:	1	2	3	.... Row-1
s^4	:	1	2	5	.... Row-2
s^3	:	0	-2		.... Row-3

$$s^3: \frac{1\times 2-2\times 1}{1} \quad \frac{1\times 3-5\times 1}{1}$$

$$s^3: \quad 0 \quad -2$$

Replace 0 by $\in$

$$s^3: \quad \in \quad -2$$

s^3	:	$\in$	-2	.... Row-3
s^2	:	$\frac{2\in+2}{\in}$	5	.... Row-4
s^1	:	$\frac{-(5\in^2+4\in+4)}{2\in+2}$		.... Row-5
s^0	:	5		.... Row-6

$$s^2: \frac{\in\times 2-(-2\times 1)}{\in} \quad \frac{\in\times 5-0\times 1}{\in}$$

$$s^2: \frac{2\in+2}{\in} \quad 5.$$

On letting $\in \to 0$, we get

s^5	:	1	2	3	.... Row-1
s^4	:	1	2	5	.... Row-2
s^3	:	0	-2		.... Row-3
s^2	:	∞	5		.... Row-4
s^1	:	-2			.... Row-5
s^0	:	5			.... Row-6

Column-1

$$s^1: \frac{\frac{2\in+2}{\in}\times(-2)-(5\times\in)}{\frac{2\in+2}{\in}}$$

$$s^1: \frac{-(5\in^2+4\in+4)}{2\in+2}$$

$$s^0: \frac{\frac{-\left(5\in^2+4\in+4\right)}{2\in+2}\times 5-0\times\frac{2\in+2}{\in}}{\frac{-\left(5\in^2+4\in+4\right)}{2\in+2}}$$

s^0: 5

RESULT

There are two sign changes in the first column.

Two roots are lying on right half of S plane.

The system is unstable.

By routh stability criterion determine the stability of the system represented by the characteristic equation, $9s^5 - 20s^4 + 10s^3 - s^2 - 9s - 10 = 0$. Comment on the location of roots of characteristic equation.

s^5	:	9	10	−9	.... Row-1
s^4	:	−20	−1	−10	.... Row-2
s^3	:	9.55	−13.5		.... Row-3
s^2	:	−29.3	−10		.... Row-4
s^1	:	−16.8			.... Row-5
s^0	:	−10			.... Row-6
		Column-1			

$$s^3: \frac{-20\times 10-(-1)\times 9}{-20} \quad \frac{-20\times(-9)-(-10)\times 9}{-20}$$

s^3: 9.55 −13.5

$$s^2: \frac{9.55\times(-1)-(-13.5)\times(-20)}{9.55} \quad \frac{9.55\times(-10)}{9.55}$$

s^2: −29.3 −10

$$s^1: \frac{-29.3\times(-13.5)-(-10)\times 9.55}{-29.3}$$

s^1: −16.8

$$s^0: \frac{-16.8\times(-10)}{-16.8}$$

s^0: −10

RESULT

There are three sign changes

Three roots are lying on right half of S plane and Two roots are lying on left half of S plane

The system is unstable.

The characteristic polynomial of a system is, $s^7+9s^6+24s^5+24s^4+24s^3+24s^2+23s+15=0$. Determine the location of roots on s-plane and hence the stability of the system.

$$s^7+9s^6+24s^5+24s^4+24s^3+24s^2+23s+15=0$$

s^7	:	1	24	24	23	.... Row-1
s^6	:	9	24	24	15	.... Row-2

Divide s^6 row by 3 to simplify the computations.

s^7	:	1	24	24	23	.... Row-1
s^6	:	3	8	8	5	.... Row-2
s^5	:	1	1	1		.... Row-3
s^4	:	1	1	1		.... Row-4
s^3	:	0	0			.... Row-5

$$s^5 : \frac{3\times24-8\times1}{3} \quad \frac{3\times24-8\times1}{3} \quad \frac{3\times23-5\times1}{3}$$

s^5 : 21.33 21.33 21.33

Divide by 21.33

s^5 : 1 1 1

$$s^4 : \frac{1\times8-1\times3}{1} \quad \frac{1\times8-1\times3}{1} \quad \frac{1\times5-0\times3}{1}$$

s^4 : 5 5 5

Divide by 5

s^4 : 1 1 1

$$s^3 : \frac{1\times1-1\times1}{1} \quad \frac{1\times1-1\times1}{1}$$

s^3 : 0 0

The auxiliary polynomial is

$$s^4 + s^2 + 1 = 0$$

$$
\begin{array}{r|l}
 & s^3 + 9s^2 + 23s + 15 \quad \text{(Quotient Polynomial)} \\
\hline
\begin{array}{c} s^4 + s^2 + 1 \\ \text{(Even Polynomial)} \end{array} & s^7 + 9s^6 + 24s^5 + 24s^4 + 24s^3 + 24s^2 + 23s + 15 \\
 & \underset{(-)}{s^7} \quad (-) + s^5 \quad (-) + s^3 \\
\cline{2-2}
 & 9s^6 + 23s^5 + 24s^4 + 23s^3 + 24s^2 + 23s + 15 \\
 & (-)\, 9s^6 + \quad (-) + 9s^4 \quad (-) + 9s^2 \\
\cline{2-2}
 & 23s^5 + 15s^4 + 23s^3 + 15s^2 + 23s + 15 \\
 & (-)\, 23s^5 \quad (-) + 23s^3 \quad (-) + 23s \\
\cline{2-2}
 & 15s^4 \quad + 15s^2 \quad + 15 \\
 & (-)\, 15s^4 \quad (-) + 15s^2 \quad (-) + 15 \\
\cline{2-2}
 & 0 \\
\cline{2-2}
\end{array}
$$

$$s^7 + 9s^6 + 24s^5 + 24s^4 + 24s^3 + 24s^2 + 23s + 15 = 0$$

$$\underbrace{(s^4 + s^2 + 1)}_{\text{Even polynomial}} \; \underbrace{(s^3 + 9s^2 + 23s + 15)}_{\text{Quotient polynomial}} = 0$$

s^3	:	1	23
s^2	:	9	15

Divide s^2 row by 3

		Column-1	
s^3	:	1	23
s^2	:	3	5
s^1	:	21.33	
s^0	:	5	

$$s^1 : \frac{3 \times 23 - 5 \times 1}{3}$$

$$s^1 : \; 21.33$$

$$s^0 : \frac{21.33 \times 5 - 0 \times 3}{21.33}$$

$$s^0 : \; 5$$

The elements of column-1 of quotient polynomial are all positive.
There is no sign change
All the roots of quotient polynomial are lying on the left half of S-plane.

To determine the stability, the roots of auxiliary polynomial should be evaluated

The auxiliary equation is, $s^4 + s^2 + 1 = 0$.

Put, $s^2 = x$ in the auxiliary equation. $s^4 + s^2 + 1 = x^2 + x + 1 = 0$

The roots of quadratic are, $x = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2} = 1\angle 120° \text{ or } 1\angle -120°$

But $s^2 = x$, $\therefore s = \pm\sqrt{x} = \pm\sqrt{1\angle 120°}$ or $\pm\sqrt{1\angle -120°}$

$= \pm\sqrt{1}\angle 120°/2$ or $\pm\sqrt{1}\angle -120°/2$

$= \pm 1\angle 60°$ or $\pm 1\angle -60°$

$= \pm(0.5 + j0.866)$ or $\pm(0.5 - j0.866)$

The roots of auxiliary equation are complex.

Two roots of auxiliary equation are lying on the right half of S plane and the other two on the left half of S plane

RESULT

The roots of auxiliary polynomial is also the roots of C.E.

Hence two roots of C.E are lying on the right half of S plane and remaining five roots are lying on left half of S plane.

The system is unstable.

Determine the range of K for stability of unity feedback system whose open loop transfer function is

$$G(s) = \frac{K}{s(s+1)(s+2)}$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\frac{K}{s(s+1)(s+2)}}{1+\frac{K}{s(s+1)(s+2)}} = \frac{K}{s(s+1)(s+2)+K}$$

The characteristic equation is $s(s+1)(s+2)+K=0$

$$s^3+3s^2+2s+K=0$$

s^3	:	1	2
s^2	:	3	K
s^1	:	$\frac{6-K}{3}$	
s^0	:	K	
		↑ Column-1	

$$s^1 : \frac{3 \times 2 - K \times 1}{3}$$

$$s^1 : \frac{6-K}{3}$$

$$s^0 : \frac{\frac{6-K}{3} \times K - 0 \times 3}{(6-K)/3}$$

$$s^0 : K$$

From s^0 row, for the system to be stable, $K > 0$

From s^1 row, for the system to be stable, $\frac{6-K}{3} > 0$

For $\frac{6-K}{3} > 0$, the value of K should be less than 6.

The range of K for the system to be stable is $0 < K < 6$

MODULE IV

Root locus

General rules for constructing Root loci

Stability from root loci

Effect of addition of poles and zeros.

ROOT LOCUS

Graphical approach

Powerful tool for adjusting the location of closed loop poles to achieve the desired system performance by varying one or more system parameters.

Consider open loop transfer function of the system

$$G(s) = \frac{K}{s\,(s+p_1)\,(s+p_2)}$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\dfrac{K}{s\,(s+p_1)\,(s+p_2)}}{1+\dfrac{K}{s\,(s+p_1)\,(s+p_2)}} = \frac{K}{s\,(s+p_1)\,(s+p_2)+K}$$

The C.E is

$$s\,(s + p_1)\,(s + p_2) + K = 0$$

Roots of characteristic equation depends on the value of ‘K’

‘K’ equal to open loop gain

The value of ‘K’ is varying from zero to infinity

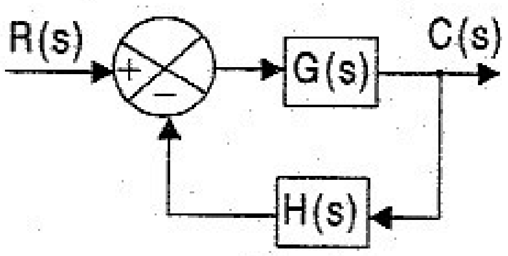
When ‘K’ equal to zero, open loop poles and closed loop poles are identical

when k is varied from zero to infinity the roots of characteristic equation will take different values

Root loci

The path taken by the roots of characteristics equation when open loop gain 'K' is varied from 0 to infinity.

For the single loop system



Characteristic equation is.

$$1+G(s)\,H(s) = 0$$

$$G(s)\,H(s) = -1$$

$G(s)\,H(s) = -1$

The equation can be converted to two

$|G(s)H(s)| = 1$ is called magnitude criterion

$\angle G(s)H(s) = \pm 180^{\circ}(2q+1)$, is called angle criterion

where $q = 0,1,2,3, \ldots$

The magnitude criterion states that $s = s_a$ will be a point on root locus if for that value of s,

$|G(s)\,H(s)| = 1$

The angle criterion states that $s = s_a$ will be a point on root locus if for that value of s

$\angle G(s)\,H(s)$ is equal to an odd multiple of 180°.

$$G(s)H(s) = K \frac{(s+z_1)(s+z_2)(s+z_3).....}{(s+p_1)(s+p_2)(s+p_3).....}$$

$$|G(s)H(s)| = K \frac{|s+z_1| \times |s+z_2| \times |s+z_3|.....}{|s+p_1| \times |s+p_2| \times |s+p_3|....}$$

$$= K \frac{\prod_{i=1}^{m} |s+z_i|}{\prod_{i=1}^{n} |s+p_i|}$$

m = Number of zeros of loop transfer function.

n = Number of poles of loop transfer function.

$$\frac{r_1 < \theta_1}{r_2 < \theta_2} = \frac{r_1}{r_2} < \theta_1 - \theta_2$$

$$(r_1 < \theta_1)(r_2 < \theta_2) = r_1 r_2 < \theta_1 + \theta_2$$

$$K \frac{\prod_{i=1}^{m} |s + z_i|}{\prod_{i=1}^{n} |s + p_i|} = 1 \qquad K = \frac{\prod_{i=1}^{n} |s + p_i|}{\prod_{i=1}^{m} |s + z_i|}$$

The open loop gain 'K' corresponding to a point S=Sa on root locus can be calculated using above equation.

$|s + p_i|$ is equal to the length of vector drawn from S=Pi to S=Sa

$|s + z_i|$ is equal to the length of vector drawn from S=Zi to S=Sa

Hence

$$K = \frac{\text{Product of length of vector from open - loop poles to the point } S = Sa}{\text{Product of length of vector from open loop zeros to the point } S = Sa}$$

$$K \frac{\prod_{i=1}^{m} |s + z_i|}{\prod_{i=1}^{n} |s + p_i|} = 1 \qquad K = \frac{\prod_{i=1}^{n} |s + p_i|}{\prod_{i=1}^{m} |s + z_i|}$$

The open loop gain 'K' corresponding to a point S=Sa on root locus can be calculated using above equation.

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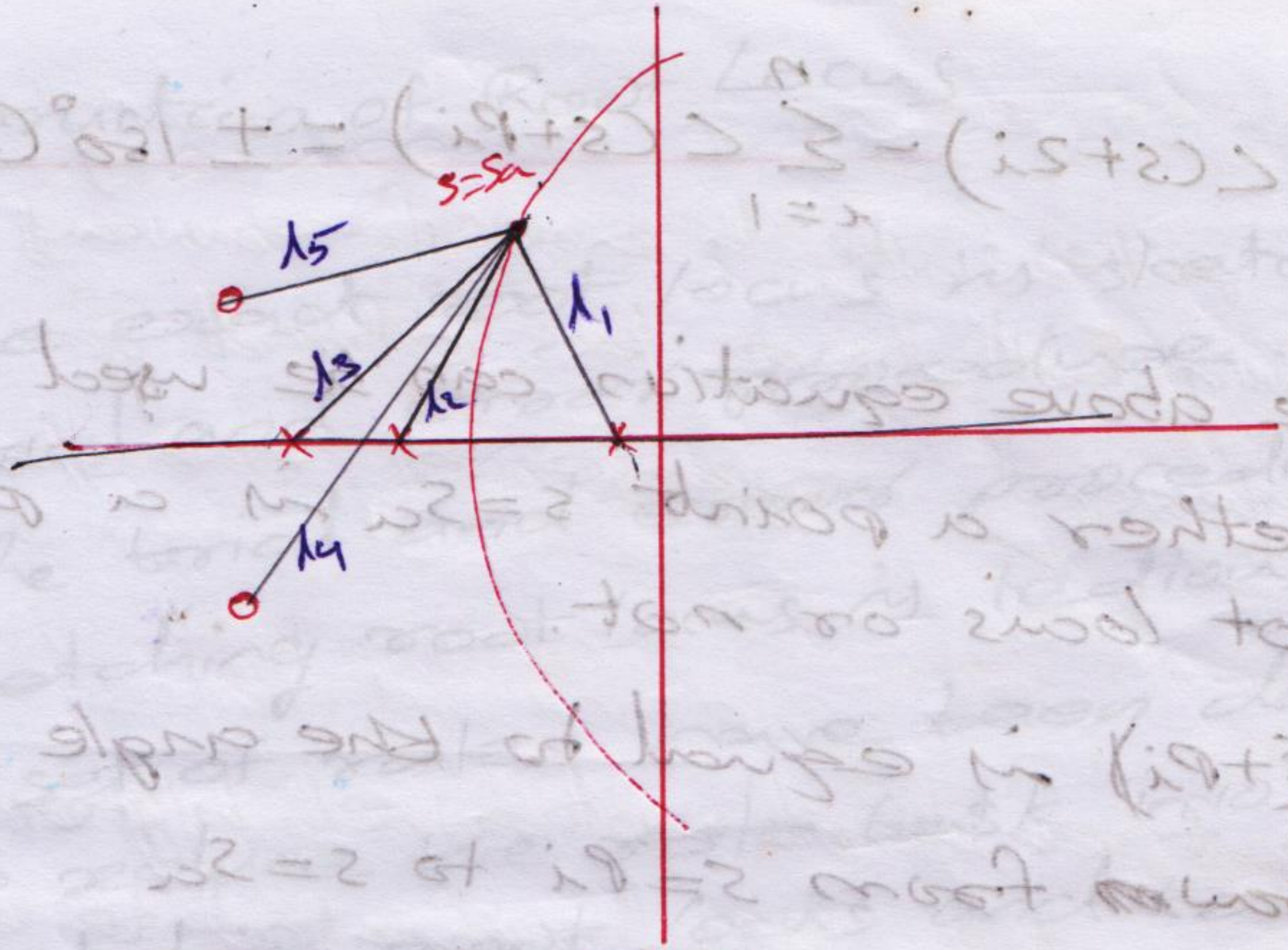
Hence

$$K = \frac{\text{Product of length of vector from open - loop poles to the point } S = Sa}{\text{Product of length of vector from open loop zeros to the point } S = Sa}$$



$$K = \frac{l_1 \cdot l_2 \cdot l_3}{l_4 \cdot l_5}$$

$$\angle G(s)H(s) = \angle(s+z_1) + \angle(s+z_2) + \angle(s+z_3) + \ldots - \angle(s+p_1) - \angle(s+p_2) - \angle(s+p_3)\ldots$$

$$= \sum_{i=1}^{m} \angle(s+z_i) - \sum_{i=1}^{n} \angle(s+p_i)$$

$$\angle G(s)H(s) = \pm 180^\circ (2q+1)$$

$$\sum_{i=1}^{m} \angle(s+z_i) - \sum_{i=1}^{n} \angle(s+p_i) = \pm 180^\circ (2q+1)$$

$$\frac{r_1 < \theta_1}{r_2 < \theta_2} = \frac{r_1}{r_2} < \theta_1 - \theta_2$$

$$(r_1 < \theta_1)(r_2 < \theta_2) = r_1 r_2 < \theta_1 + \theta_2$$

The above equation can be used to check whether a point S=Sa is a point on the root locus or not.

<(S+Pi) is equal to the angle of vector drawn from S=Pi to S=Sa

<(S+Zi) is equal to angle of vector drawn from S=Zi to S=Sa

$$\angle G(s)H(s) = \begin{pmatrix} \text{Sum of angles of vector} \\ \text{from open loop zeros} \\ \text{to the point } s = s_a \end{pmatrix} - \begin{pmatrix} \text{Sum of angles of vector} \\ \text{from open loop poles} \\ \text{to the point } s = s_a \end{pmatrix} = \pm 180° \ (2q + 1)$$

Determination of open loop gain for a specified damping of the dominant roots

The dominant Pole is a pair of complex conjugate pole which decides the transient response of the system.

In higher order systems the dominant poles are given by the poles which are very close to origin, provided all other poles are lying far away from the dominant poles.

The poles which are far away from the origin will have less effect on the transient response of the system.

The transfer function of higher order system can be approximated to a second order transfer function whose standard form of closed loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

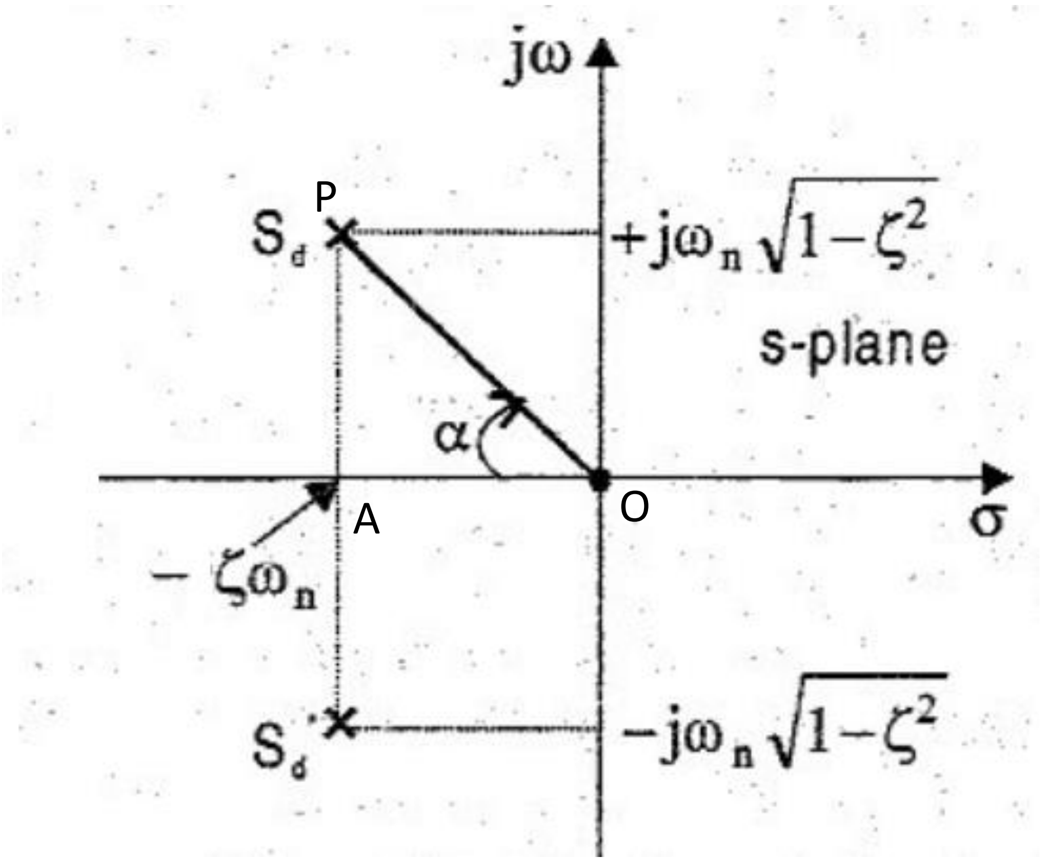
The dominant poles (Sd & Sd*) are given by the roots of quadratic factor

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$s_d = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

The dominant pole can be plotted on the ‘S’ plane as shown below

The dominant pole can be plotted on the 'S' plane as shown below



$$-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

In the right angle triangle or OAP

$$\cos\alpha = \frac{\zeta\,\omega_n}{\omega_n} = \zeta$$

$$\alpha = \cos^{-1}\zeta$$

To fix a dominant pole on root locus, draw a line at an right angle of $\cos^{-1}\zeta$ with respect to negative real axis .

The meeting point of this line with root locus will give the location of dominant pole.

The value of ‘K’ corresponding to the dominant pole can be obtained magnitude condition

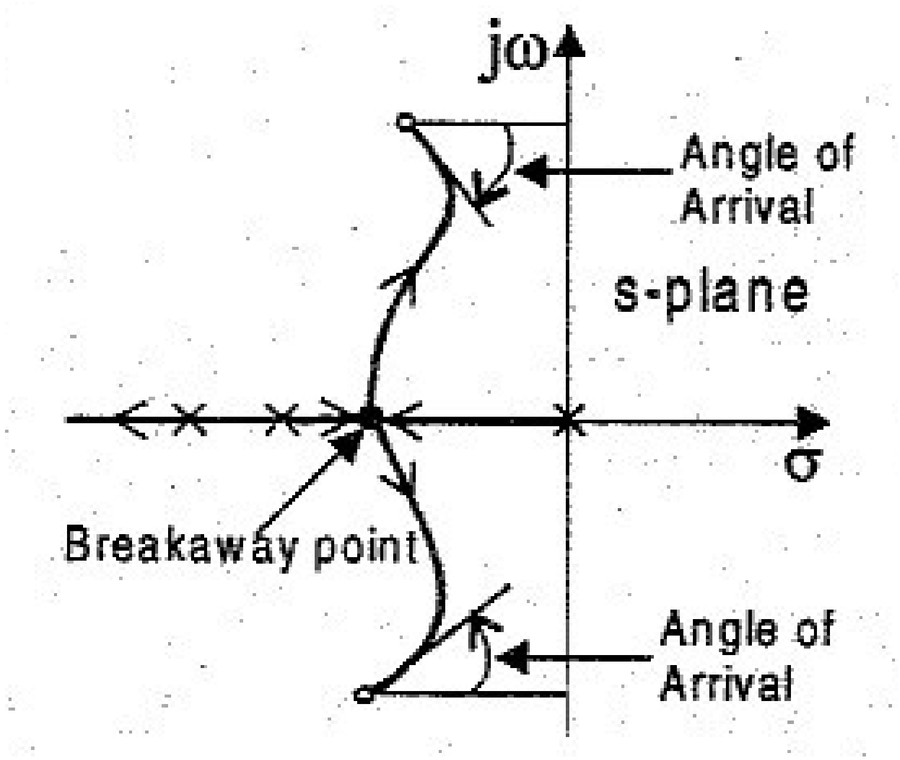
$$\left(\begin{array}{l} \text{The gain 'K' corresponding} \\ \text{to dominant pole, Sd} \end{array} \right) = \frac{\text{Product of length of vectors from open loop poles to dominant pole}}{\text{Product of length of vectors from open loop zeros to dominant pole}}$$

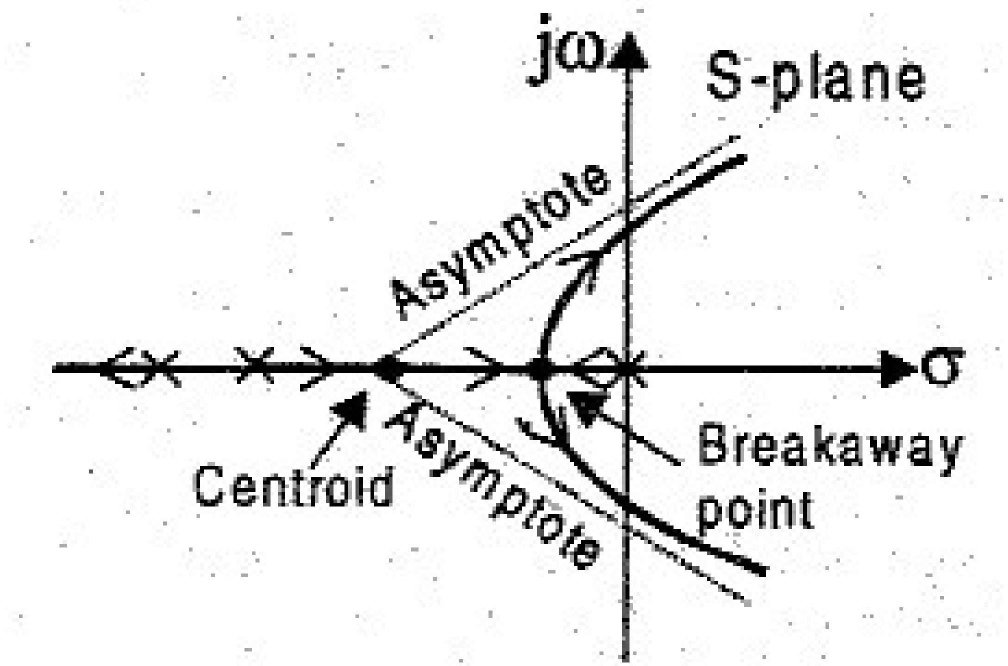
RULES FOR CONSTRUCTION OF ROOT LOCUS

Rule 1

The root locus is symmetrical about the real axis

The root locus on real axis is shown as a bold line





Rule 2: Location of poles and zeros

Locate the poles and zeros of G(S)H(S) on the ‘S’ plane.

The poles are marked by cross “**X**” and zeros are marked by small circle “**o**”.

The number of root locus branches is equal to number of poles of open loop transfer function.

The root locus branch start from open loop poles and terminate at zeros.

If n=number of poles and m=number of finite zeros,

then ‘m’ root locus branches ends at finite zeros.

The remaining (n-m) root locus branches will end at zeros at infinity.

Rule 3: The root locus on real axis

To decide the part of root locus on real axis, take a test point on real axis.

If the total number of poles and zeros on the real axis to the right of this test point is odd number then the test point lies on the root locus.

If it is even then the test point does not lie on the root locus.

The root locus on real axis is shown as a bold line

Rule 4: Angles of asymptotes and centroid

If n is number of poles and m is number of finite zeros, then $n-m$ root locus branches will terminate at zeros at infinity.

These $n-m$ root locus branches will go along an asymptotic path and meets the asymptotes at infinity.

Hence number of asymptotes is equal to number of root locus branches going to infinity.

The angles of asymptotes and the centroid are given by the following formula.

$$\text{Angles of asymptotes} = \frac{\pm 180\,(2q+1)}{n-m}$$

where, $q = 0, 1, 2, 3, \ldots\ldots(n-m)$

Centroid (meeting point of asymptote with real axis)

$$= \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m}$$

Rule 5: Breakaway and Breakin points

The breakaway or breakin points either lie on real axis or exist as complex conjugate pairs.

If there is a root locus on real axis between 2 poles then there exist a breakaway point.

If there is a root locus on real axis between 2 zeros then there exist a breakin point.

Let the C.E. be in the form

$$B(s) + K\,A(s) = 0$$

$$K = \frac{-B(s)}{A(s)} \longrightarrow 1$$

The breakaway and breakin point is given by roots of the equation $\frac{dK}{dS} = 0$

Substitute the value of 'S' in equation -1

If the gain 'K' is positive and real, then there exist a breakaway or breakin point

Rule 5: Breakaway and Breakin points

The breakaway or breakin points either lie on real axis or exist as complex conjugate pairs.

If there is a root locus on real axis between 2 poles then there exist a breakaway point.

If there is a root locus on real axis between 2 zeros then there exist a breakin point.

Let the C.E. be in the form

$$B(s) + K\,A(s) = 0$$

$$K = \frac{-B(s)}{A(s)} \longrightarrow 1$$

The breakaway and breakin point is given by roots of the equation $\frac{dK}{dS} = 0$

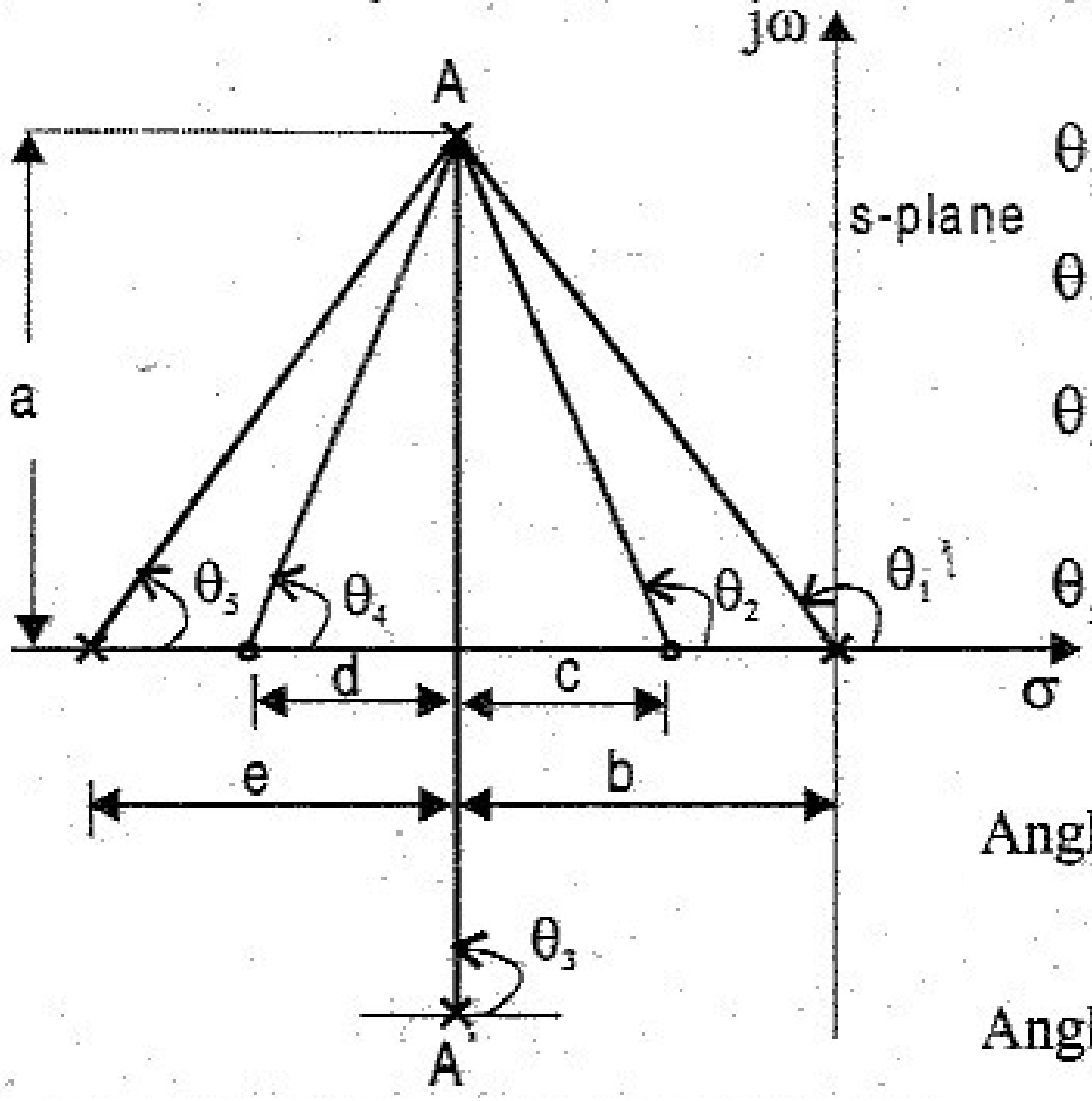
Substitute the value of 'S' in equation -1

If the gain 'K' is positive and real, then there exist a breakaway or breakin point

Rule 6: Angle of Departure and angle of arrival

$$\left.\begin{matrix}\text{Angle of departure}\\ \text{from a complex pole A}\end{matrix}\right\rangle = 180^0 - \begin{pmatrix}\text{sum of angles of vector}\\ \text{to the complex pole A from other poles}\end{pmatrix} + \begin{pmatrix}\text{sum of angles of vectors}\\ \text{to the complex pole A from zeros}\end{pmatrix}$$

$$\left.\begin{matrix}\text{Angle of arrival}\\ \text{at a complex zero A}\end{matrix}\right\rangle = 180^0 - \begin{pmatrix}\text{sum of angles of vectors}\\ \text{to the complex zero A from all other zeros}\end{pmatrix} + \begin{pmatrix}\text{sum of angles of vectors}\\ \text{to the complex zero A from poles}\end{pmatrix}$$



$$\theta_1 = 180° - \tan^{-1}\frac{a}{b}$$

$$\theta_2 = 180° - \tan^{-1}\frac{a}{c}$$

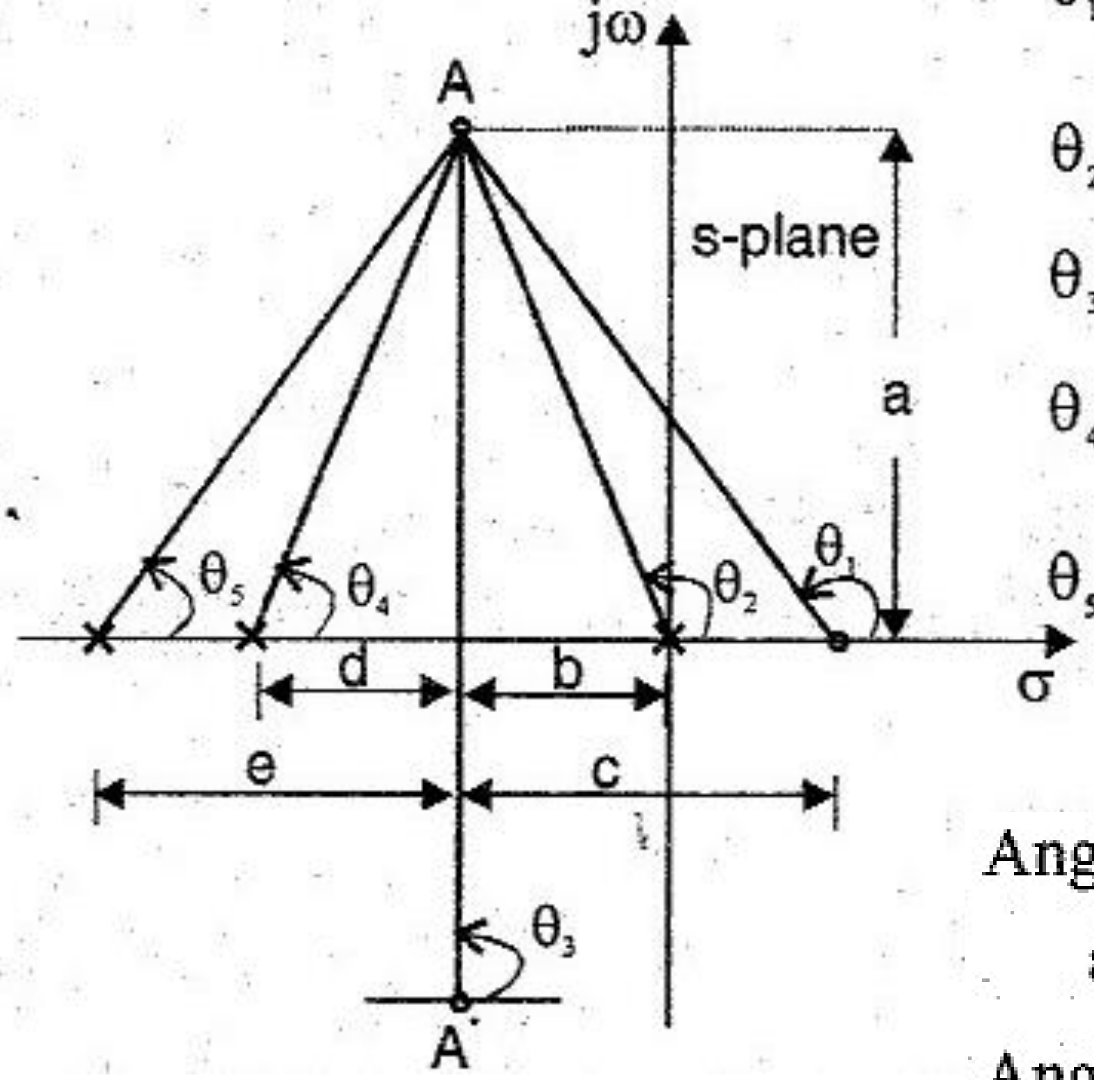
$$\theta_3 = 90°$$

$$\theta_4 = \tan^{-1}\frac{a}{d}$$

$$\theta_5 = \tan^{-1}\frac{a}{e}$$

$$\left.\begin{array}{c}\text{Angle of departure}\\ \text{at pole A}\end{array}\right\} = 180° - (\theta_1 + \theta_3 + \theta_5) + (\theta_2 + \theta_4)$$

$$\left.\begin{array}{c}\text{Angle of departure}\\ \text{at pole A *}\end{array}\right\} = -[\text{Angle of departure at pole A}]$$



$$\theta_1 = 180° - \tan^{-1}\frac{a}{b}$$

$$\theta_2 = 180° - \tan^{-1}\frac{a}{c}$$

$$\theta_3 = 90°$$

$$\theta_4 = \tan^{-1}\frac{a}{d}$$

$$\theta_5 = \tan^{-1}\frac{a}{e}$$

$$\left.\begin{array}{r}\text{Angle of arrival}\\ \text{at zero B}\end{array}\right\} = 180° - (\theta_1 + \theta_3) + (\theta_2 + \theta_4 + \theta_5)$$

$$\left.\begin{array}{r}\text{Angle of arrival}\\ \text{at zero B*}\end{array}\right\} = -[\text{Angle of arrival at zero B}]$$

Rule 7: Point of intersection of root locus with imaginary axis

Letting $s = j\omega$ in the C.E and separate the real part and imaginary part.

Equate real part to zero.

Equate imaginary part to zero.

Solve the equation for w and K.

The value of 'ω' gives the point where the root locus crosses imaginary axis.

The value of K gives the value of gain K at the crossing point.

This value of K is the limiting value of K for stability of the system

Rule 8: Test points and root locus

Take a series of test point in the broad neighbourhood of the origin of the ‘S’ plane and adjust the test point to satisfy angle criterion.

Sketch the root locus by joining the test point by smooth curve

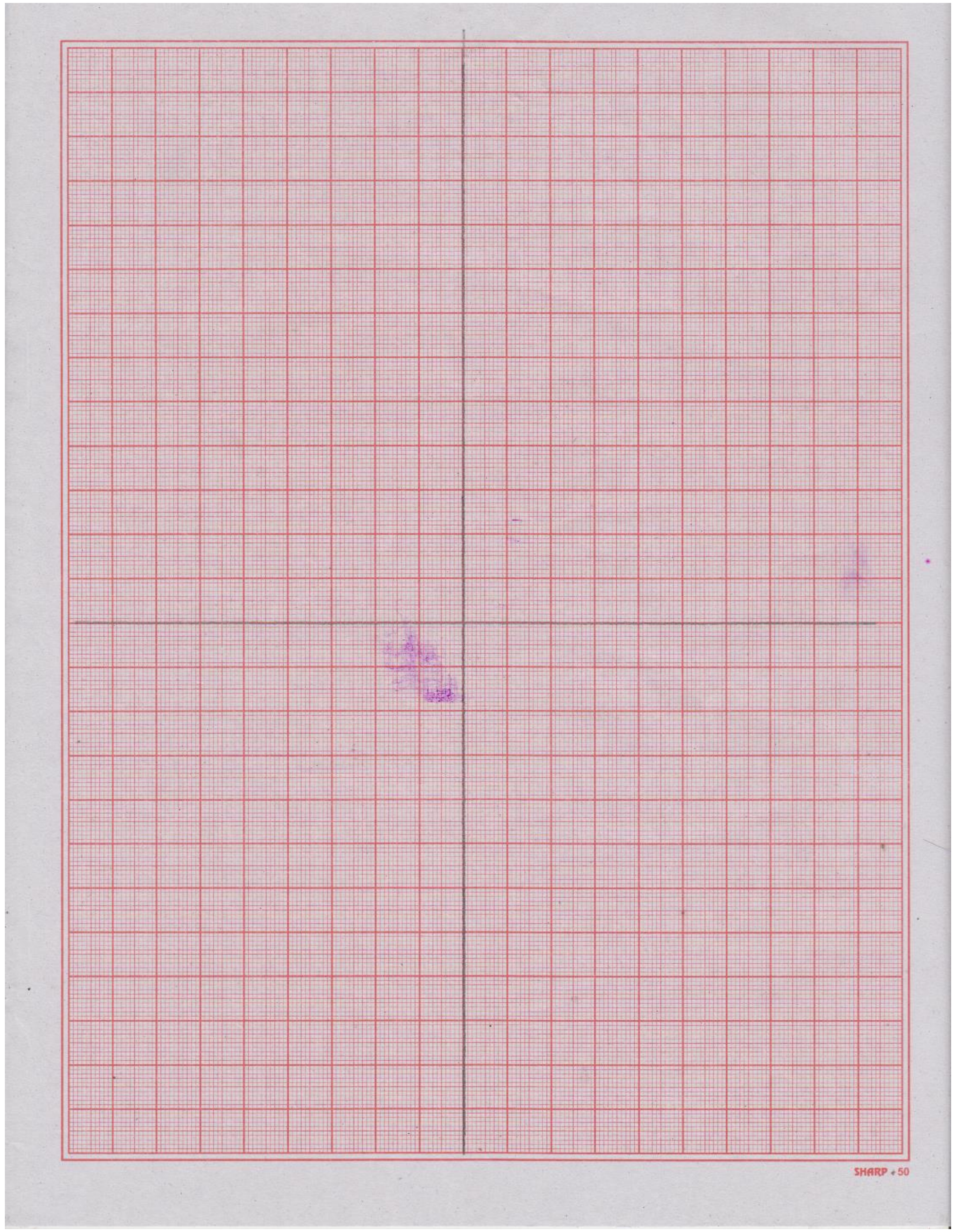
A unity feedback control system has an open loop transfer function, $G(s) = \frac{K}{s\,(s^2 + 4s + 13)}$. Sketch the root locus.

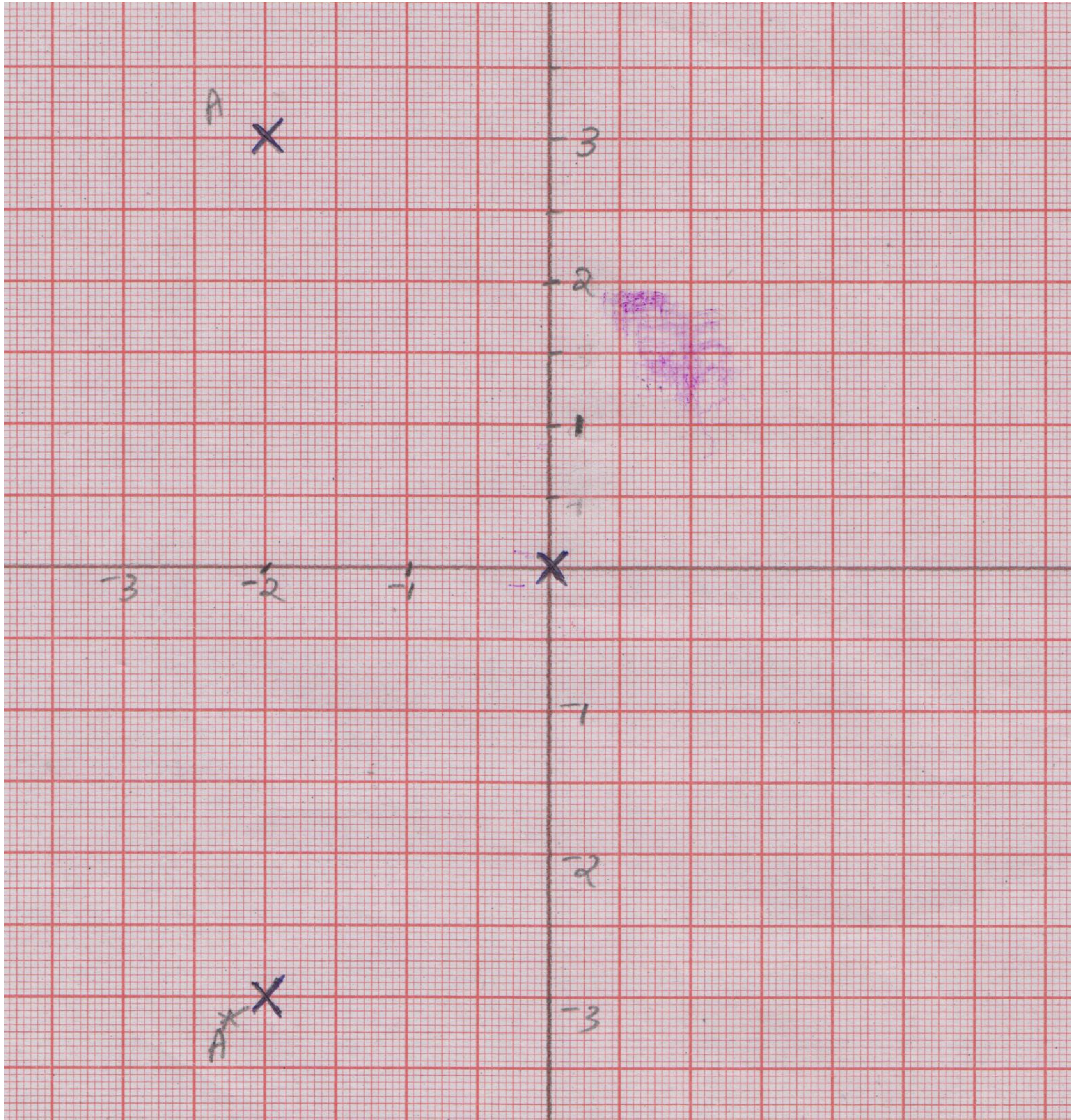
To locate poles and zeros

The poles of open loop transfer function are the roots of the equation $s\,(s^2+4s+13)=0$.

The roots of the quadratic are, $s = \frac{-4 \pm \sqrt{4^2 - 4 \times 13}}{2} = -2 \pm j3$

The poles are lying at $s = 0, -2 + j3$ and $-2 - j3$


SHARP • 50


A
3
2
1
-3
-2
-1
-1
-2
-3
A'

To find the root locus on real axis

the entire negative real axis will be part of root locus

To find angles of asymptotes and centroid

$$\text{Centroid} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n - m} = \frac{0 - 2 + j3 - 2 - j3 - 0}{3} = \frac{-4}{3} = -1.33$$

$$\text{Angles of asymptotes} = \frac{\pm 180^{\circ}\ (2q + 1)}{n - m} \quad ; \quad q = 0,\ 1, \ldots\ldots\ldots n - m$$

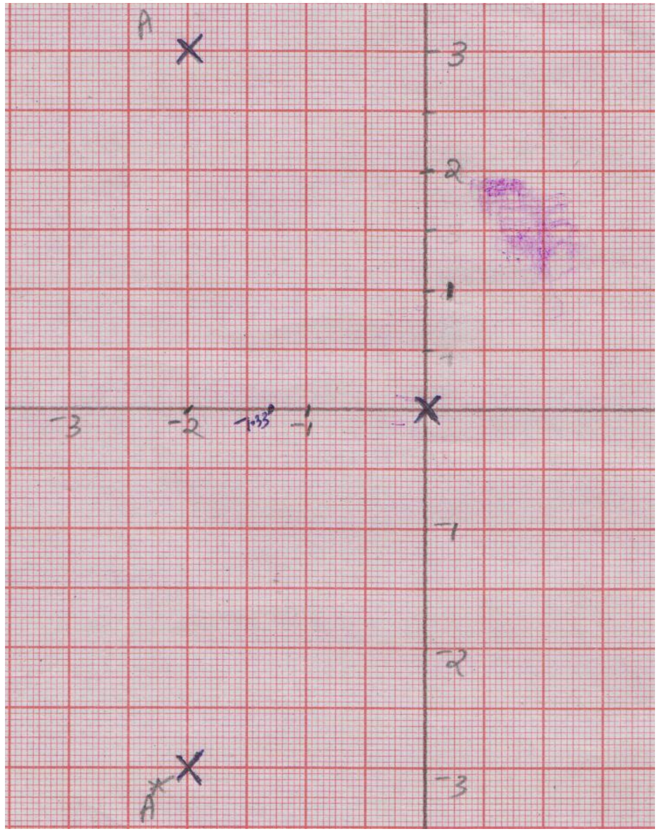
Here $n = 3$, and $m = 0$. $\therefore q = 0, 1, 2, 3$.

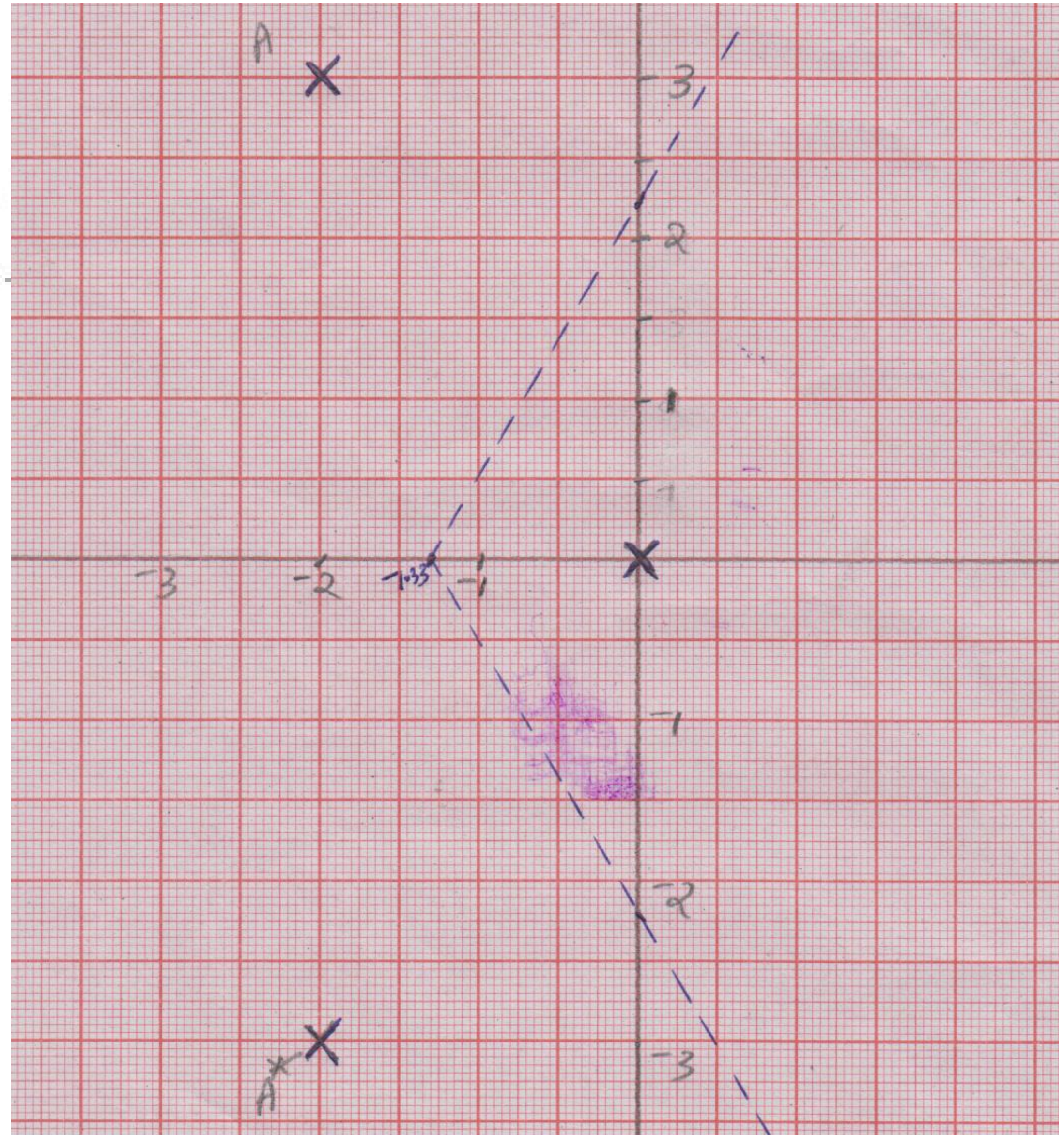
When $q = 0$, $\quad \text{Angles} = \pm \dfrac{180^{\circ}}{3} = \pm 60^{\circ}$

When $q = 1$, $\quad \text{Angles} = \pm \dfrac{180° \times 3}{3} = \pm 180°$

When $q = 2$, $\quad \text{Angles} = \pm \dfrac{180° \times 5}{3} = \pm 300° = \mp 60°$

When $q = 3$, $\quad \text{Angles} = \pm \dfrac{180° \times 7}{3} = \pm 420° = \pm 60°$





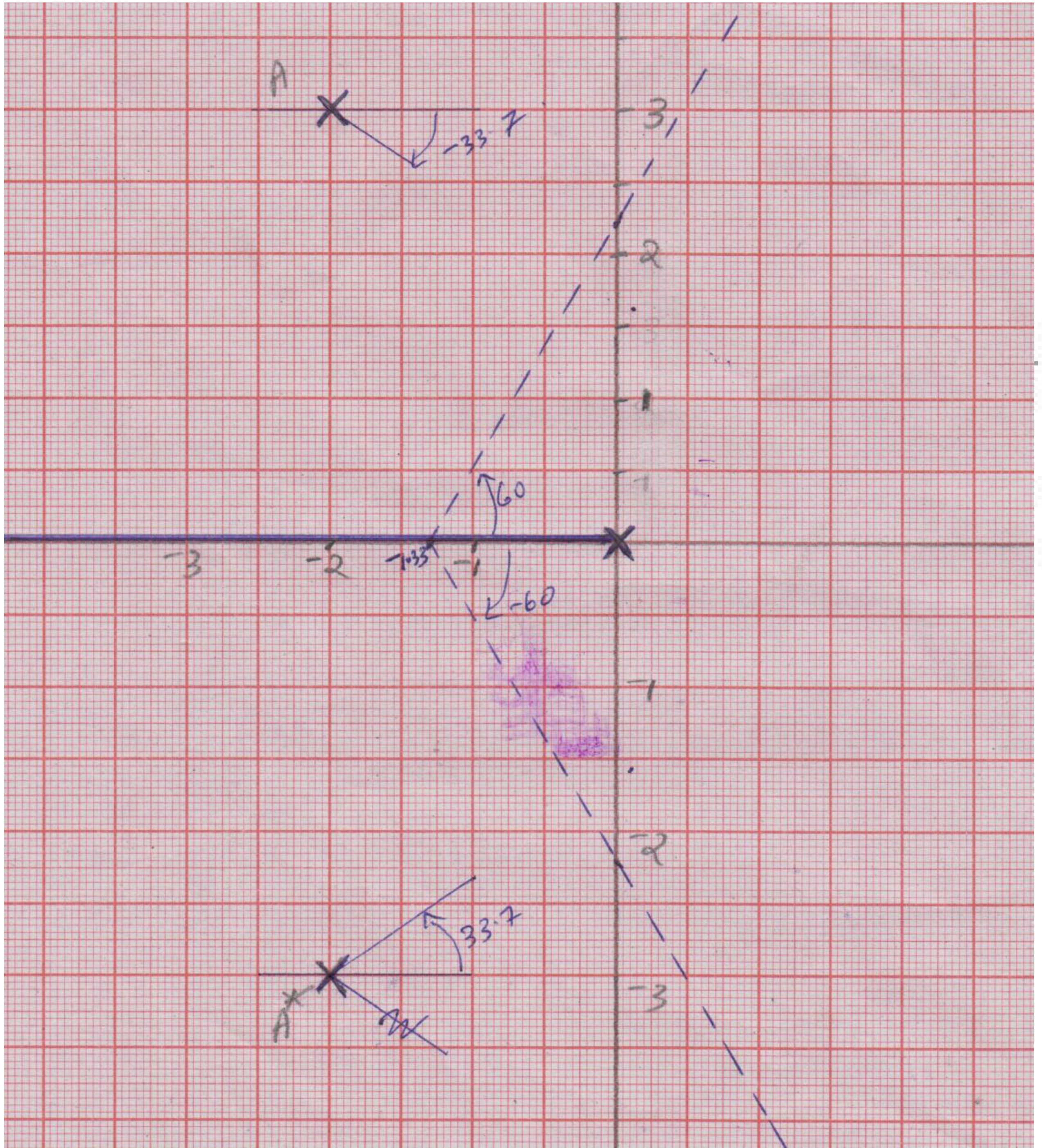
To find the breakaway and breakin points

There is no root locus exist between two poles or two zeros. So the root locus has neither breakaway nor breakin point

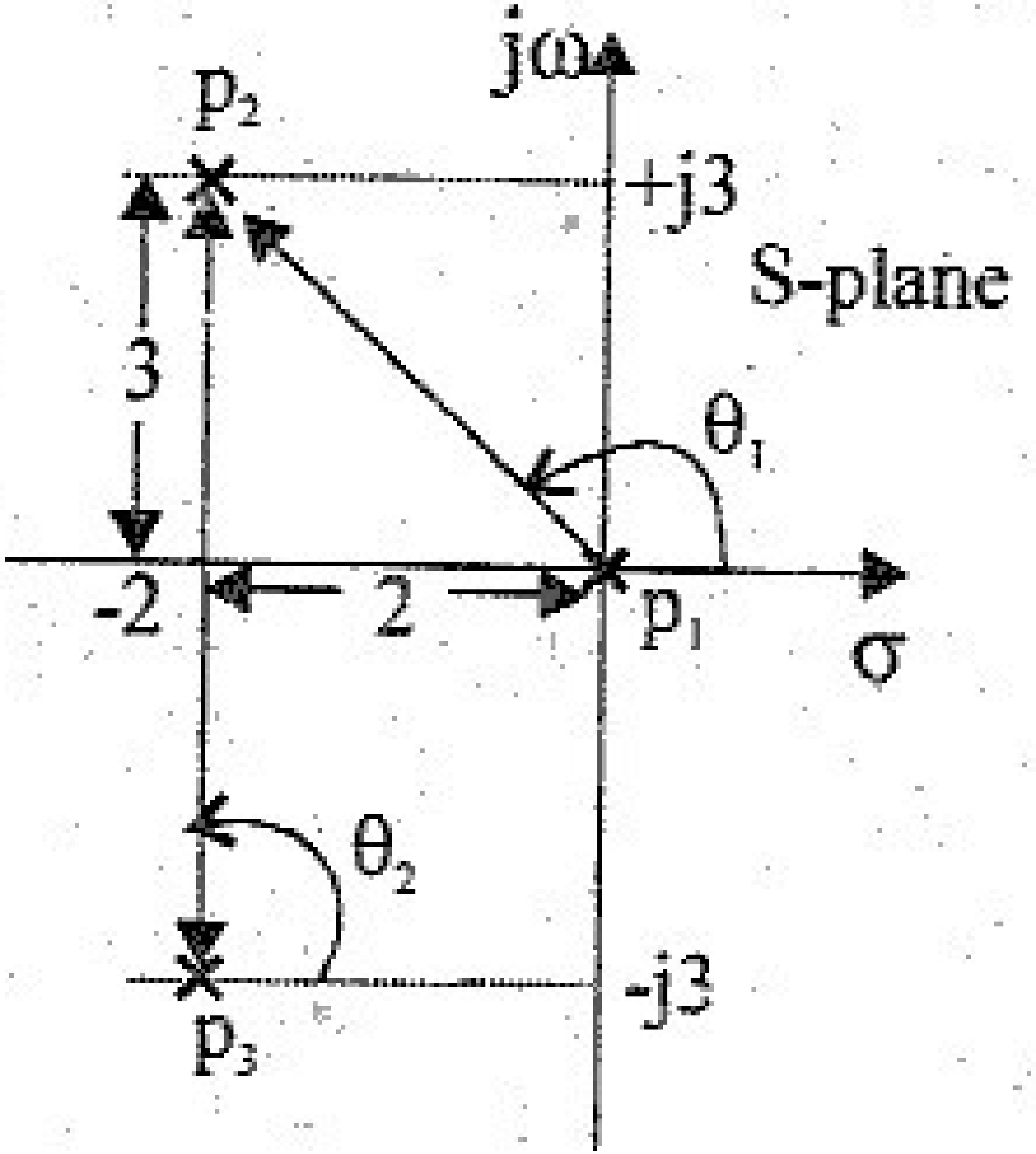
To find the angle of departure

$$\text{Here, } \theta_1 = 180° - \tan^{-1}(3/2) = 123.7° \quad ; \quad \theta_2 = 90°$$

$$\begin{aligned}\text{Angle of departure from the complex pole } p_2 &= 180° - (\theta_1 + \theta_2) \\ &= 180° - (123.7° + 90°) \\ &= -33.7°\end{aligned}$$



A
-33.7
3
2
1
60
-3
-2
-1.33
-1
-60
-1
-2
33.7
-3
A



jω
p2
+j3
S-plane
3
θ1
-2
2
p1
σ
θ2
-j3
p3

To find the crossing point on imaginary axis

The characteristic equation is given by,

$$s^3 + 4s^2 + 13s + K = 0$$

Put $s = j\omega$

$$(j\omega)^3 + 4(j\omega)^2 + 13(j\omega) + K = 0 \quad \Rightarrow -j\omega^3 - 4\omega^2 + 13j\omega + K = 0$$

On equating imaginary part to zero, we get,

$$-\omega^3 + 13\omega = 0$$

$$-\omega^3 = -13\omega$$

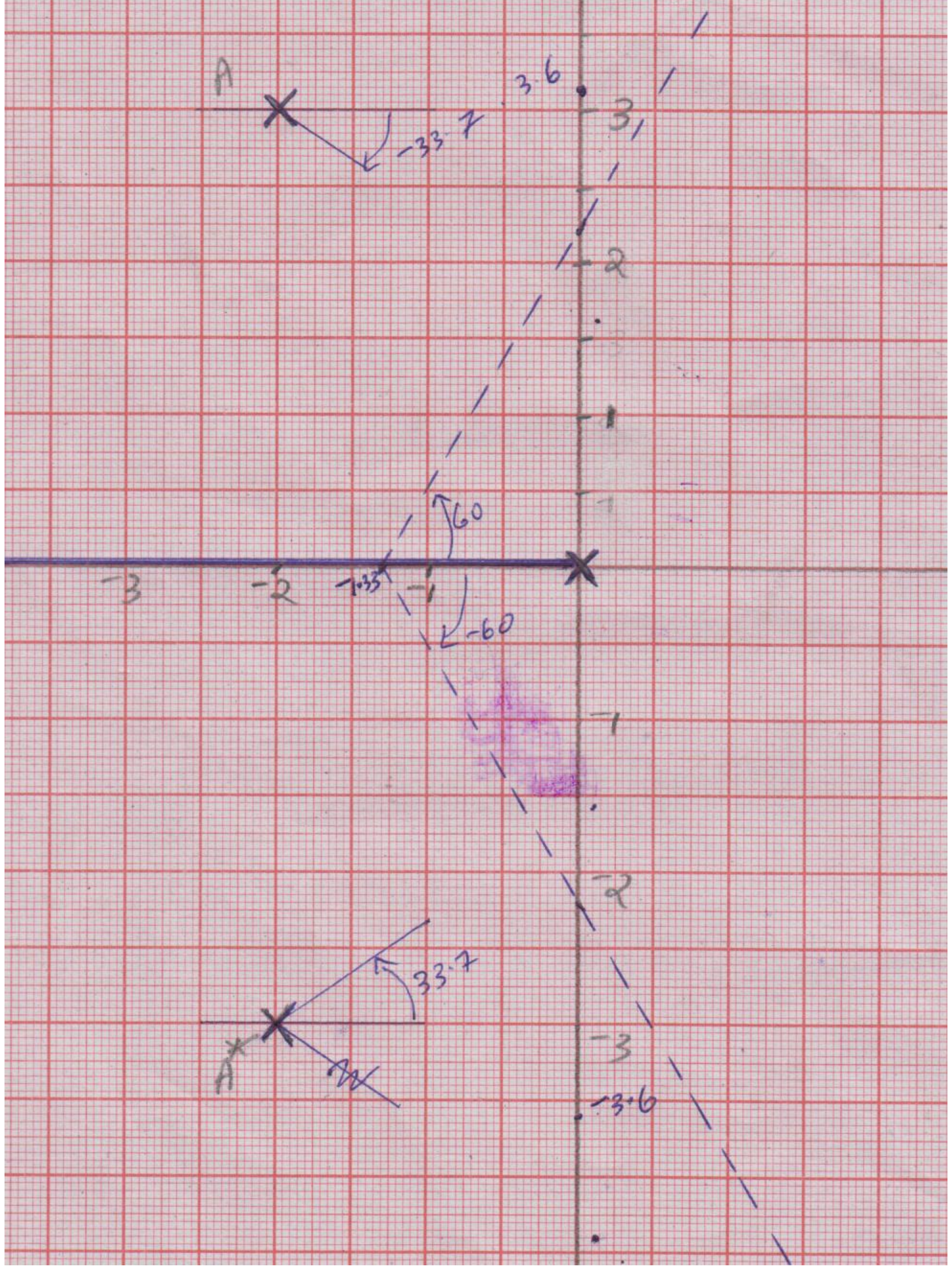
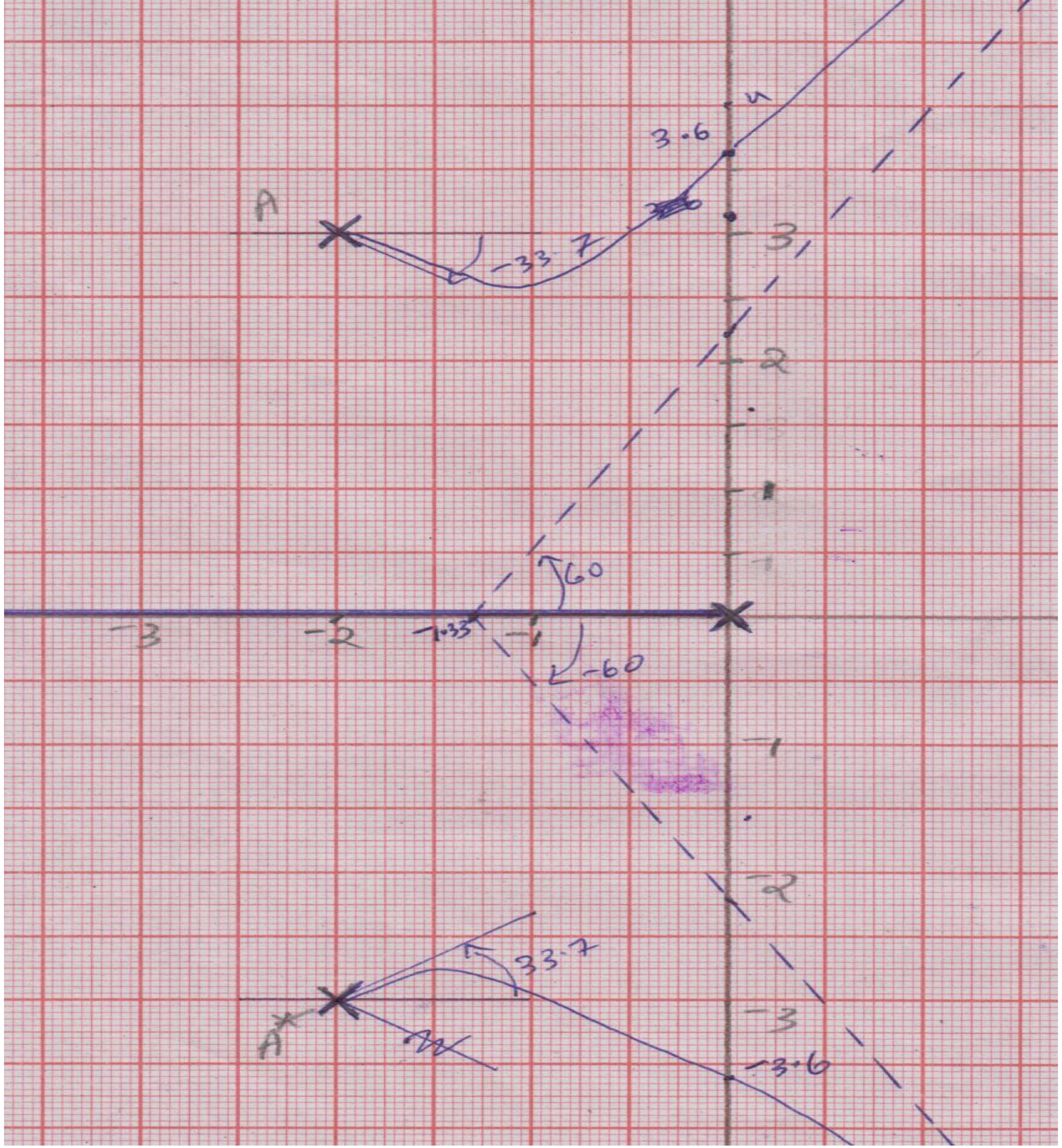
$$\omega^2 = 13 \quad \Rightarrow \quad \omega = \pm\sqrt{13} = \pm 3.6$$

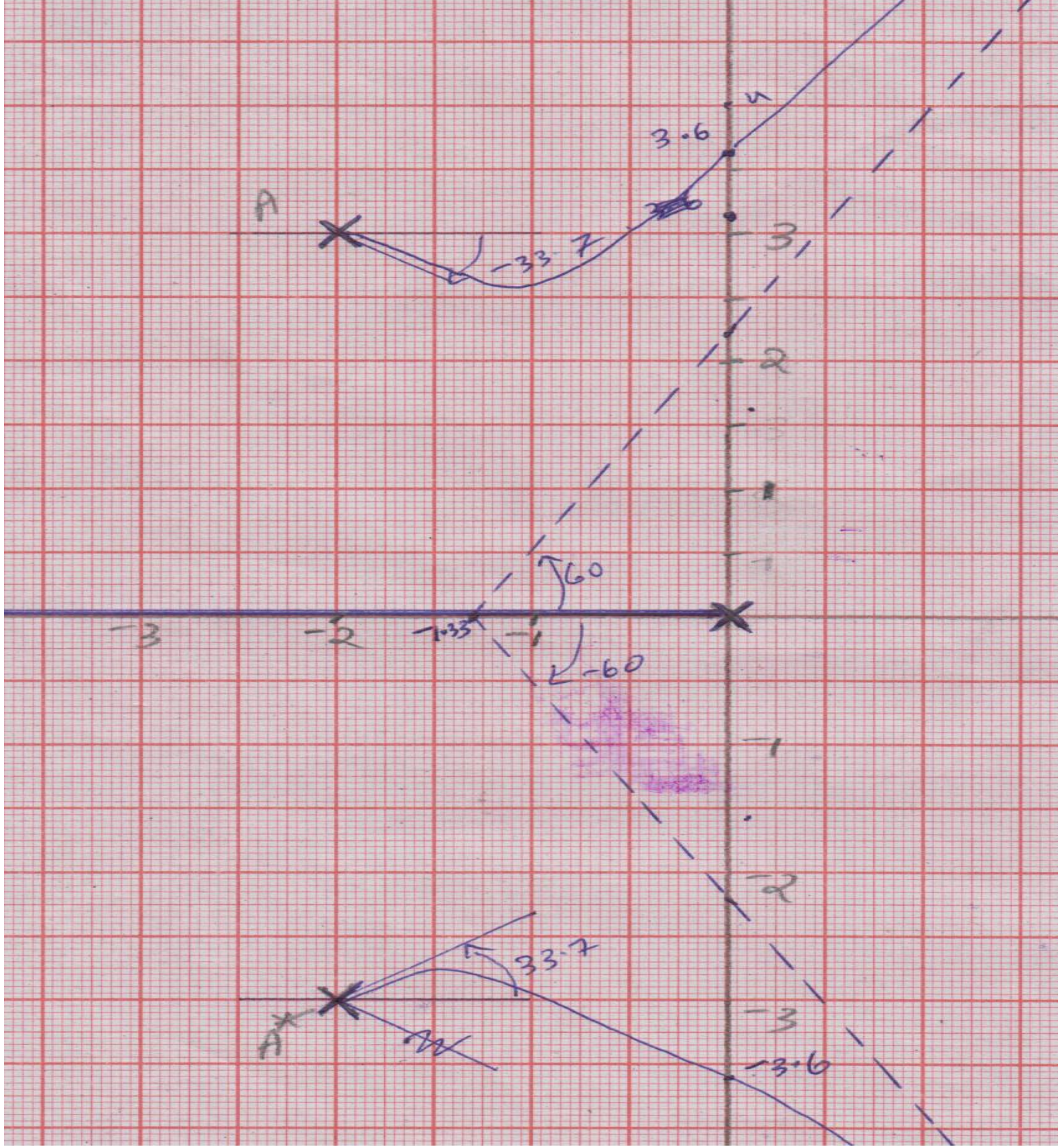
On equating real part to zero, we get,

$$-4\omega^2 + K = 0$$

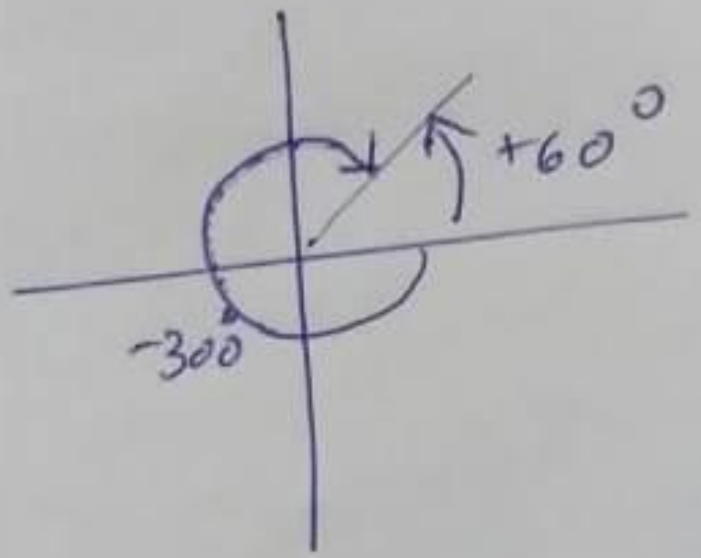
$$K = 4\omega^2$$

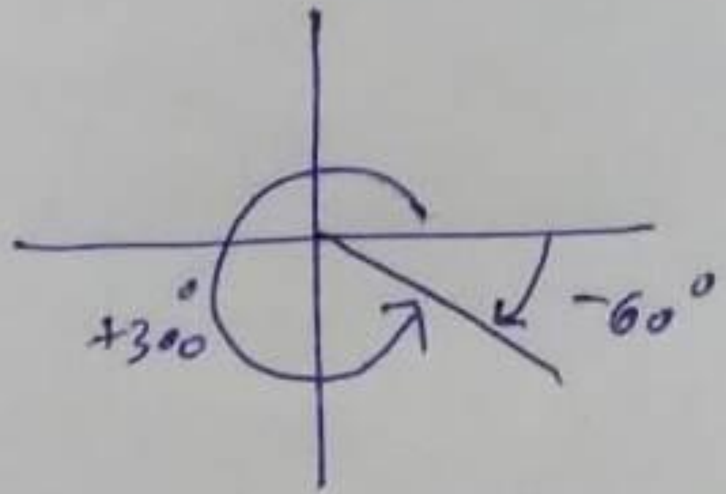
$$= 4 \times 13 = 52$$


A
-33.7
3.6
3
2
1
60
-3
-2
-1.33
-1
-60
-1
-2
33.7
A*
-3
-3.6

3.6
A
-33.7
3
2
1
60
-3
-2
-1.33
-1
-60
-1
-2
33.7
A*
-3
-3.6



A
3.6
3
2
1
-33.7
60
-60
-1.33
-3
-2
-1
-2
-3
-3.6
33.7
A'


+60°
-300°


+300°
-60°

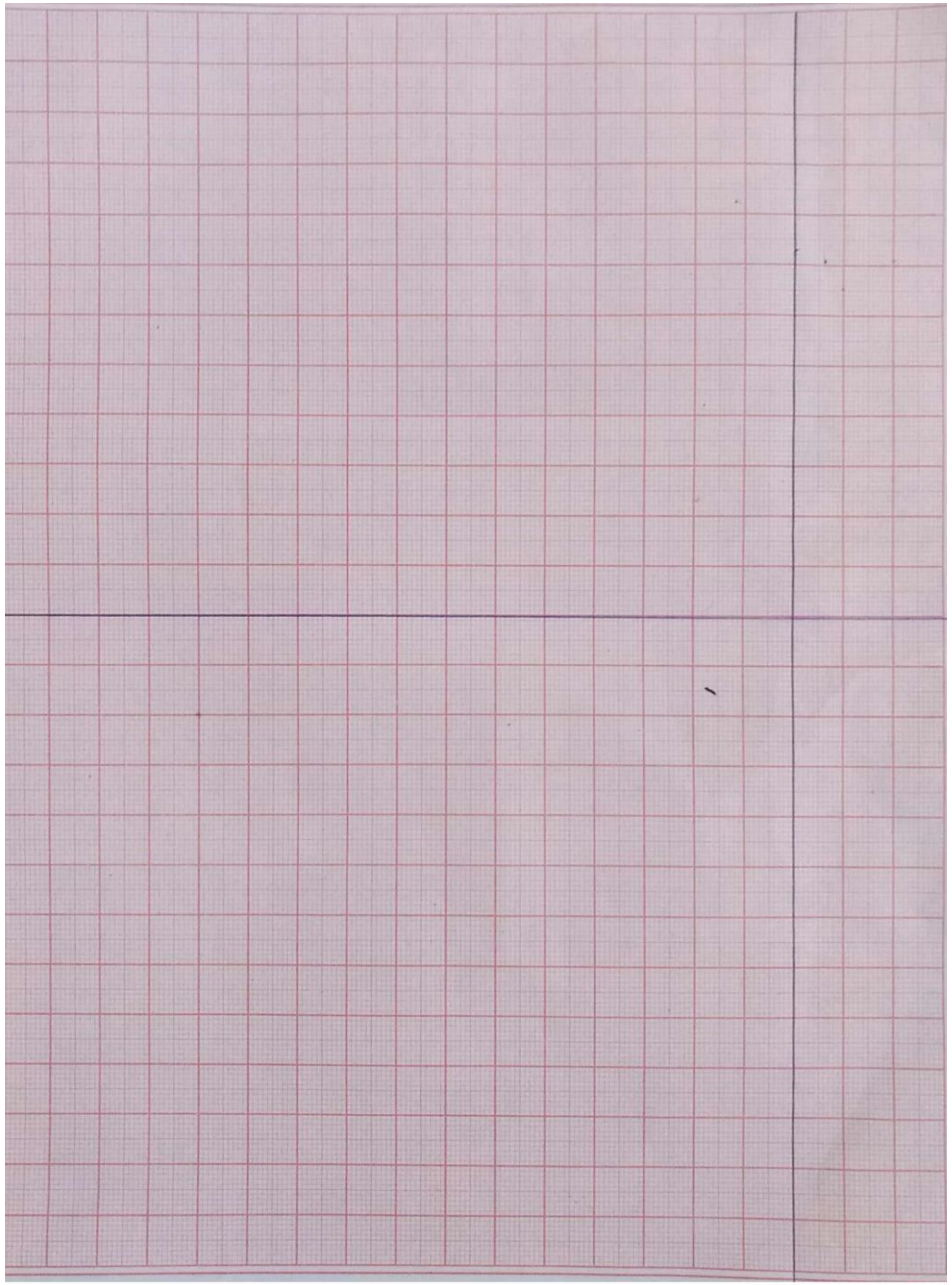
Sketch the root locus of the system whose open loop transfer function is, $G(s) = \frac{K}{s(s+2)(s+4)}$

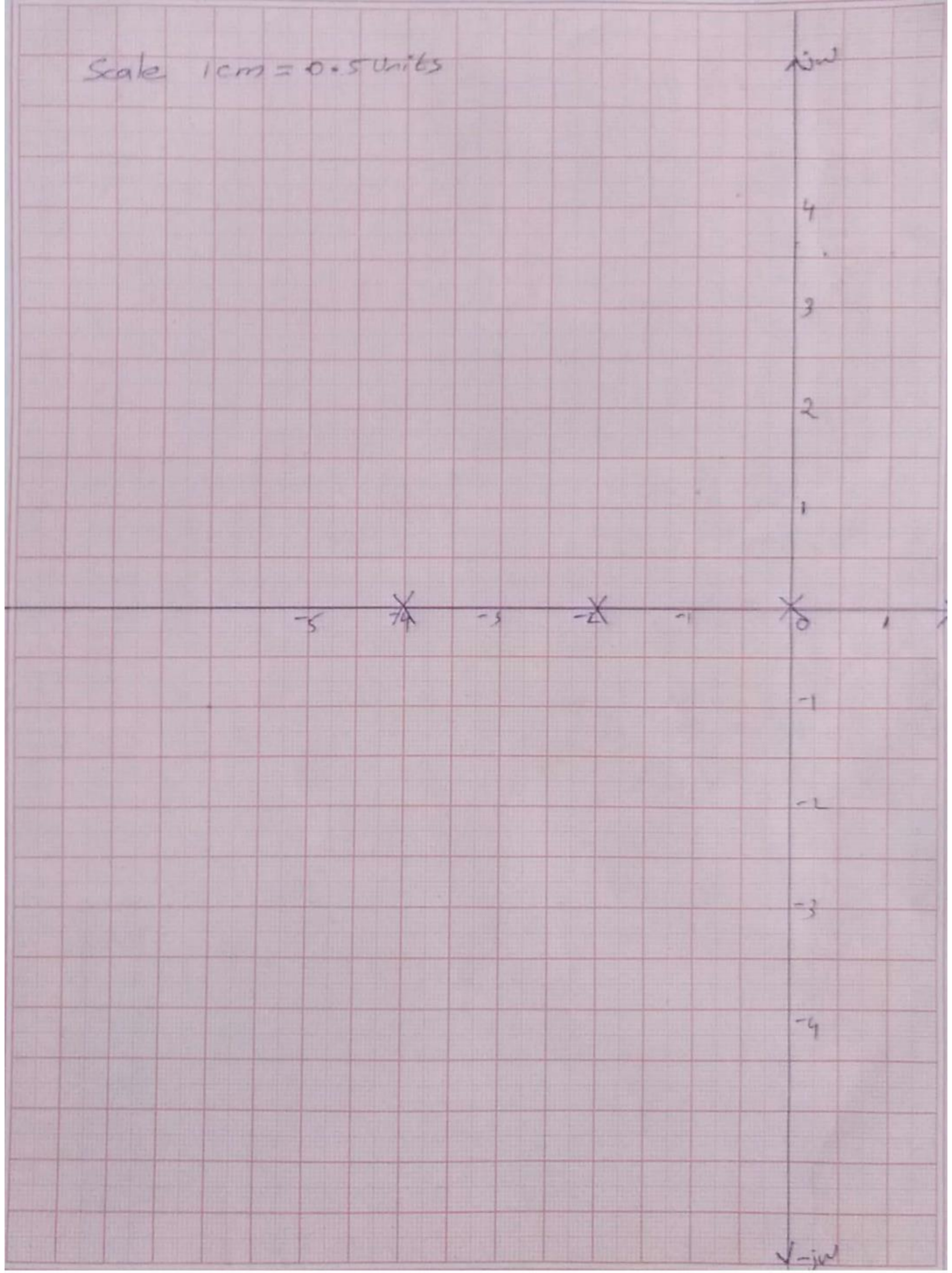
Find the value of K so that the damping ratio of the closed loop system is 0.5.

To locate poles and zeros

The poles of open loop transfer function are the roots of the equation $s(s+2)(s+4) = 0$.

The poles are lying at, $s = 0, -2, -4$



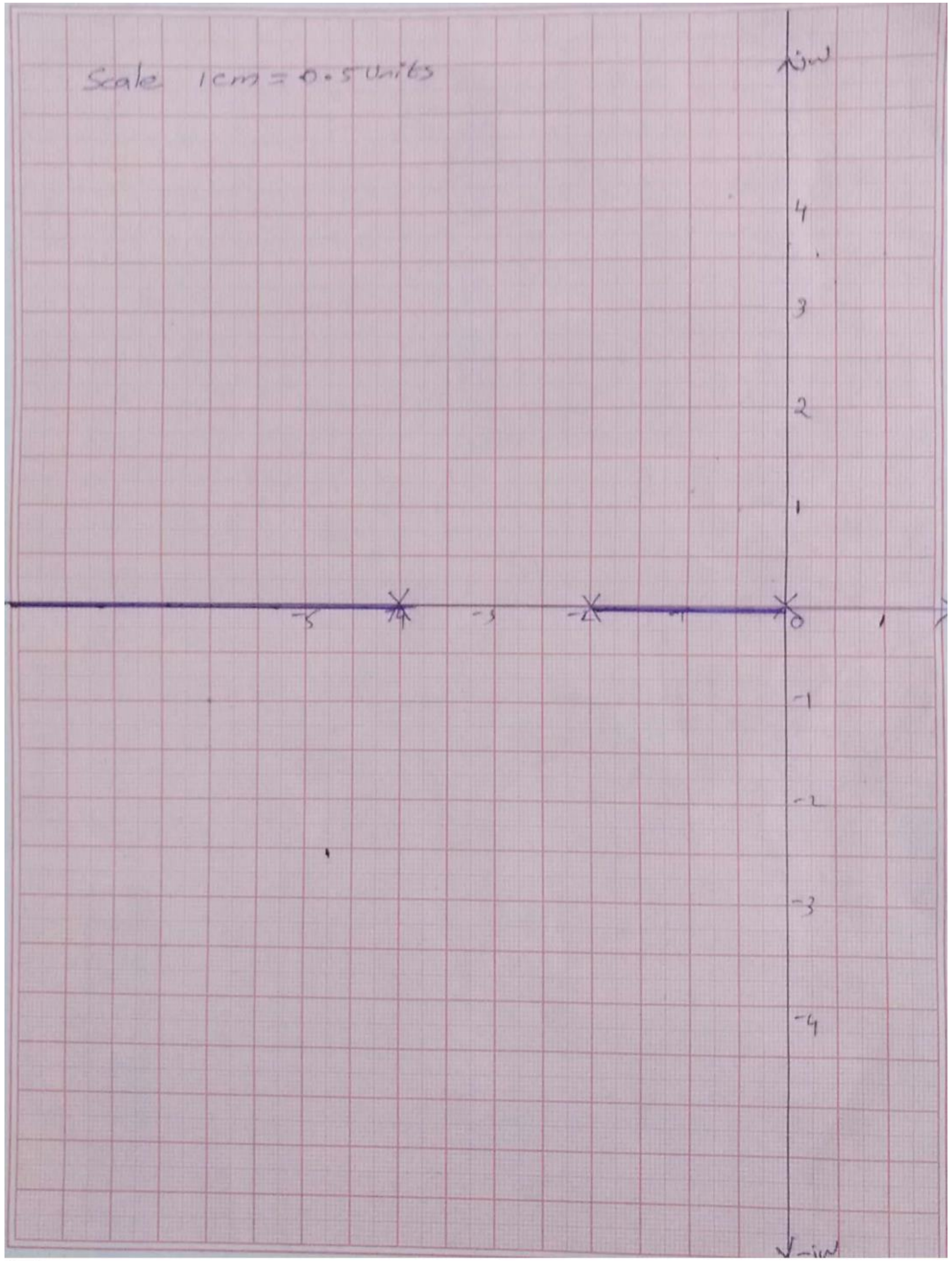

Scale 1cm = 0.5 units
jω
4
3
2
1
-5
-4
-3
-2
-1
0
1
-1
-2
-3
-4
-jω

To find the root locus on real axis

real axis between $s = 0$ and $s = -2$ will be a part of root locus

real axis between $s = -2$ and $s = -4$ will not be a part of root locus.

entire negative real axis from $s = -4$ to $-\infty$ will be a part of root locus.


Scale 1cm = 0.5 units
jω
4
3
2
1
-1
-2
-3
-4
-jω
-5
-4
-3
-2
-1
0
1

asymptotes and centroid

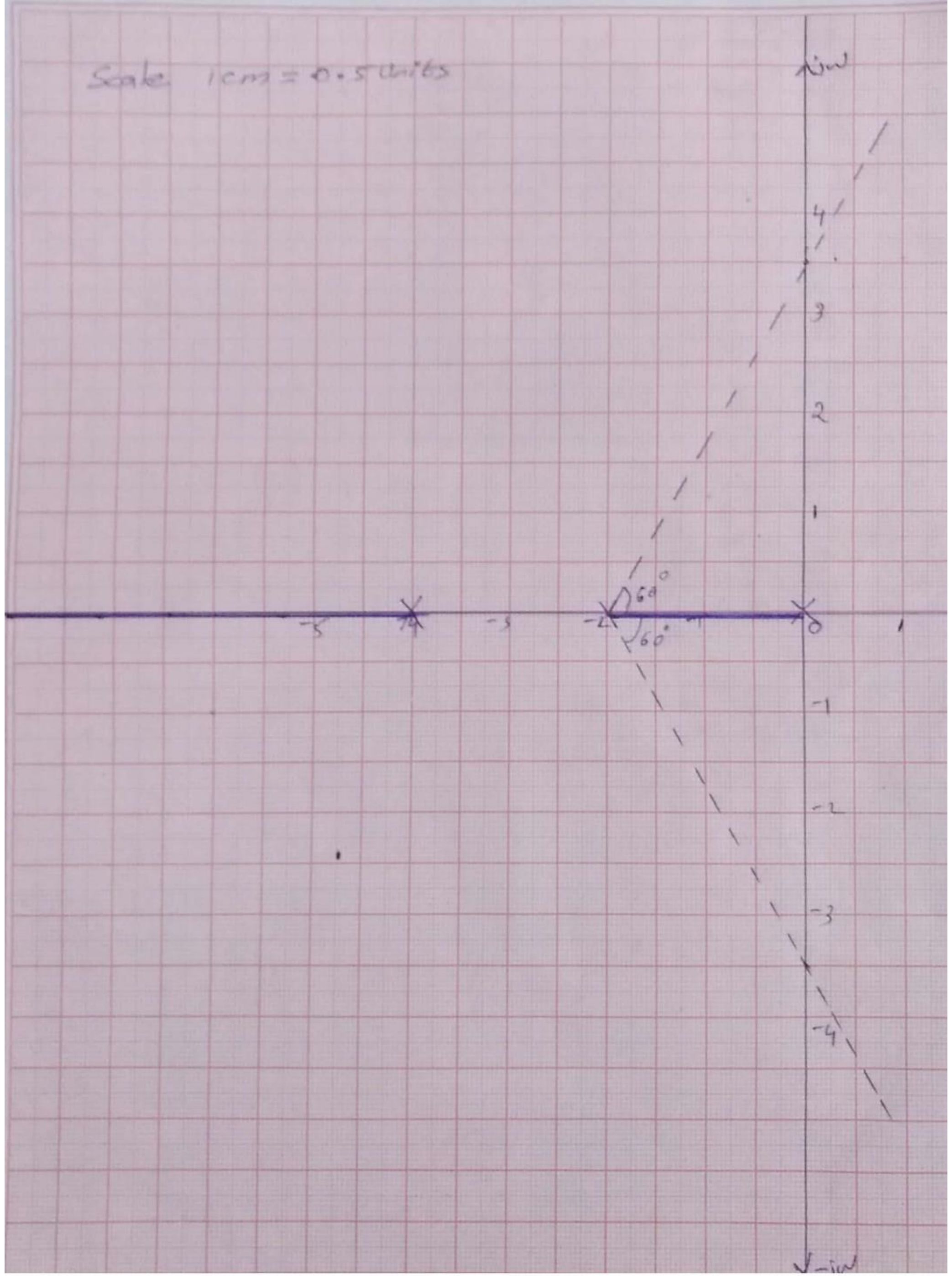
$$\text{Angles of asymptotes} = \frac{\pm 180^\circ (2q+1)}{n-m} \qquad q = 0, 1, 2, \ldots\ldots n-m$$

Here, $n = 3$ and $m = 0$. $q = 0, 1, 2, 3$

When $q = 0$, $\text{Angles} = \pm\frac{180^\circ}{3} = \pm 60^\circ$

When $q = 1$, $\text{Angles} = \pm\frac{180^\circ \times 3}{3} = \pm 180^\circ$

$$\text{Centroid} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m} = \frac{0-2-4-0}{3} = -2$$


Scale 1cm = 0.5 units
jw
-jw
60°
60°

breakaway and breakin points

The closed loop transfer function

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\dfrac{K}{s(s+2)(s+4)}}{1+\dfrac{K}{s(s+2)(s+4)}} = \frac{K}{s(s+2)(s+4)+K}$$

The characteristic equation is given by

$$s(s+2)(s+4)+K=0$$

$$s(s^2+6s+8)+K=0 \quad \Rightarrow \quad s^3+6s^2+8s+K=0$$

$$K=-s^3-6s^2-8s$$

$$\frac{dK}{ds} = -(3s^2 + 12s + 8)$$

$$\frac{dK}{ds} = 0$$

$$. -(3s^2 + 12s + 8) = 0$$

$$(3s^2 + 12s + 8) = 0$$

$$s = \frac{-12 \pm \sqrt{12^2 - 4 \times 3 \times 8}}{2 \times 3} = -0.845 \quad \text{or} \quad -3.154$$

When s = –0.845

$$K = -[(-0.845)^3 + 6(-0.845)^2 + 8(-0.845)] = 3.08$$

Since K, is positive and real for, s = –0.845, this point is actual breakaway point.

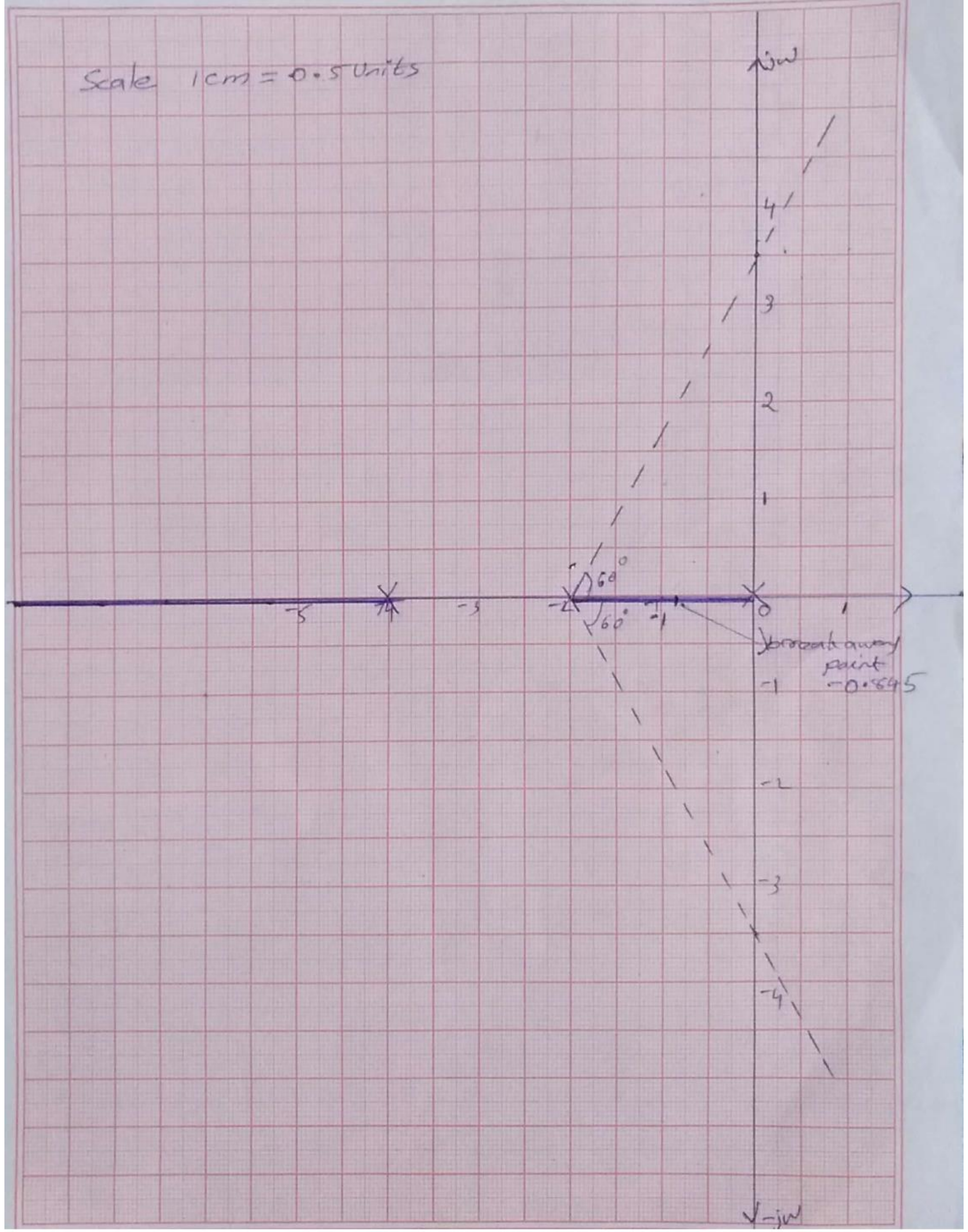
When s = –3.154, the value of K is given by,

$$K = -[(-3.154)^3 + 6(-3.154)^2 + 8(-3.154)] = -3.08$$

Since K, is negative for, s = –3.154, this is not a actual breakaway point.

ANGLE OF DEPARTURE OR ANGLE OF ARRIVAL

Since there is no complex pole or complex zero, there will be no angle of departure or angle of arrival


Scale 1cm = 0.5 units
jw
60°
60°
breakaway point
-0.845
-jw

crossing point of imaginary axis

The characteristic equation is given by,

$$s^3 + 6s^2 + 8s + K = 0$$

Put $s = j\omega$

$$(j\omega)^3 + 6(j\omega)^2 + 8(j\omega) + K = 0$$

$$-j\omega^3 - 6\omega^2 + j8\omega + K = 0$$

Equating imaginary part to zero

$$-j\omega^3 + j8\omega = 0$$

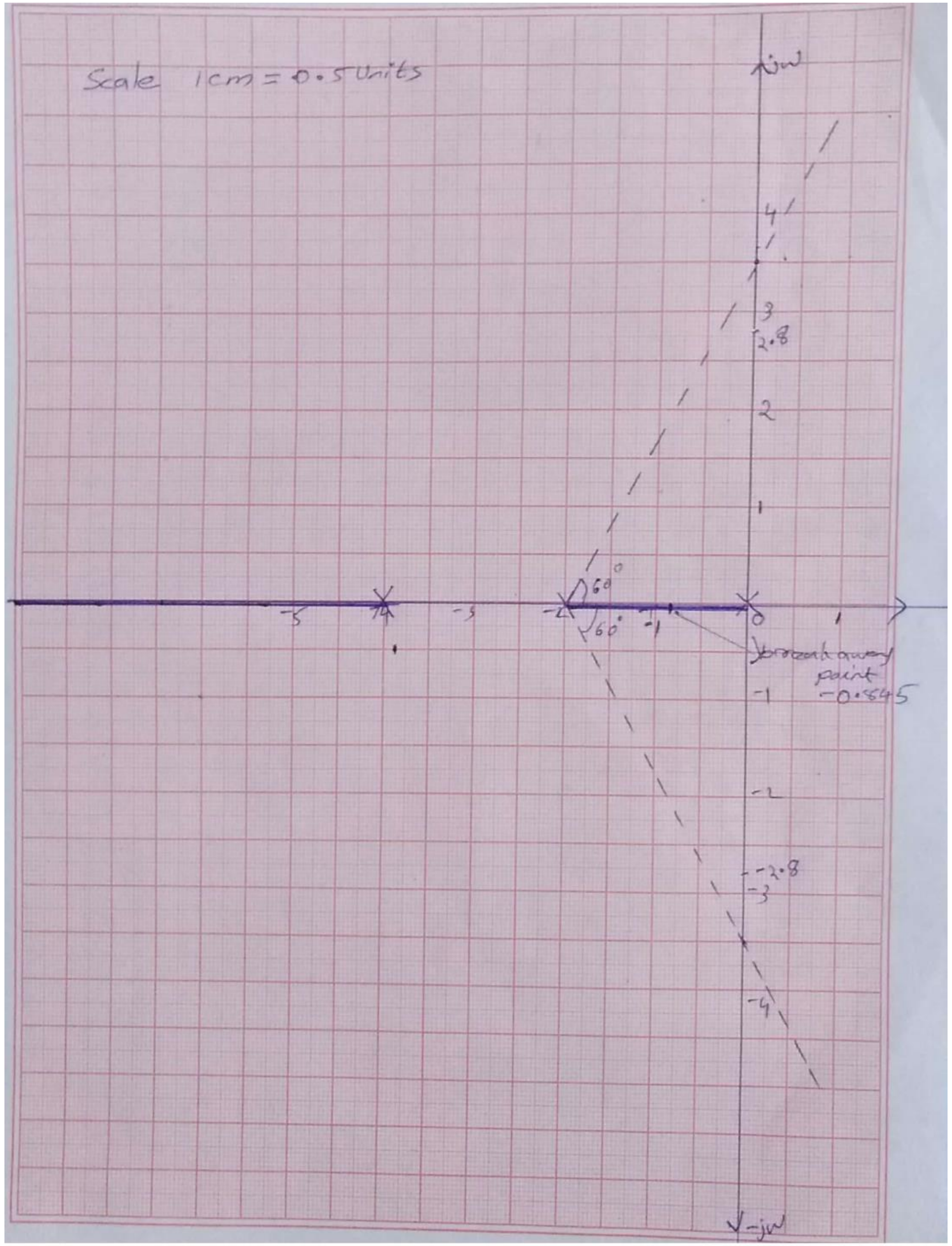
$$-j\omega^3 = -j8\omega$$

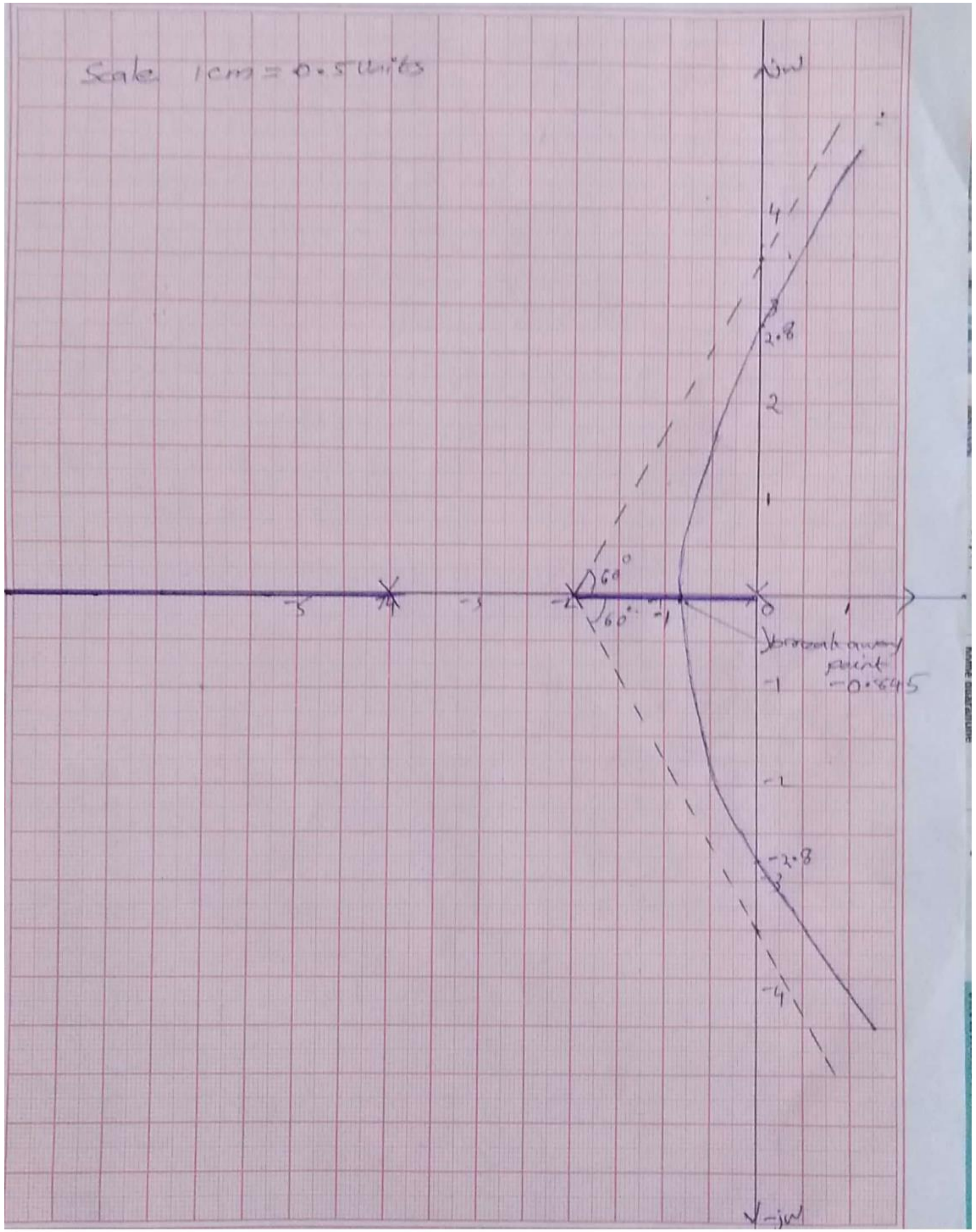
$$\omega^2 = 8 \quad \Rightarrow \quad \omega = \pm\sqrt{8} = \pm 2.8$$

Equating real part to zero

$$-6\omega^2 + K = 0$$

$$K = 6\omega^2 = 6 \times 8 = 48$$


Scale 1cm = 0.5 units
jw
-jw
60°
60°
breakaway point -0.845
2.8
-2.8


Scale 1cm = 0.5 units
jw
60°
60°
breakaway point
-0.845
2.8
-2.8
-jw

The crossing point of root locus is $\pm j2.8$. The value of K corresponding to this point is K = 48

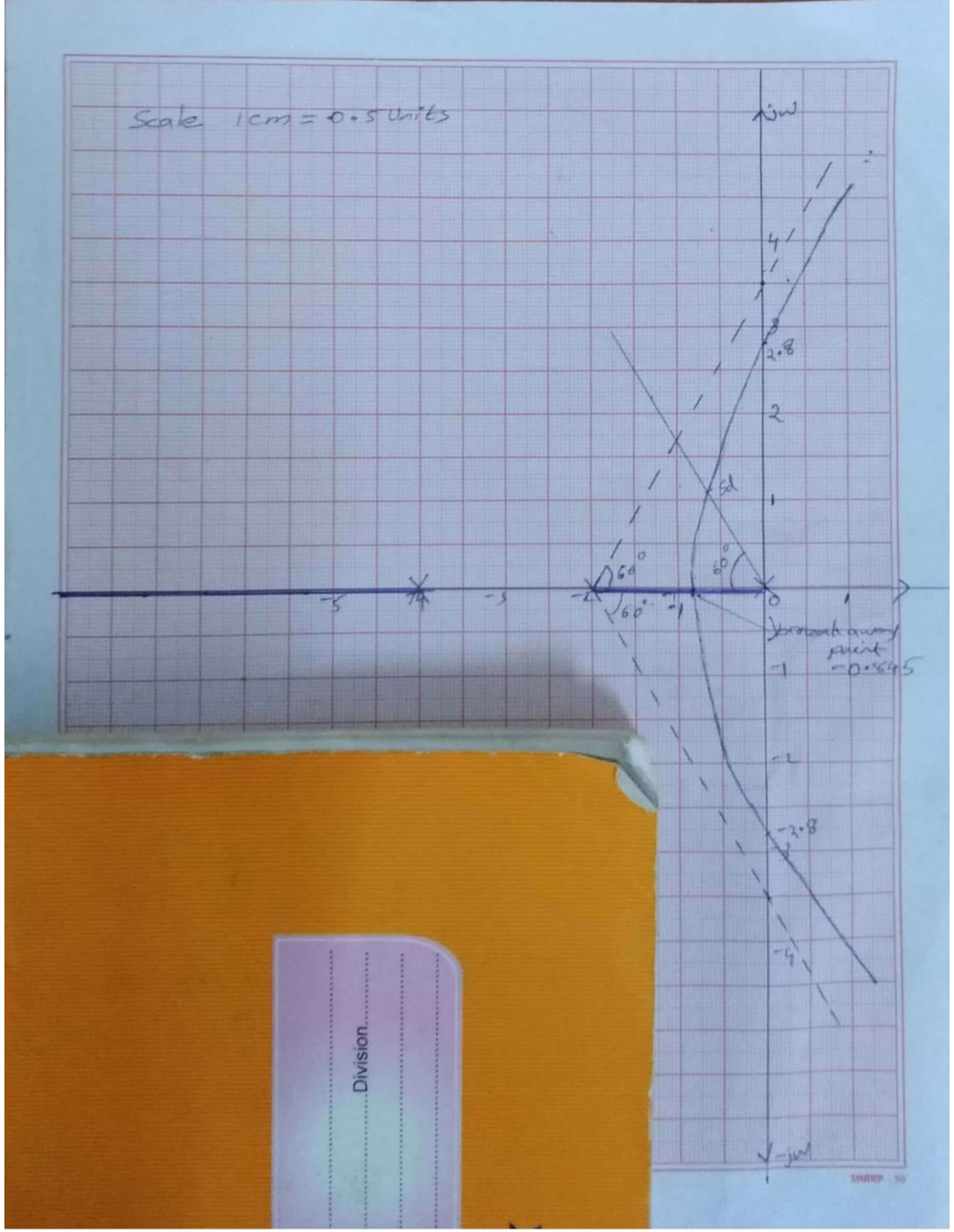
This is the limiting value of K for the stability of the system

the value of K corresponding to $\zeta = 0.5$

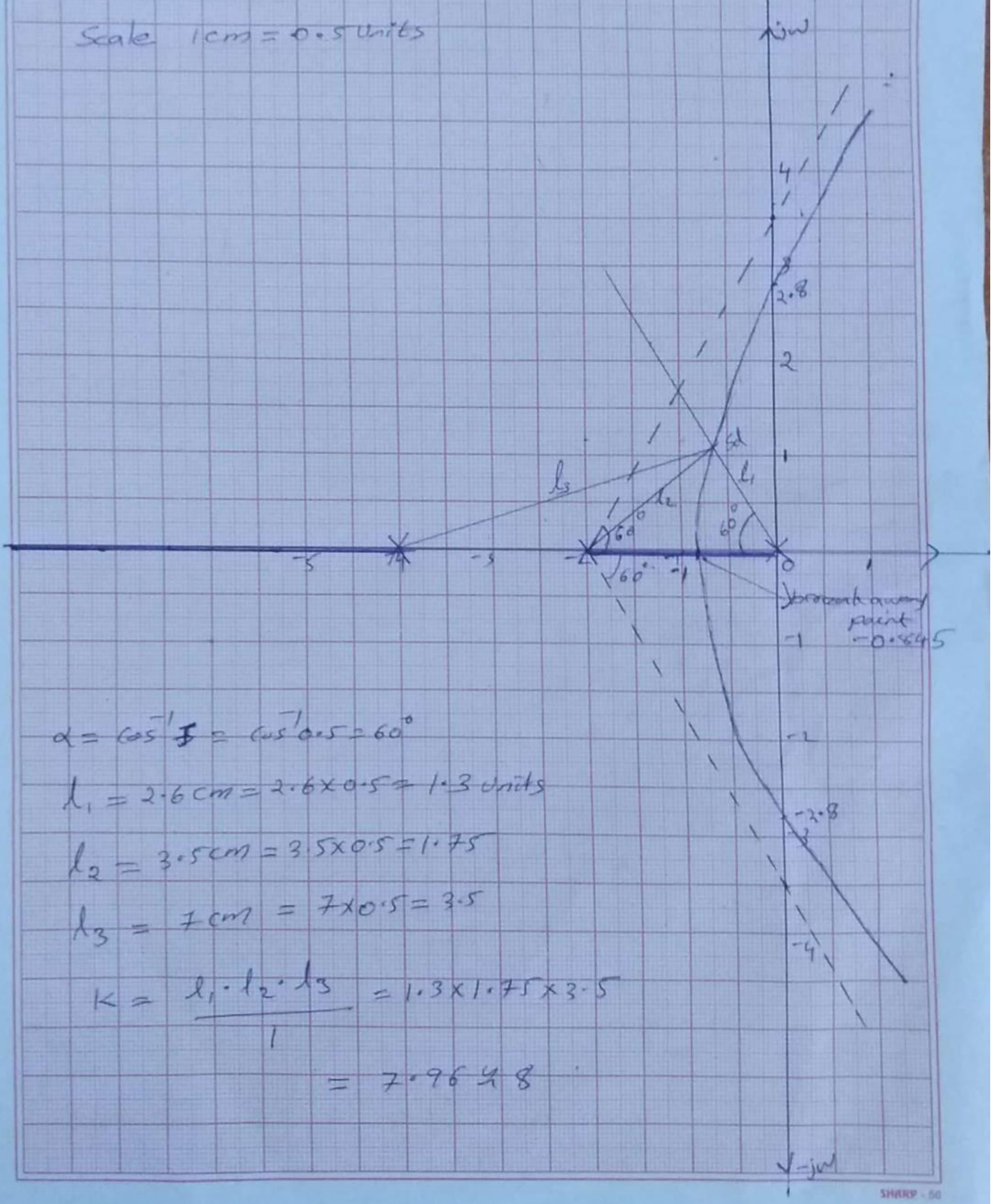
$\zeta = 0.5$

$\alpha = \cos^{-1}\zeta = \cos^{-1}0.5 = 60°$

$$K_i = \frac{l_1 \times l_2 \times l_3}{1} = 1.3 \times 1.75 \times 3.5 = 7.96 \approx 8$$


Scale 1cm = 0.5 units
jw
-jw
2.8
-2.8
60°
60°
60°
breakaway point
-0.845
Division

Scale 1cm = 0.5 units



$$\alpha = \cos^{-1}\xi = \cos^{-1}0.5 = 60^\circ$$

$$l_1 = 2.6\,cm = 2.6 \times 0.5 = 1.3 \text{ units}$$

$$l_2 = 3.5\,cm = 3.5 \times 0.5 = 1.75$$

$$l_3 = 7\,cm = 7 \times 0.5 = 3.5$$

$$K = \frac{l_1 \cdot l_2 \cdot l_3}{1} = 1.3 \times 1.75 \times 3.5$$

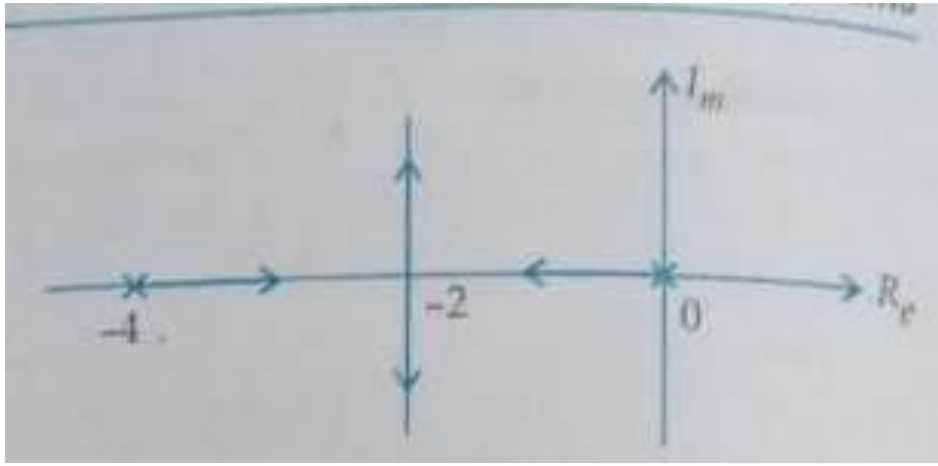
$$= 7.9648$$

EFFECT OF ADDITION OF POLES AND ZEROS ON ROOT LOCUS

Consider

$$G(S)H(S) = \frac{K}{S(S+4)}$$

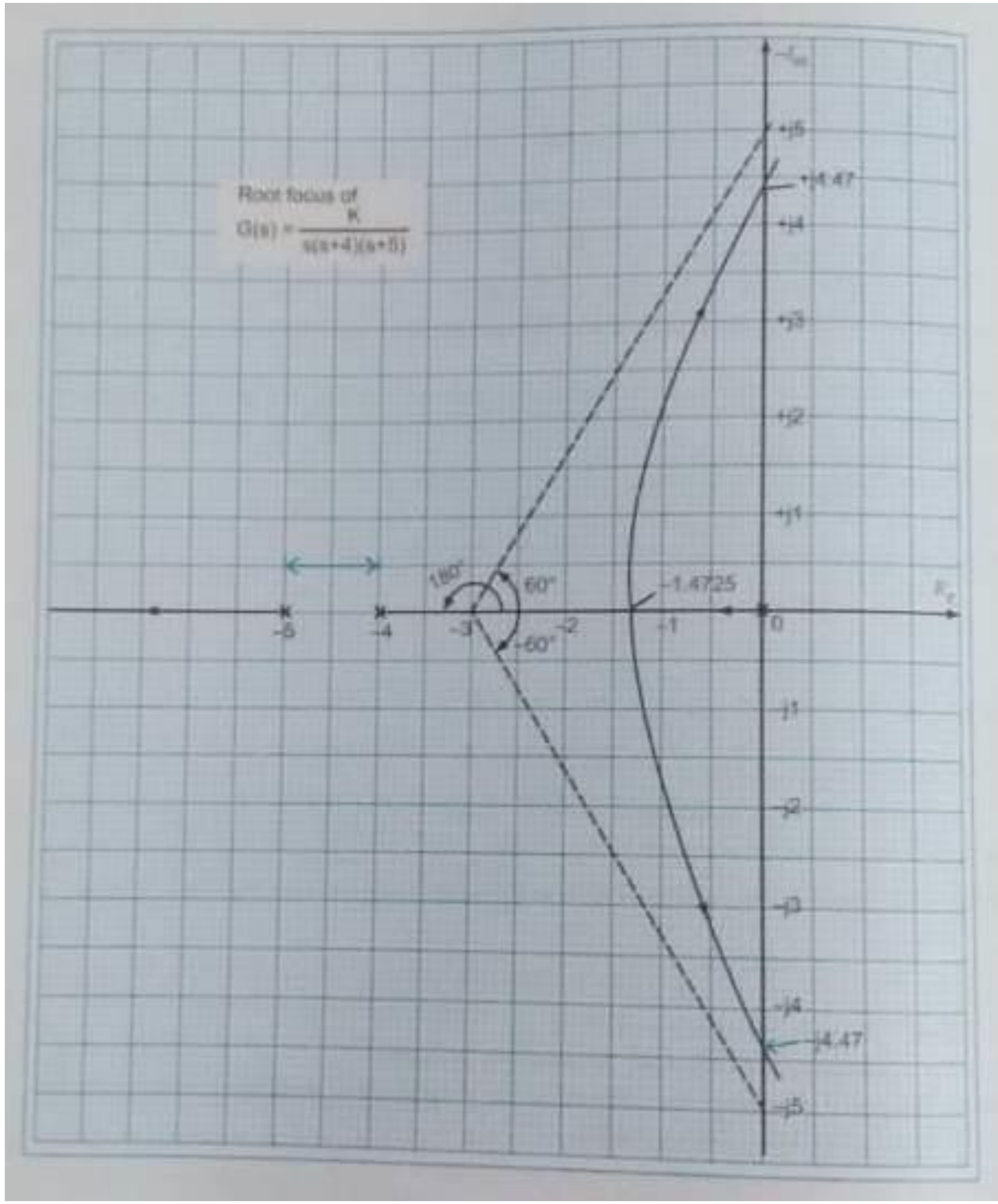
The root locus is



now add one pole at S= --5

$$G(S)H(S) = \frac{K}{S(S+4)(s+5)}$$

The corresponding root locus is given by



Before addition of pole for any value of ‘K’ the system is stable

After addition of pole to the left half, the two branches of root locus moves to the right half for some value of ‘K’.

The system will be stable for this value of ‘K’, after this value of ‘K’ the system becomes unstable.

The stability of the system gets restricted.

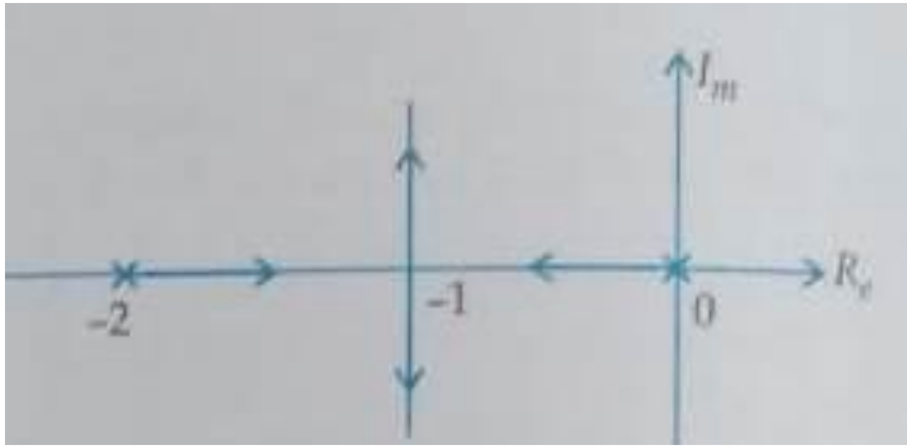
Further addition of poles to the left half, the breaking point moves towards right.

So by addition of poles to the left half, the root locus shifted towards right half side and stability of the system decreases.

Consider

$$G(S)H(S) = \frac{K}{S(S+2)}$$

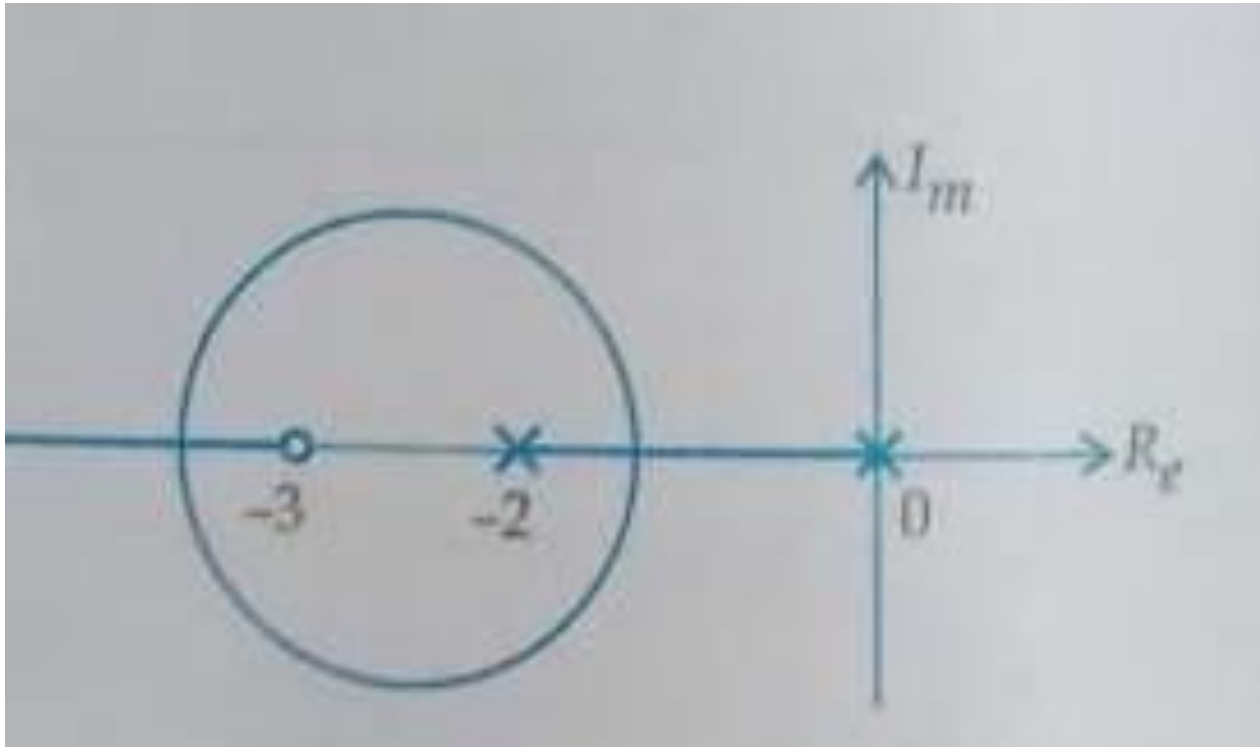
The root locus is



now add one zero at S= --3

$$G(S)H(S) = \frac{K(s+3)}{S(S+2)}$$

The corresponding root locus is given by



By addition of zeros towards left, the root locus shifts towards left half.

Since root locus shifts towards left half, the relative stability increases.

In conclusion

1. by addition of poles, the root locus shifts towards imaginary axis and system stability decreases
2. by addition of zeros towards left half, the root locus moves away from the imaginary axis and system stability increases

The open loop transfer function of a unity feedback system is given by, $G(s) = \frac{K(s+9)}{s(s^2+4s+11)}$

Sketch the root locus of the system

locate poles and zeros

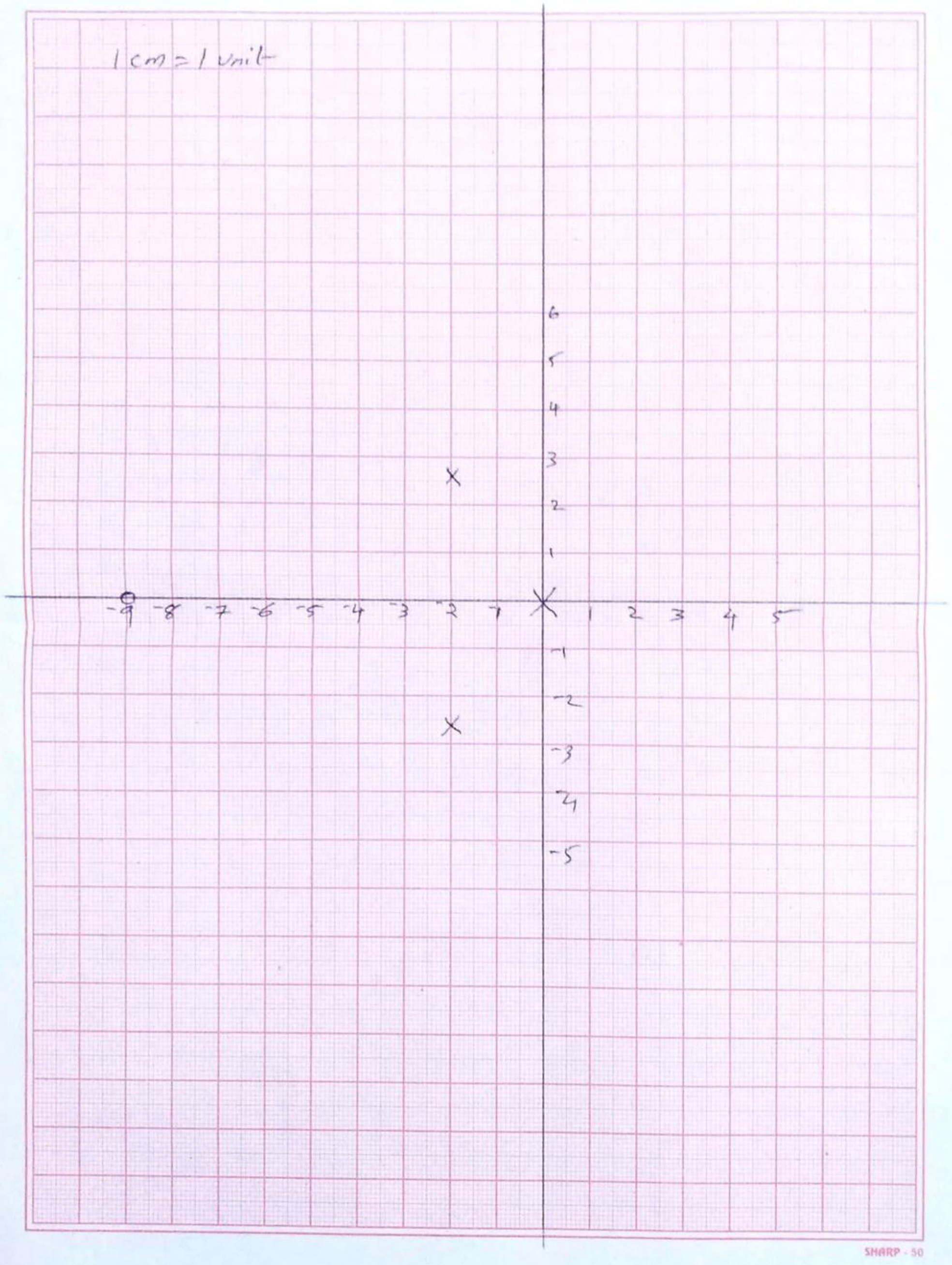
The roots of the quadratic are, $s = \frac{-4 \pm \sqrt{4^2 - 4 \times 11}}{2} = -2 \pm j2.64$

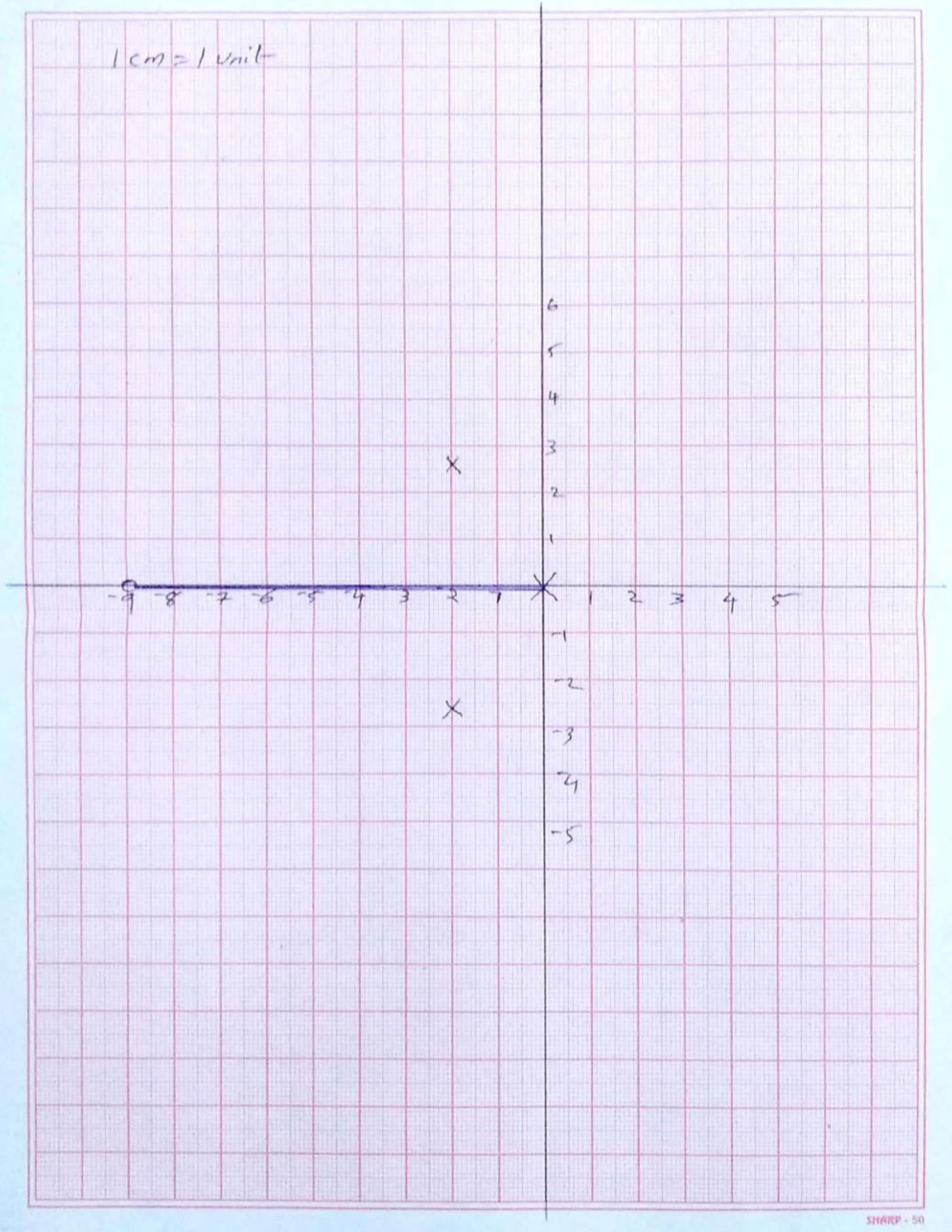
The poles are lying at, $s = 0, -2 + j2.64, -2 - j2.64$

The zeros are lying at, $s = -9$ and infinity.

find the root locus on real axis

One pole and one zero lie on real axis.





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from s = 0 to s = –9 will be a part of root locus.

from s = –9 to –∞ will not be a part of root locus.

To find angles of asymptotes and centroid

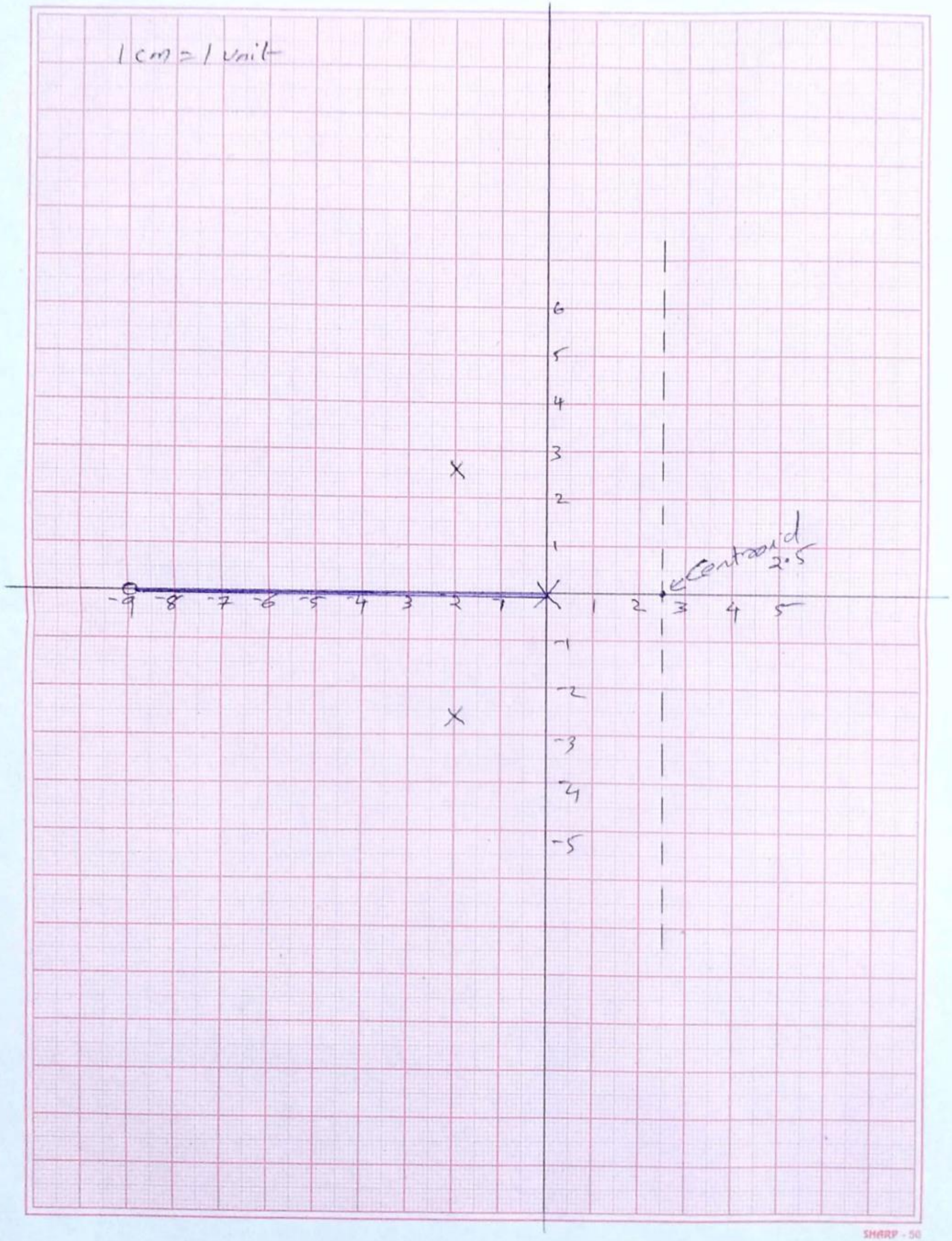
$$\text{Angles of asymptotes} = \frac{\pm 180^\circ (2q+1)}{n-m}; \qquad q = 0, 1, 2, \ldots\ldots n-m.$$

Here, n = 3 and m = 0. $\therefore$ q = 0, 1, 2, 3 .

When q = 0, $\text{Angles} = \pm\frac{180^\circ}{3} = \pm 60^\circ$

When q = 1, $\text{Angles} = \pm\frac{180^\circ \times 3}{2} = \pm 270^\circ = \mp 90^\circ$

When q = 2, $\text{Angles} = \pm\frac{180^\circ \times 5}{2} = \pm 450^\circ = \pm 90^\circ$



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$$\text{Centroid} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m} = \frac{0-2+j2.64-2-j2.64-(-9)}{2} = 2.5$$

the breakaway and breakin points

From the location of poles and zero there is no possibility of breakaway or breakin points.

the angle of departure

$$\text{Here, } \theta_1 = 180° - \tan^{-1}\frac{2.64}{2} = 127.1°$$

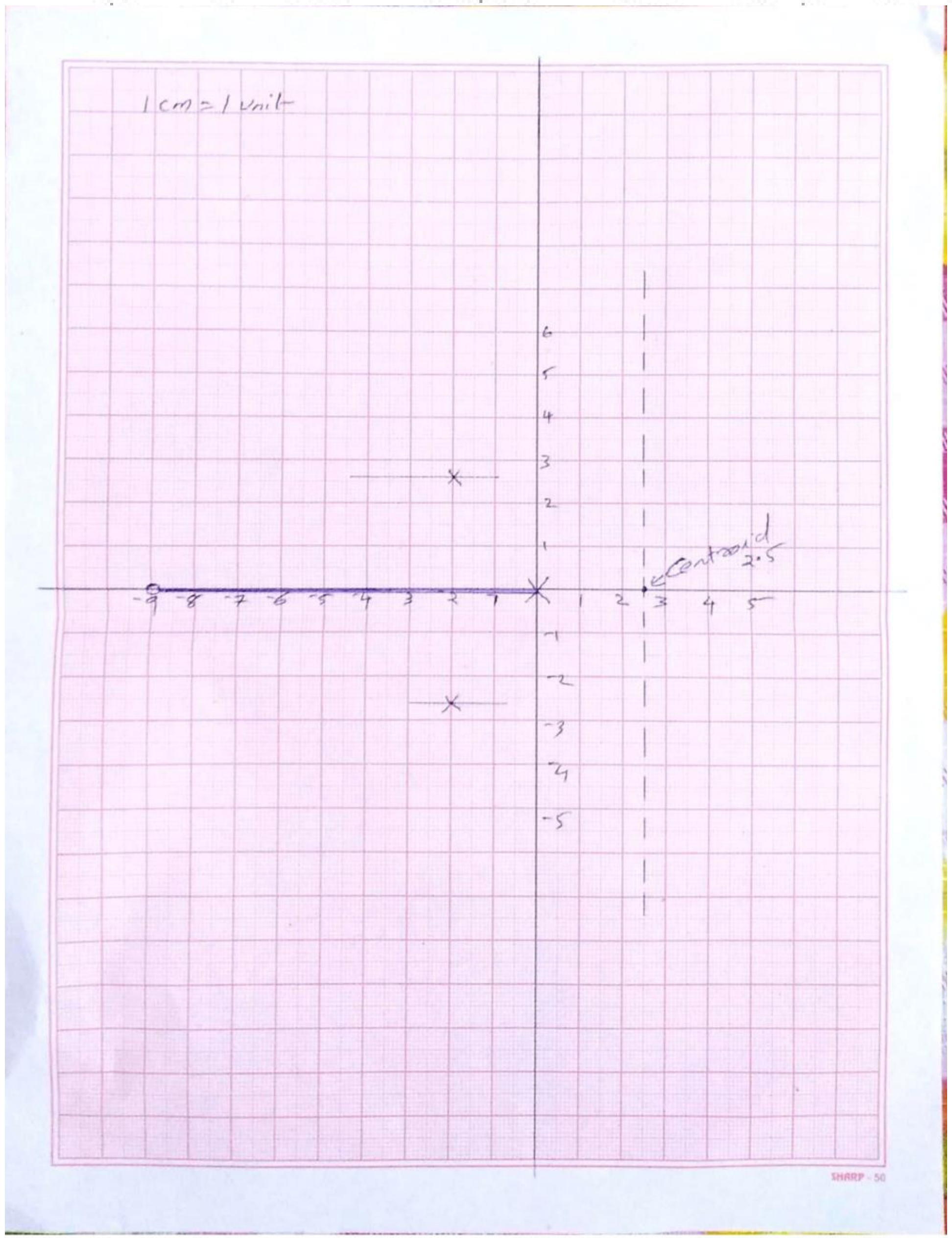
$$\theta_2 = 90°$$

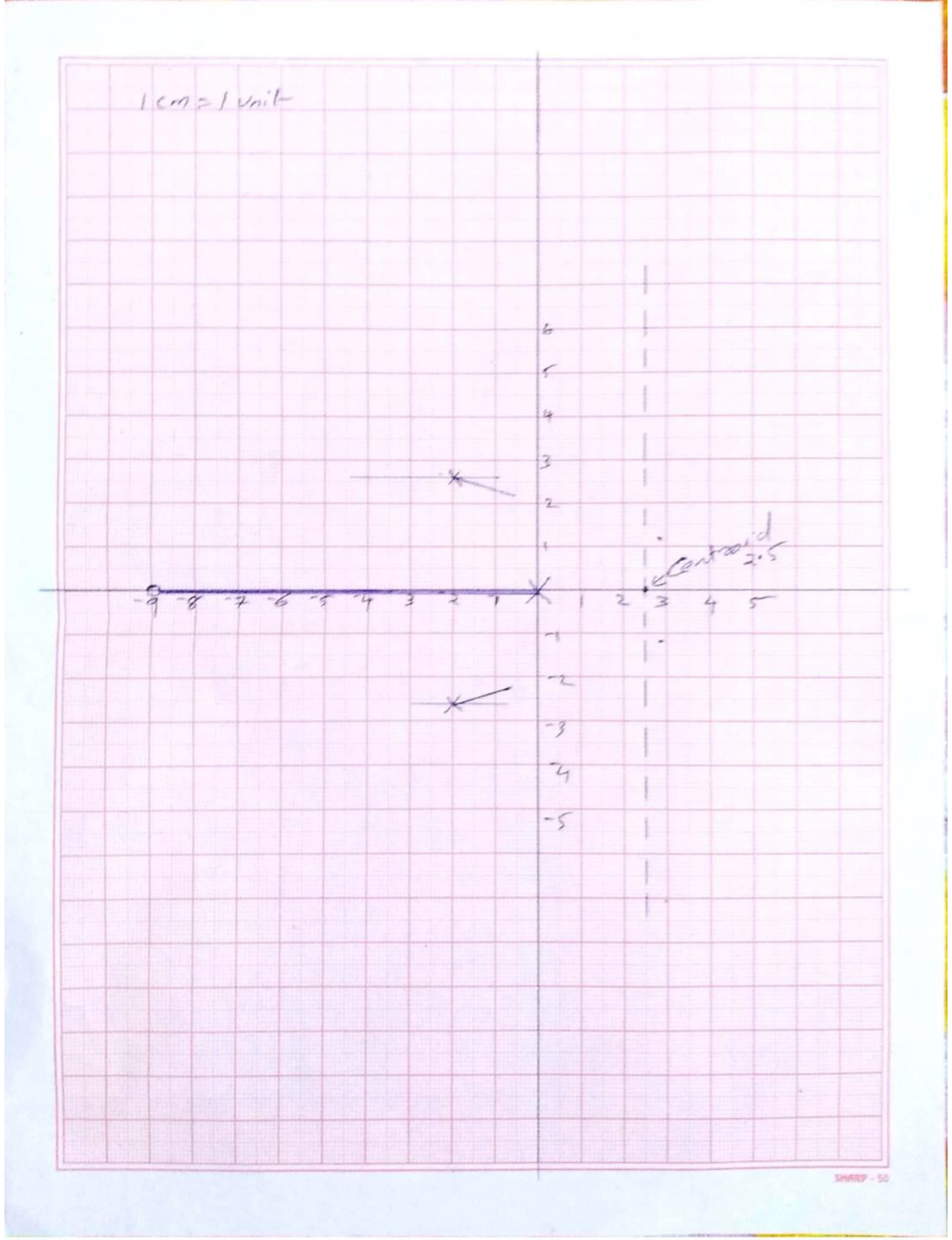
$$\theta_3 = \tan^{-1}\frac{2.64}{7} = 20.7°$$

$$\left.\begin{array}{l}\text{Angle of departure from}\\ \text{the complex pole } p_2\end{array}\right\} = 180^{\circ} - (\theta_1 + \theta_2) + \theta_3$$

$$= 180^{\circ} - (127.1^{\circ} + 90^{\circ}) + 20.7^{\circ} = -16.4°$$

Angle of departure at pole $p_3 = -(-16.4) = +16.4°$





the crossing point of imaginary axis

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\frac{K(s+9)}{s(s^2+4s+11)}}{1+\frac{K(s+9)}{s(s^2+4s+11)}} = \frac{K(s+9)}{s(s^2+4s+11)+K(s+9)}$$

$\therefore$ The characteristic equation is,

$$s(s^2+4s+11)+K(s+9)=0 \quad \Rightarrow \quad (s^3+4s^2+11s)+Ks+9K=0$$

put $s = j\omega$

$$(j\omega)^3 + 4(j\omega)^2 + 11(j\omega) + K(j\omega) + 9K = 0$$

$$-j\omega^3 - 4\omega^2 + j11\omega + jK\omega + 9K = 0$$

On equating imaginary part to zero,

$$-j\omega^3 + j11\omega + jK\omega = 0 \Rightarrow -j\omega^3 = -j11\omega - jK\omega$$

$$\therefore \omega^2 = 11 + K$$

Put $K = 8.8$, $\therefore \omega^2 = 11 + 8.8 = 19.8$

$$\omega = \pm\sqrt{19.8} = \pm 4.4$$

On equating real part to zero,

$$-4\omega^2 + 9K = 0 \quad \Rightarrow \quad 9K = 4\omega^2$$

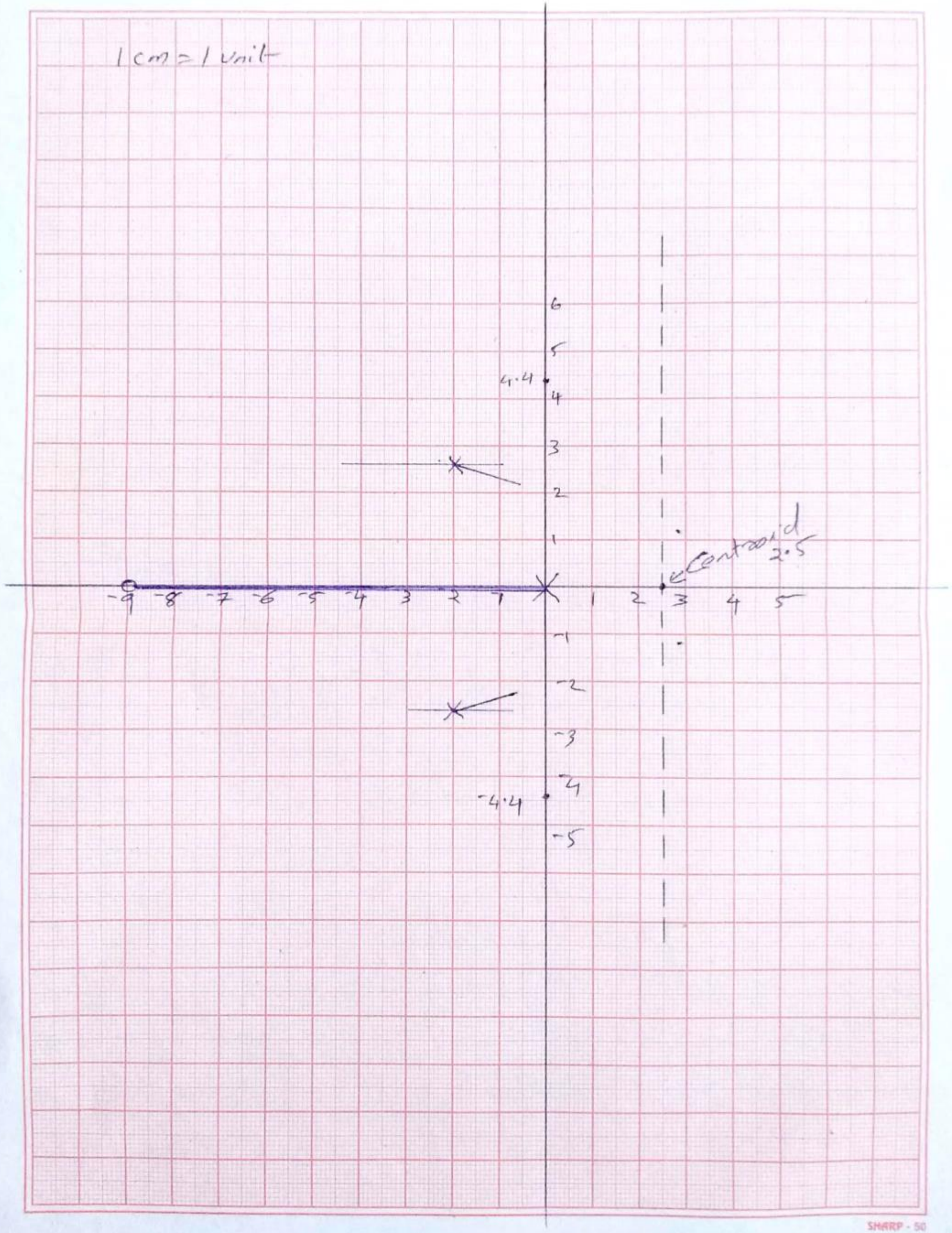
Put, $\omega^2 = 11 + K$ $\therefore 9K = 4(11 + K) = 44 + 4K$

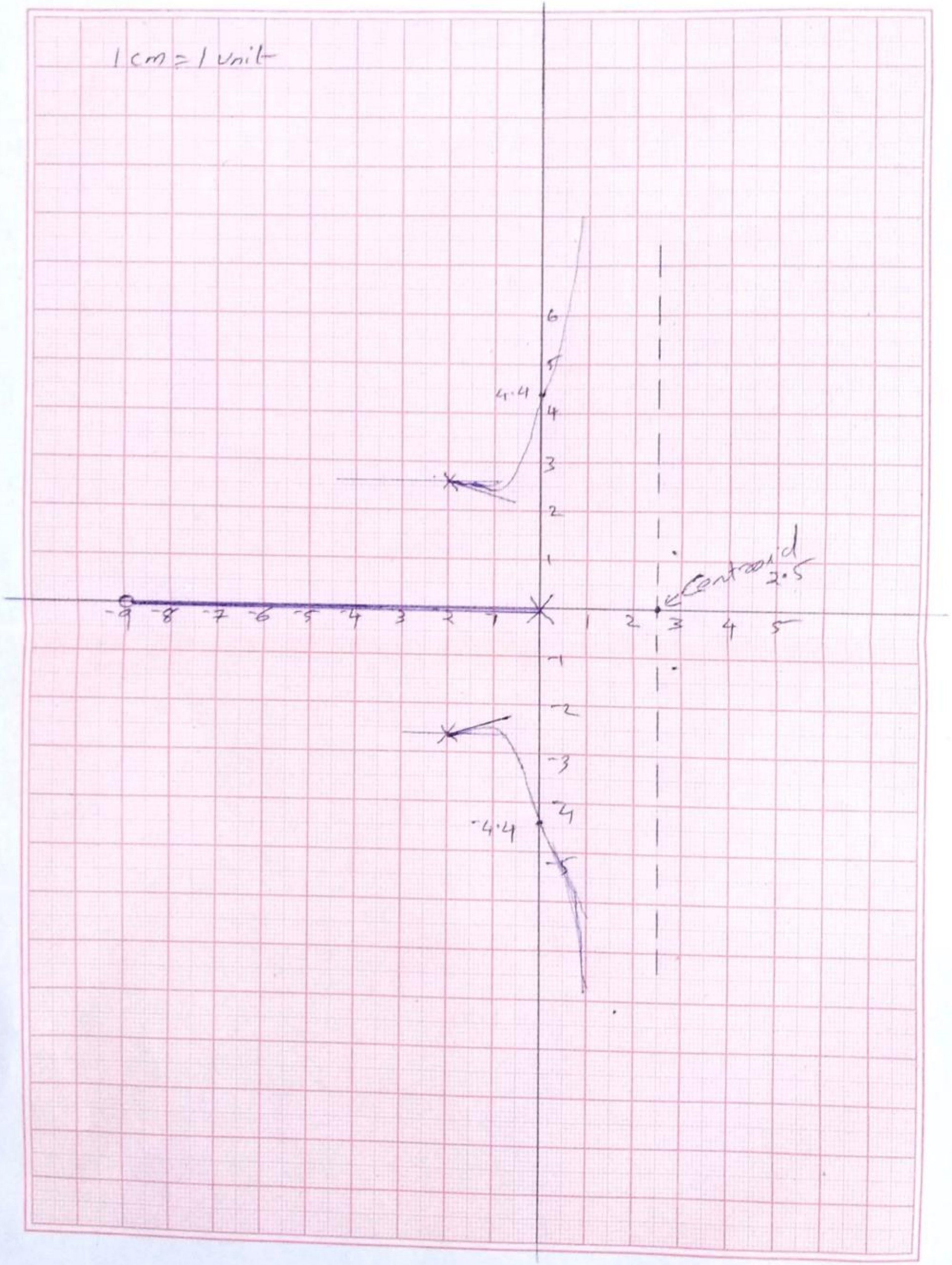
$$\therefore 9K - 4K = 44$$

$$\therefore 5K = 44 \quad \Rightarrow \quad K = \frac{44}{5} = 8.8$$

The crossing point of root locus is $\pm j4.4$. The value of K at this crossing point is $K = 8.8$

This is the limiting value of K for the stability of the system


1cm = 1 unit
centroid
2.5
4.4
-4.4
6
5
4
3
2
1
-1
-2
-3
-4
-5
-9
-8
-7
-6
-5
-4
3
2
1
1
2
3
4
5
SHARP - 50



1 cm = 1 unit
Centroid
2.5
4.4
-4.4

MODULE V

FREQUENCY DOMAIN ANALYSIS

Frequency domain specifications

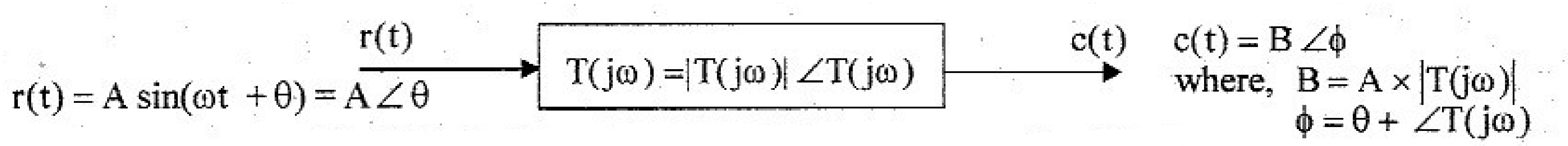
Bode plot

Log magnitude vs. phase plot,

FREQUENCY DOMAIN ANALYSIS

Frequency response is the steady state response of a system when the input to the system is a sinusoidal signal.

Consider a LTI system
let r(t) be an input sinusoidal signal.
The response or output y(t) is also a sinusoidal signal of the same frequency but with different magnitude and phase angle.



The magnitude and phase relationship between the sinusoidal input and the steady state output of the system is termed as frequency response.

In the system transfer function T(S), if 'S' is replaced by j-Omega ($j\omega$) then the resulting transfer function T($j\omega$) is called sinusoidal transfer function.

The frequency response of the system can be directly obtained from the sinusoidal transfer function T($j\omega$) of the system.

Open loop transfer function : $G(s) \xrightarrow{s=j\omega} G(j\omega) = |G(j\omega)| \angle G(j\omega)$

Loop transfer function : $G(s)H(s) \xrightarrow{s=j\omega} G(j\omega)H(j\omega) = |G(j\omega)H(j\omega)| \angle G(j\omega)H(j\omega)$

Closed loop transfer function: $M(s) \xrightarrow{s=j\omega} M(j\omega) = |M(j\omega)| \angle M(j\omega)$

where, $|G(j\omega)|$, $|M(j\omega)|$, $|G(j\omega)H(j\omega)|$ are Magnitude functions

$\angle G(j\omega)$, $\angle M(j\omega)$, $\angle G(j\omega)H(j\omega)$ are Phase functions.

The advantage of frequency response analysis

The absolute and relative stability of the closed loop system can be estimated from the knowledge of their open loop frequency response.

The practical testing of the system can be easily carried with available sinusoidal signal generators and precise measurement equipment

The transfer function of complicated systems can be determined experimentally by frequency response tests.

The design and parameter adjustment of the open loop transfer function of a system for specified closed-loop performance is carried out more easily in frequency domain.

When the system is designed by use of the frequency response analysis the effect of noise disturbances and parameters variations are relatively easy to visualise and incorporate corrective measures.

The frequency response analysis and response can be extended to certain nonlinear control systems.

Frequency domain specifications

1. resonant peak Mr
2. resonant frequency
3. bandwidth
4. cutoff rate
5. gain margin
6. phase margin

Resonant peak

The maximum value of the magnitude of closed loop transfer function is called resonant peak (Mr)

$$\text{Resonant peak, } M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

Resonant frequency ω_r

The frequency at which resonant peak occurs is called resonant frequency

$$\text{Normalized resonant frequency, } u_r = \frac{\omega_r}{\omega_n} = \sqrt{1-2\zeta^2}$$

$$\text{The resonant frequency, } \omega_r = \omega_n\sqrt{1-2\zeta^2}$$

Bandwidth ω_b

The bandwidth is the range of frequencies for which the system gain is more than -3 db.

The frequency at which the gain -3 db is called cutoff frequency.

The bandwidth is a measure of the ability of a feedback system to reproduce the input signal, noise rejection characteristics and rise time.

$$\text{Normalized bandwidth, } u_b = \frac{\omega_b}{\omega_n}$$

$$\text{Bandwidth, } \omega_b = \omega_n u_b = \omega_n \left[1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + 4\zeta^4}\right]^{\frac{1}{2}}$$

Cut off rate

The slope of the log magnitude curve near the cutoff frequency is called the cutoff rate.

The cut off rate indicates the ability of the system to distinguish the signal from the noise.

Gain margin k_g

The gain margin $\mathbf{k_g}$ is defined as the reciprocal of the magnitude of open loop transfer function at phase crossover frequency.

The frequency at which the phase of open loop transfer function is '- 180 degree' is called the phase crossover frequency ω_{pc}.

$$\text{Gain Margin, } K_g = \frac{1}{|G(j\omega_{pc})|}$$

the gain margin in db can be expressed as

$$K_g \text{ in db} = 20 \log K_g = 20 \log \frac{1}{|G(j\omega_{pc})|}$$

$$= -20 \log \left|G(j\omega_{pc})\right|$$

Gain margin of a second order system is infinity.

Phase margin γ

The phase margin is that amount of additional phase lag to be added at the gain crossover frequency in order to bring the system to the verge of instability.

The gain crossover frequency is the frequency at which the magnitude of the open loop transfer function is unity. (It is the frequency at which the db magnitude is zero)

Phase margin, $\gamma = 180° + \phi_{gc}$,

where, $\phi_{gc} = \angle G(j\omega_{gc})$

Note : $\angle G(j\omega_{gc})$ *is the phase angle of* $G(j\omega)$ at $\omega = \omega_{gc}$

At the gain cross-over frequency ω_{gc}, the magnitude of $G(j\omega)$ is unity.

Let normalized gain cross over frequency, $u_{gc} = \omega_{gc}/\omega_n$

$$u_{gc} = \left[-2\zeta^2 + \sqrt{4\zeta^4 + 1}\right]^{\frac{1}{2}}$$

$$\gamma = 180 + \left(-90^\circ - \tan^{-1}\frac{u_{gc}}{2\zeta}\right)$$

Frequency response plots

1. Bode plot
2. Polar plot (nyquist plot)
3. Nichols plot
4. M and N circles
5. Nichols chart

Bode plot

A bode plot consist of two graph

One is a product of the magnitude of a sinusoidal transfer function versus log Omega ($\log \omega$)

The other is a plot of the phase angle of a sinusoidal transfer function versus log Omega ($\log \omega$)

The bode plot can be drawn for both open loop and closed loop transfer function

Usually the bode plot is drawn for open loop system.

Consider the open loop transfer function, $G(s) = \dfrac{K(1+sT_1)}{s(1+sT_2)(1+sT}$

$$G(j\omega) = \frac{K(1+j\omega T_1)}{j\omega(1+j\omega T_2)(1+j\omega T_3)}$$

$$= \frac{K\angle 0^\circ \sqrt{1+\omega^2 T_1^2} \angle \tan^{-1} \omega T_1}{\omega\angle 90^\circ \sqrt{1+\omega^2 T_2^2} \angle \tan^{-1} \omega T_2 \sqrt{1+\omega^2 T_3^2} \angle \tan^{-1} \omega T_3}$$

The magnitude of G(jω) = $|G(j\omega)| = \dfrac{K\sqrt{1+\omega^2T_1^2}}{\omega\sqrt{1+\omega^2T_2^2}\sqrt{1+\omega^2T_3^2}}$

The phase angle of the G(jω) = $\angle G(j\omega) = \tan^{-1}\omega T_1 - 90° - \tan^{-1}\omega T_2 - \tan^{-1}\omega T_3$

$|G(j\omega)|$ in db = $20\log|G(j\omega)|$

$$= 20\log\left[\frac{K\sqrt{1+\omega^2T_1^2}}{\omega\sqrt{1+\omega^2T_2^2}\sqrt{1+\omega^2T_3^2}}\right]$$

$$= 20\log\left[\frac{K}{\omega}\times\sqrt{1+\omega^2T_1^2}\times\frac{1}{\sqrt{1+\omega^2T_2^2}}\times\frac{1}{\sqrt{1+\omega^2T_3^2}}\right]$$

$$= 20\log\frac{K}{\omega} + 20\log\sqrt{1+\omega^2T_1^2} + 20\log\frac{1}{\sqrt{1+\omega^2T_2^2}} + 20\log\frac{1}{\sqrt{1+\omega^2T_3^2}}$$

$$= 20\log\frac{K}{\omega} + 20\log\sqrt{1+\omega^2T_1^2} - 20\log\sqrt{1+\omega^2T_2^2} - 20\log\sqrt{1+\omega^2T_3^2}$$

When the magnitude is expressed in db, the multiplication is converted to addition.

In magnitude plot, the db magnitudes of individual factors of $G(j\omega)$ can be added.

Individual factors of $G(j\omega)$

1. constant gain K
2. integral factor $\frac{K}{j\omega}$ or $\frac{K}{(j\omega)^n}$
3. derivative factor $K \times j\omega$ or $K \times (j\omega)^n$
4. first order factor in denominator $\frac{1}{1+j\omega T}$ or $\frac{1}{(1+j\omega T)^m}$
5. first order factor in numerator $(1+j\omega T)$ or $(1+j\omega T)^m$
6. quadratic factor in denominator $\left[\frac{1}{1+2\zeta(j\omega/\omega_n)+(j\omega/\omega_n)^2}\right]$
7. quadratic factor in numerator $\left[1+2\zeta\left(\frac{j\omega}{\omega_n}\right)+\left(\frac{j\omega}{\omega_n}\right)^2\right]$

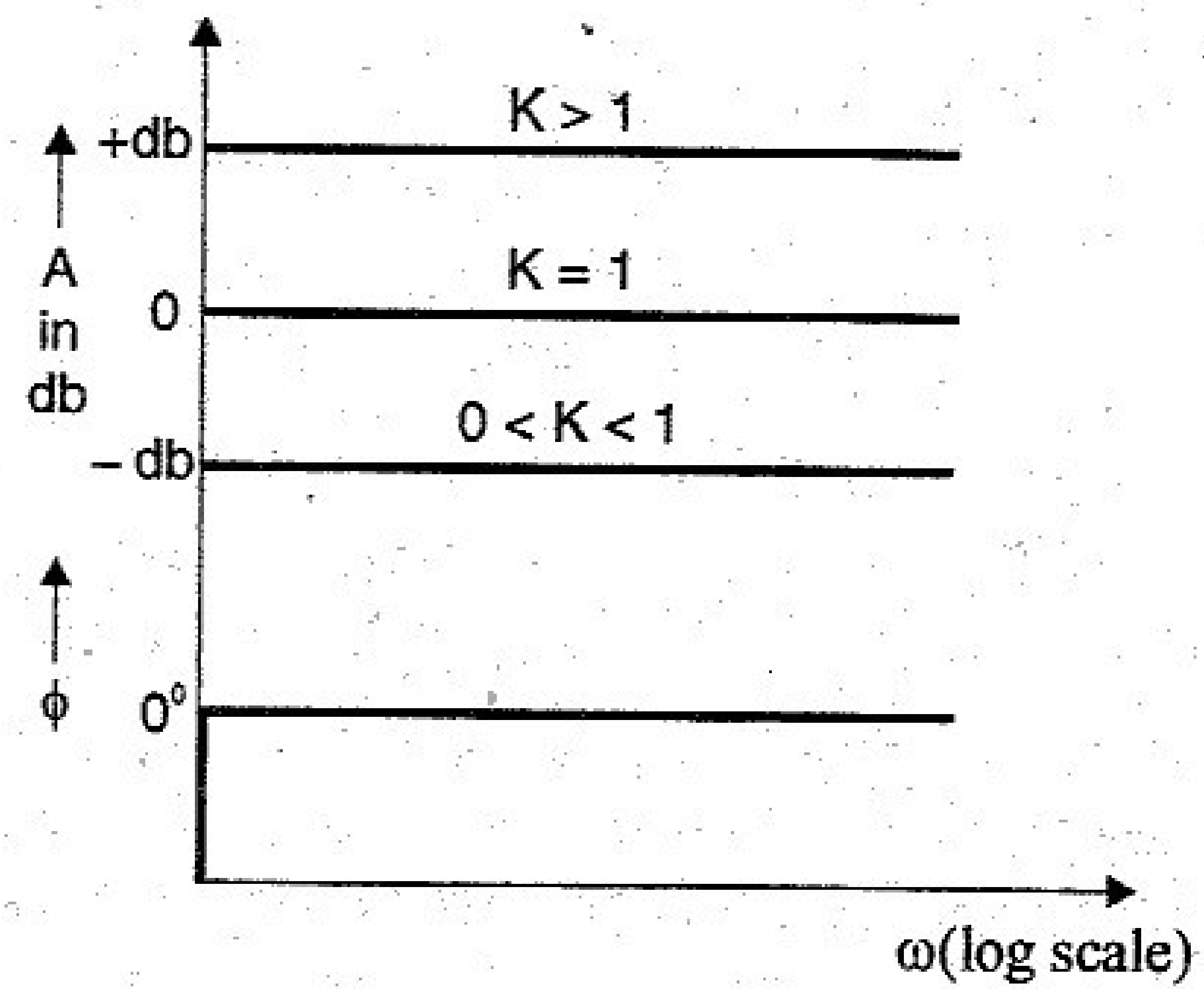
1. constant gain

Let, $G(s) = K$

$$\therefore G(j\omega) = K = K\angle 0^\circ$$

$$A = |G(j\omega)| \text{ in db} = 20 \log K$$

$$\phi = \angle G(j\omega) = 0^\circ$$



Bode plot of constant gain, K.

2. integral factor

Let, $G(s) = \frac{K}{s}$

$$\therefore G(j\omega) = \frac{K}{j\omega} = \frac{K}{\omega}\angle -90^\circ$$

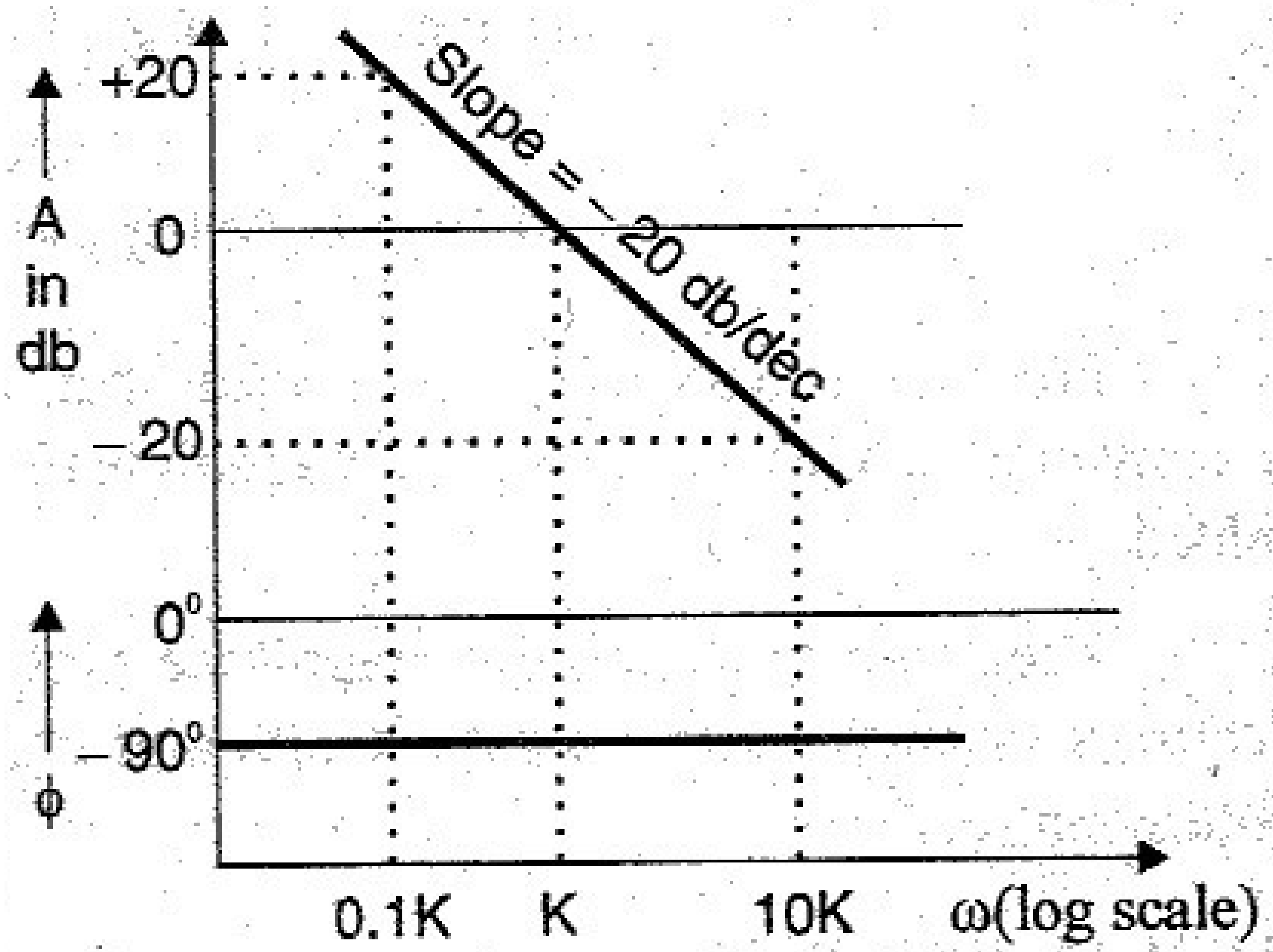
$$A = |G(j\omega)| \text{ in db} = 20 \log (K/\omega)$$

$$\phi = \angle G(j\omega) = -90^\circ$$

When $\omega = 0.1$ K, $\quad A = 20 \log (1/0.1) = 20$ db

When $\omega = K$, $\quad A = 20 \log 1 = 0$ db

When $\omega = 10$ K, $\quad A = 20 \log (1/10) = -20$ db



magnitude plot of the integral factor is a straight line with a slope of –20 db/dec and passing through zero db, when ω = K. Since the ∠G(jω) is a constant and independent of ω the phase plot is a straight line at –90°.

When an integral factor has multiplicity of n, then.

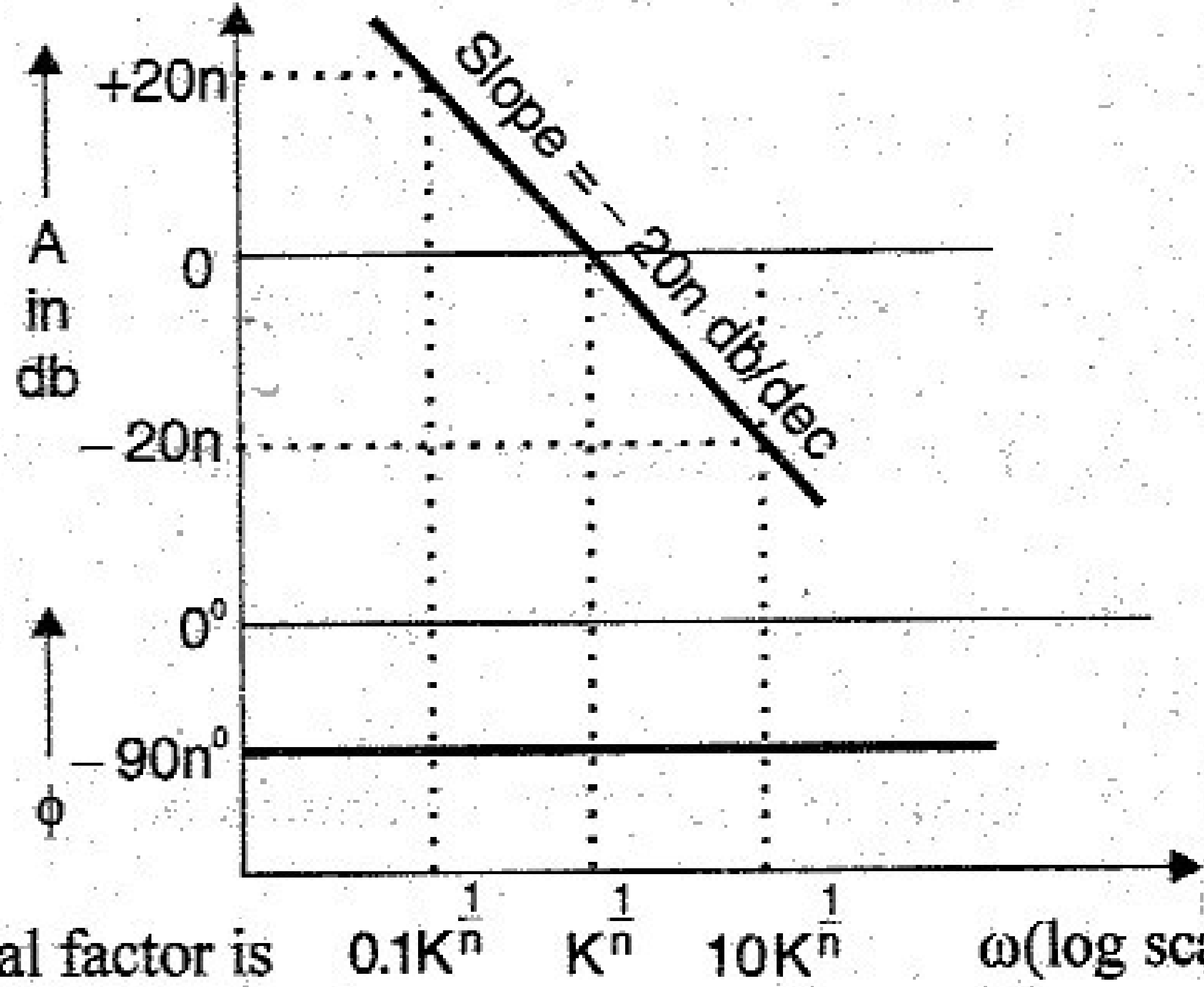
$$G(s) = K/s^n$$

$$G(j\omega) = K/(j\omega)^n = K/\omega^n \angle -90n^\circ$$

$$A = |G(j\omega)| \text{ in db } = 20 \log \frac{K}{\omega^n}$$

$$= 20\log \left(\frac{K^{\frac{1}{n}}}{\omega}\right)^n = 20\, n \log \left(\frac{K^{\frac{1}{n}}}{\omega}\right)$$

$$\phi = \angle G(j\omega) = -90\, n^\circ$$



Now the magnitude plot of the integral factor is a straight line with a slope of $-20n$ db/dec and passing through zero db when $\omega = K^{1/n}$. The phase plot is a straight line at $-90n^\circ$.

3. derivative factor

Let, $G(s) = Ks$

$$\therefore G(j\omega) = K\ j\omega = K\omega\ \angle 90^\circ$$

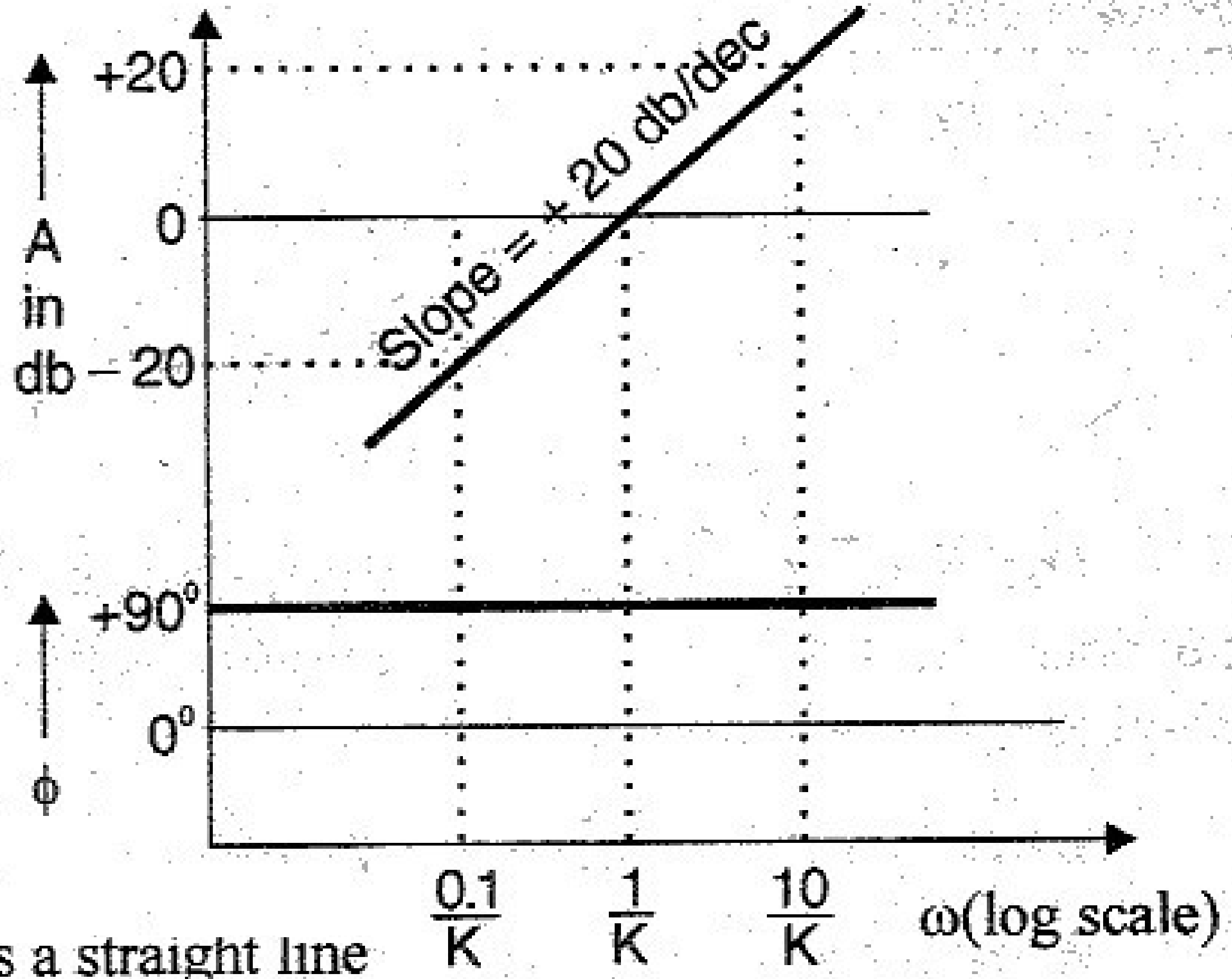
$$A = |G(j\omega)| \text{ in db} = 20 \log (K\omega)$$

$$\phi = \angle G(j\omega) = +90^\circ$$

When $\omega = 0.1/K$, $A = 20 \log (0.1) = -20$ db

When $\omega = 1/K$, $A = 20 \log 1 = 0$ db

When $\omega = 10/K$, $A = 20 \log 10 = +20$ db



magnitude plot of the derivative factor is a straight line with a slope of +20 db/dec and passing through zero db when $\omega = 1/K$. Since the $\angle G(j\omega)$ is a constant and independent of ω, the phase plot is a straight line at $+90^\circ$.

When derivative factor has multiplicity of n then,

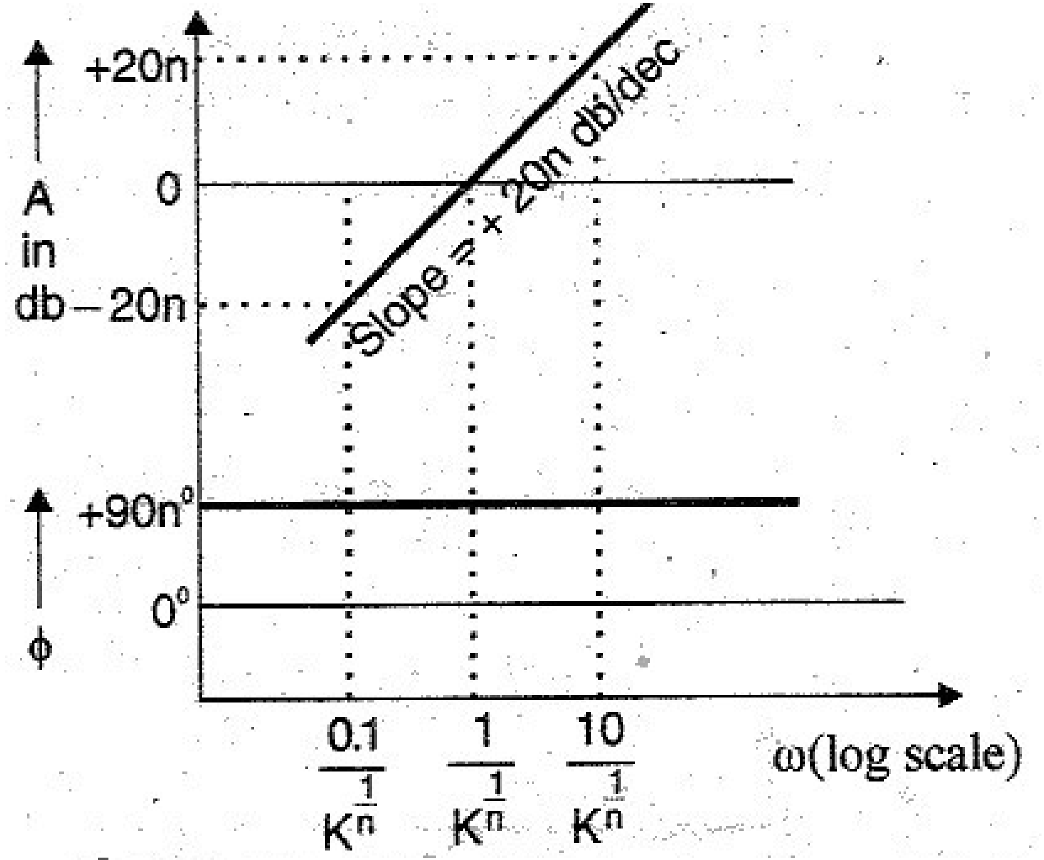
$$G(s) = K\, s^n$$

$$\therefore\ G(j\omega) = K(j\omega)^n = K\omega^n \angle 90n°$$

$$A = |G(j\omega)| \text{ in db} = 20 \log (K\omega^n)$$

$$= 20 \log (K^{1/n}\, \omega)^n = 20\, n \log (K^{1/n}\, \omega)$$

$$\phi = \angle G(j\omega) = 90n°$$



Now the magnitude plot of the derivative factor is a straight line with a slope of +20n db/dec and passing through zero db when $\omega = 1/K^{1/n}$. The phase plot is a straight line at +90n°.

4. first order factor in denominator

$$G(s) = \frac{1}{1+sT}$$

$$\therefore G(j\omega) = \frac{1}{1+j\omega T} = \frac{1}{\sqrt{1+\omega^2T^2}} \angle -\tan^{-1}\omega T$$

Let, $A = |G(j\omega)|$ in db.

$$\therefore \ A = |G(j\omega)|_{\text{in db}} = 20 \log \frac{1}{\sqrt{1+\omega^2T^2}} = -20 \log \sqrt{1+\omega^2T^2}$$

At very low frequencies, $\omega T << 1$; $\quad \therefore A = -20 \log \sqrt{1+\omega^2T^2} \approx -20 \log 1 = 0$

At very high frequencies, $\omega T >> 1$; $\quad \therefore A = -20 \log \sqrt{1+\omega^2T^2} \approx -20 \log \sqrt{\omega^2T^2} = -20 \log \omega T$

$$\text{At } \omega = \frac{1}{T}, \quad A = -20 \log 1 = 0$$

$$\text{At } \omega = \frac{10}{T}, \quad A = -20 \log 10 = -20 \text{ db}$$

Magnitude plot can be approximated by two straight line

One is a straight line at 0db for the frequency range $0 < \omega < 1/T$.

Second one is a straight line with slope –20 db/dec for the frequency range, $1/T < \omega < \infty$

The two straight lines are asymptotes of the exact curve.

The frequency at which the two asymptotes meet is called corner frequency or break frequency

For factor $1/(1+j\omega T)$ the frequency $\omega = 1/T$ is the corner frequency

It divides the frequency response curve into two region, a curve for low frequency region and a curve for high frequency region

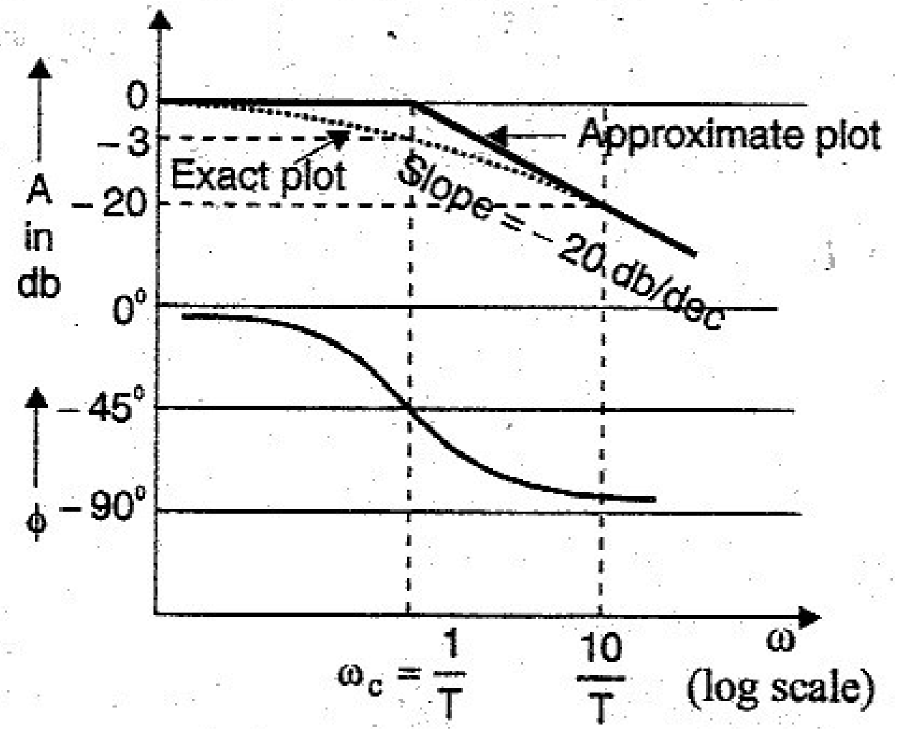
The actual magnitude at the corner frequency, $\omega_c = \frac{1}{T}$ is, $A = -20 \log\sqrt{1+1} = -3$ db

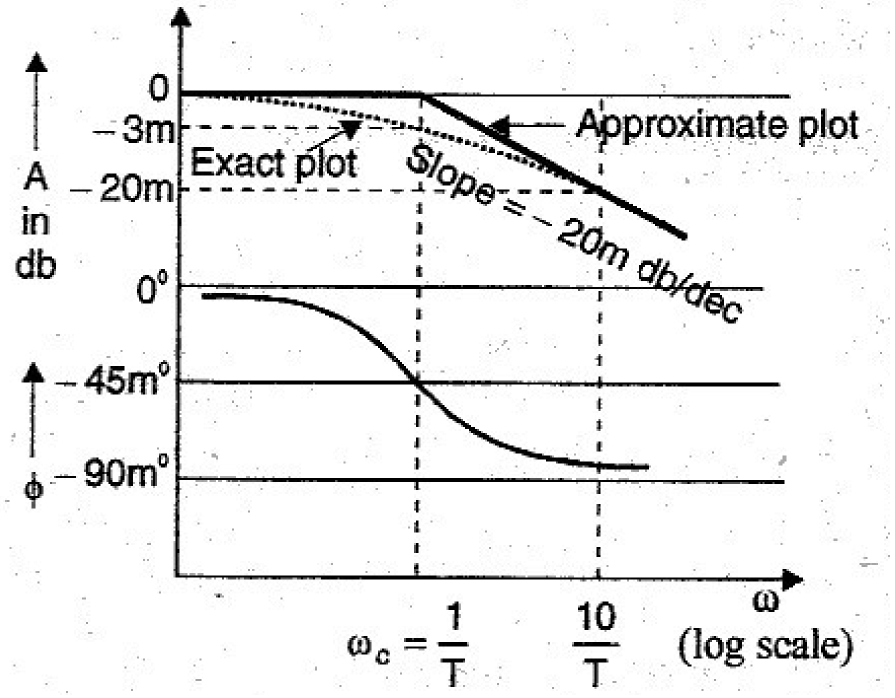
Phase angle, $\phi = \angle G(j\omega) = -\tan^{-1} \omega T$

At the corner frequency, $\omega = \omega_c = \frac{1}{T}$, $\phi = -\tan^{-1}\omega T = -\tan^{-1} 1 = -45°$

As $\omega \to 0$, $\phi \to 0°$

As $\omega \to \infty$, $\phi \to -90°$





$$G(s) = \frac{1}{(1+sT)^m} \; ; \qquad \therefore G(j\omega) = \frac{1}{(1+j\omega T)^m} = \frac{1}{\left(\sqrt{1+\omega^2 T^2}\right)^m \angle m \tan^{-1}\omega T}$$

$$A = |G(j\omega)| \text{ in db} = 20 \log \frac{1}{\left(\sqrt{1+\omega^2 T^2}\right)^m} = -20\, m \log\sqrt{1+\omega^2 T^2}$$

$$\phi = \angle G(j\omega) = -m \tan^{-1}\omega T$$

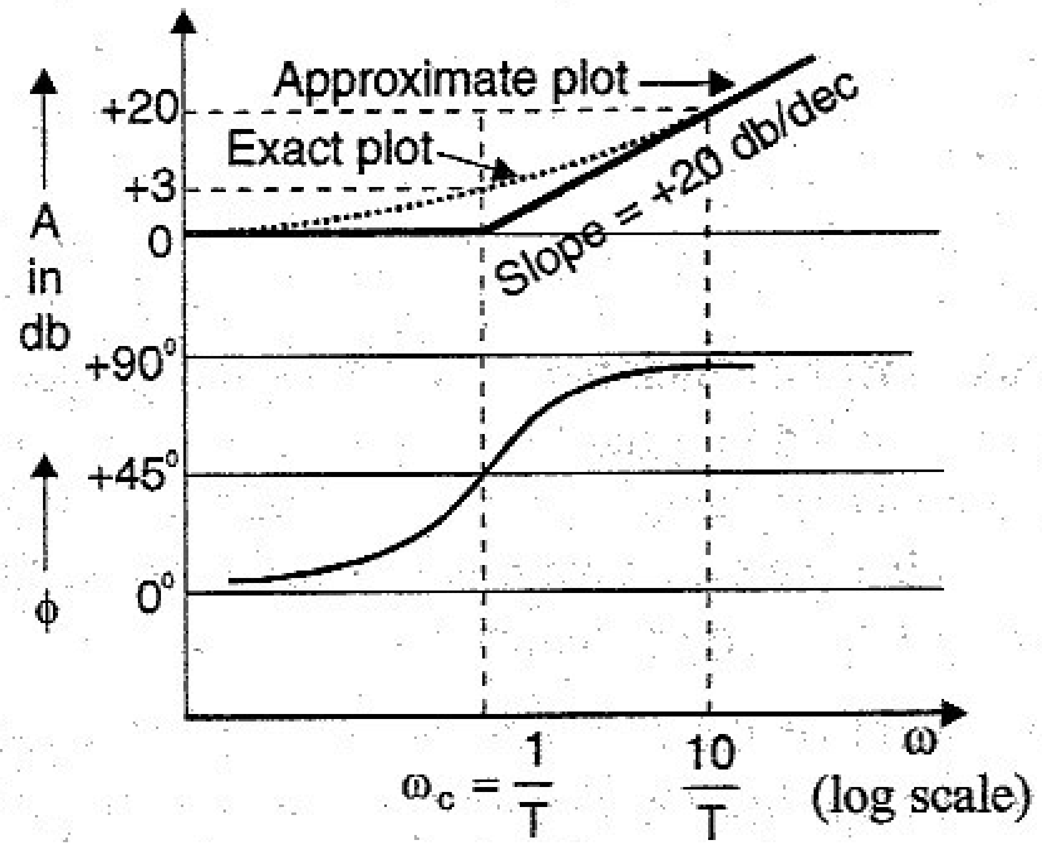
5. first order factor in numerator

$$G(s) = 1 + sT$$

$$G(j\omega) = 1 + j\omega T \quad = \sqrt{1+\omega^2T^2}\ \angle \tan^{-1}\omega T$$

$$A = |G(j\omega)| \text{ in db} = 20\log\sqrt{1+\omega^2T^2}$$

$$\phi = \angle G(j\omega) = \tan^{-1}\omega T$$



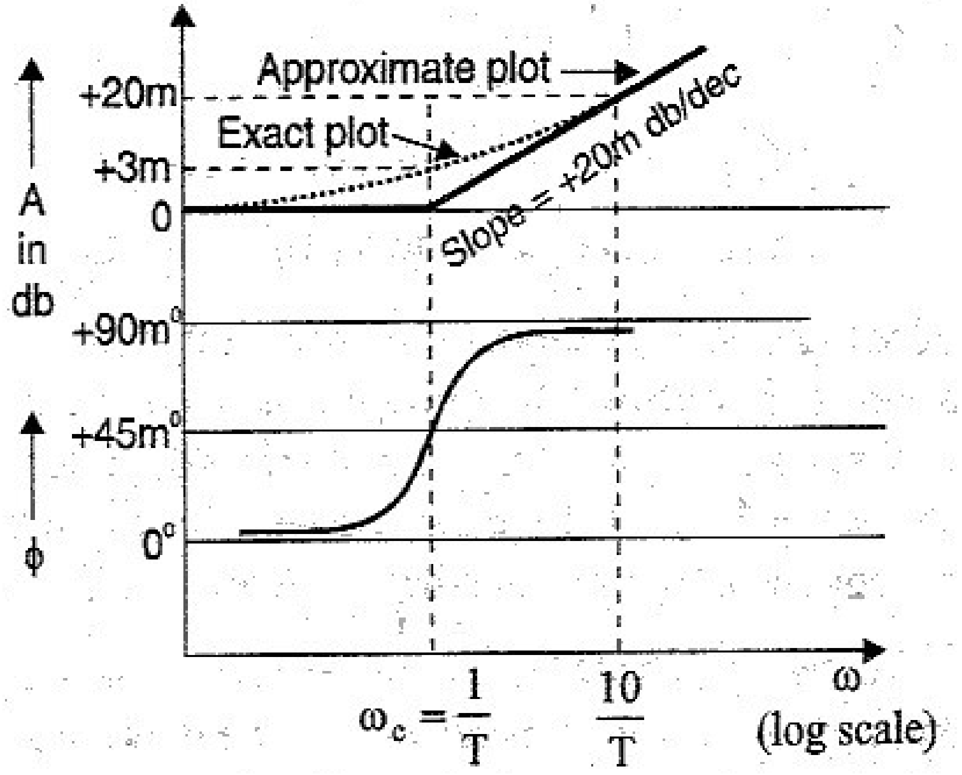
first order factor in the numerator has a multiplicity of m

$$G(s) = (1 + sT)^m$$

$$G(j\omega) = (1 + j\omega T)^m \quad = \left(\sqrt{1+\omega^2T^2}\right)^m \angle m\tan^{-1}\omega T$$

$$A = |G(j\omega)| \text{ in db} = 20\log\left(\sqrt{1+\omega^2T^2}\right)^m = 20m\log\sqrt{1+\omega^2T^2}$$

$$\phi = \angle G(j\omega) = m\tan^{-1}\omega T$$

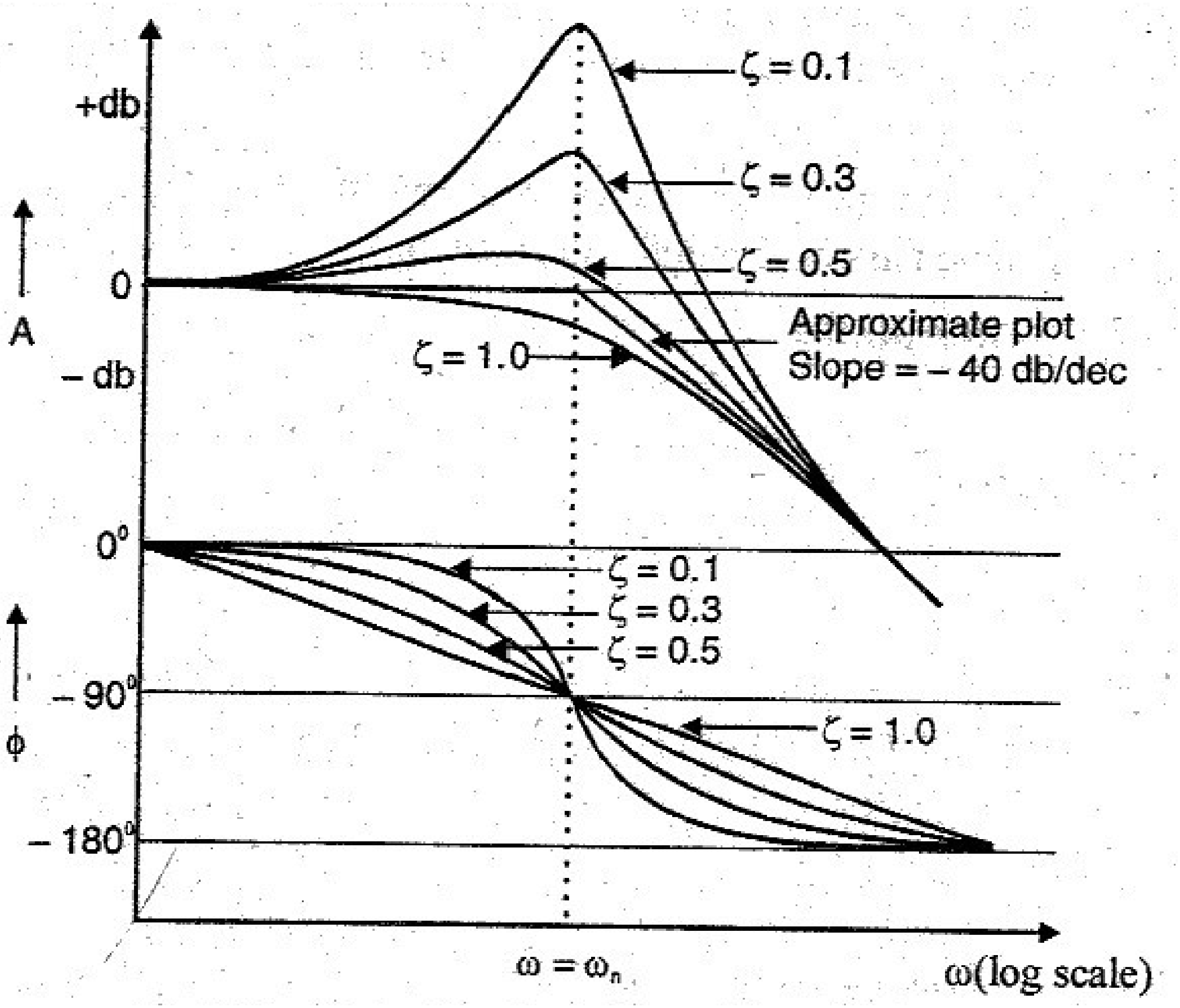


6. quadratic factor in denominator

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{1 + 2\zeta\frac{s}{\omega_n} + \left(\frac{s}{\omega_n}\right)^2}$$

$$G(j\omega) = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\zeta^2\frac{\omega^2}{\omega_n^2}}} \angle -\tan^{-1}\frac{2\zeta\frac{\omega}{\omega_n}}{1 - \frac{\omega^2}{\omega_n^2}}$$

Magnitude plot can be approximated by two straight line
One is a straight line at 0db for the frequency range $0 < \omega < \omega_n$
Second one is a straight line with slope –40 db/dec for the frequency range, $\omega_n < \omega < \infty$
The two straight lines are asymptotes of the exact curve.
The frequency at which the two asymptotes meet is called corner frequency or break frequency
For quadratic factor the frequency ω_n is the corner frequency ω_c.



At $\omega = \omega_n$, $A = -40 \log 1 = 0$ db

At $\omega = 10\omega_n$, $A = -40 \log 10 = -40$ db

As $\omega = \omega_n$, $\phi = -\tan^{-1} \frac{2\zeta}{0} = -\tan^{-1} \infty = -90^\circ$

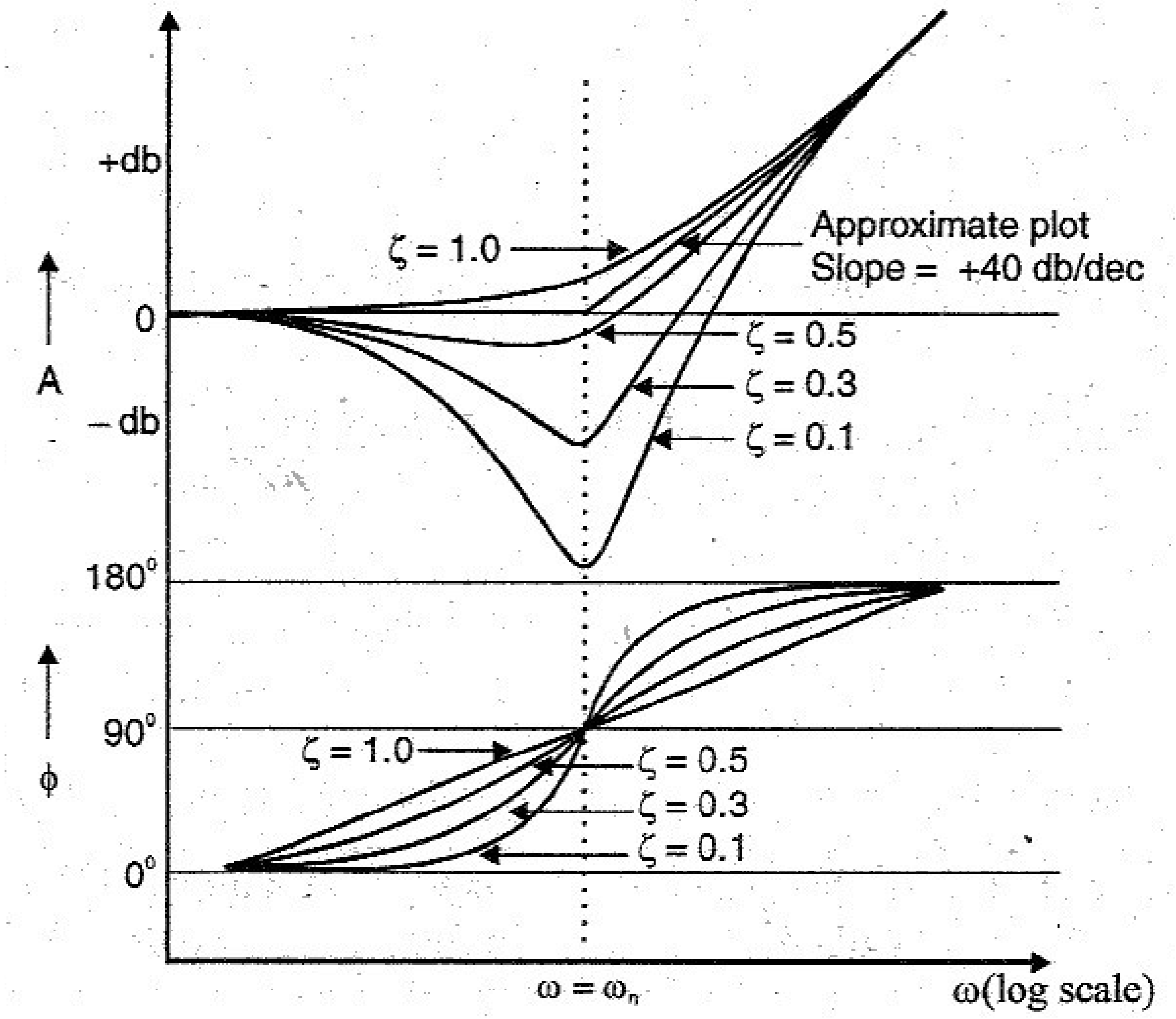
As $\omega \to 0$, $\phi \to 0$

As $\omega \to \infty$, $\phi \to -180^\circ$

7. quadratic factor in numerator

$$G(s) = \frac{s^2 + 2\zeta\omega_n s + \omega_n^2}{\omega_n^2} = 1 + 2\zeta\left(\frac{s}{\omega_n}\right) + \left(\frac{s}{\omega_n}\right)^2$$

$$G(j\omega) = 1 + j2\zeta\,\frac{\omega}{\omega_n} + \left(\frac{j\omega}{\omega_n}\right)^2 = \sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\zeta^2\,\frac{\omega^2}{\omega_n^2}}\ \angle\tan^{-1}\frac{2\zeta\dfrac{\omega}{\omega_n}}{1 - \dfrac{\omega^2}{\omega_n^2}}$$



Magnitude plot can be approximated by two straight line

One is a straight line at 0db for the frequency range $0 < \omega < \omega_n$
Second one is a straight line with slope +40 db/dec for the frequency range, $\omega_n < \omega < \infty$

The two straight lines are asymptotes of the exact curve.

For quadratic factor ω_n is the corner frequency

	Individual factors	Angle	Slope
1.	K	0	0
2.	$\frac{K}{(j\omega)^n}$	$-90n$	$-n20$
3.	$K(j\omega)^n$	$90n$	$n20$
4.	$\frac{1}{(1+j\omega T)^m}$	$-m\tan^{-1}\omega T$	$-20m$
5.	$(1+j\omega T)^m$	$m\tan^{-1}\omega T$	$20m$
6.	$\frac{1}{1+j2\xi\frac{\omega}{\omega_n}+\left(\frac{j\omega}{\omega_n}\right)^2}$	$-\tan-1\frac{y}{x}$	-40
7.	$1+j2\xi\frac{\omega}{\omega_n}+\left(\frac{j\omega}{\omega_n}\right)^2$	$\tan-1\frac{y}{x}$	40

BODE PLOT

Procedure for plotting magnitude plot

1. Put $s \to j\omega$ in open loop TF
2. Find corner frequencies for $(1 + j\omega T)$, $\omega_c = \frac{1}{T}$

list the corner frequencies in the increasing order and prepare a table as shown below

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec

3. find Lower frequency and higher frequencies $\omega_l < \omega_c$ and $\omega_h > \omega_c$

calculate the db magnitude of $K, \ \frac{K}{(j\omega)^n}, \ K(j\omega)^n$ at ωl and at the lowest corner frequency

4. calculate the remaining db magnitude one by one using the formula

$$\text{Gain at } \omega_y = \text{change in gain from } \omega_x \text{ to } \omega_y + \text{Gain at } \omega_x$$

$$= \left[\text{Slope from } \omega_x \text{ to } \omega_y \times \log \frac{\omega_y}{\omega_x}\right] + \text{Gain at } \omega_x$$

5. plot the magnitude in semi log graph sheet

Procedure for plotting phase plot

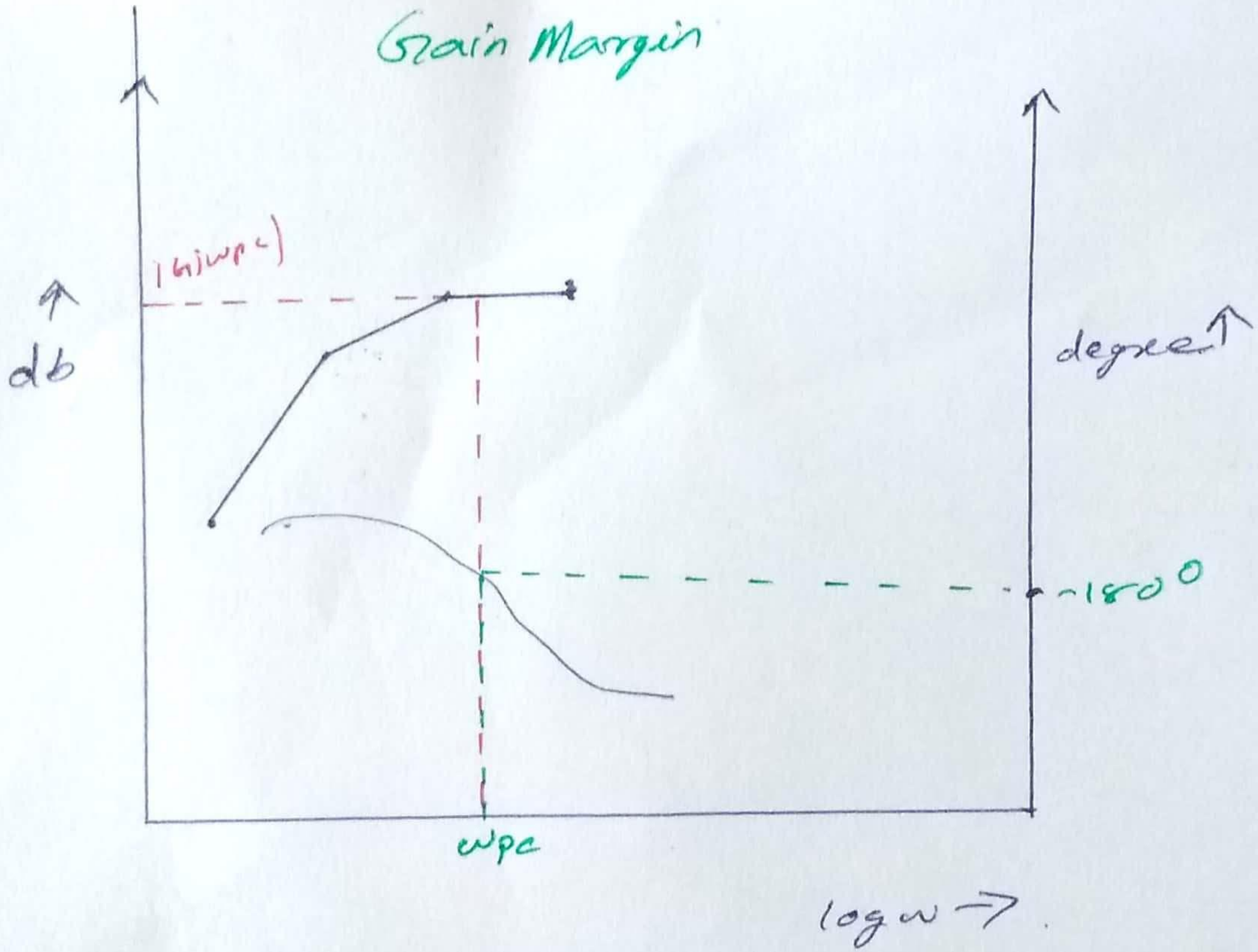
1. Phase angles are computed for different values of Omega and tabulated.
2. Plot the phase angle in semi log graph sheet

Gain adjustment in bode plot

Let, x = change in db (x is positive if the plot is shifted up and vice versa)

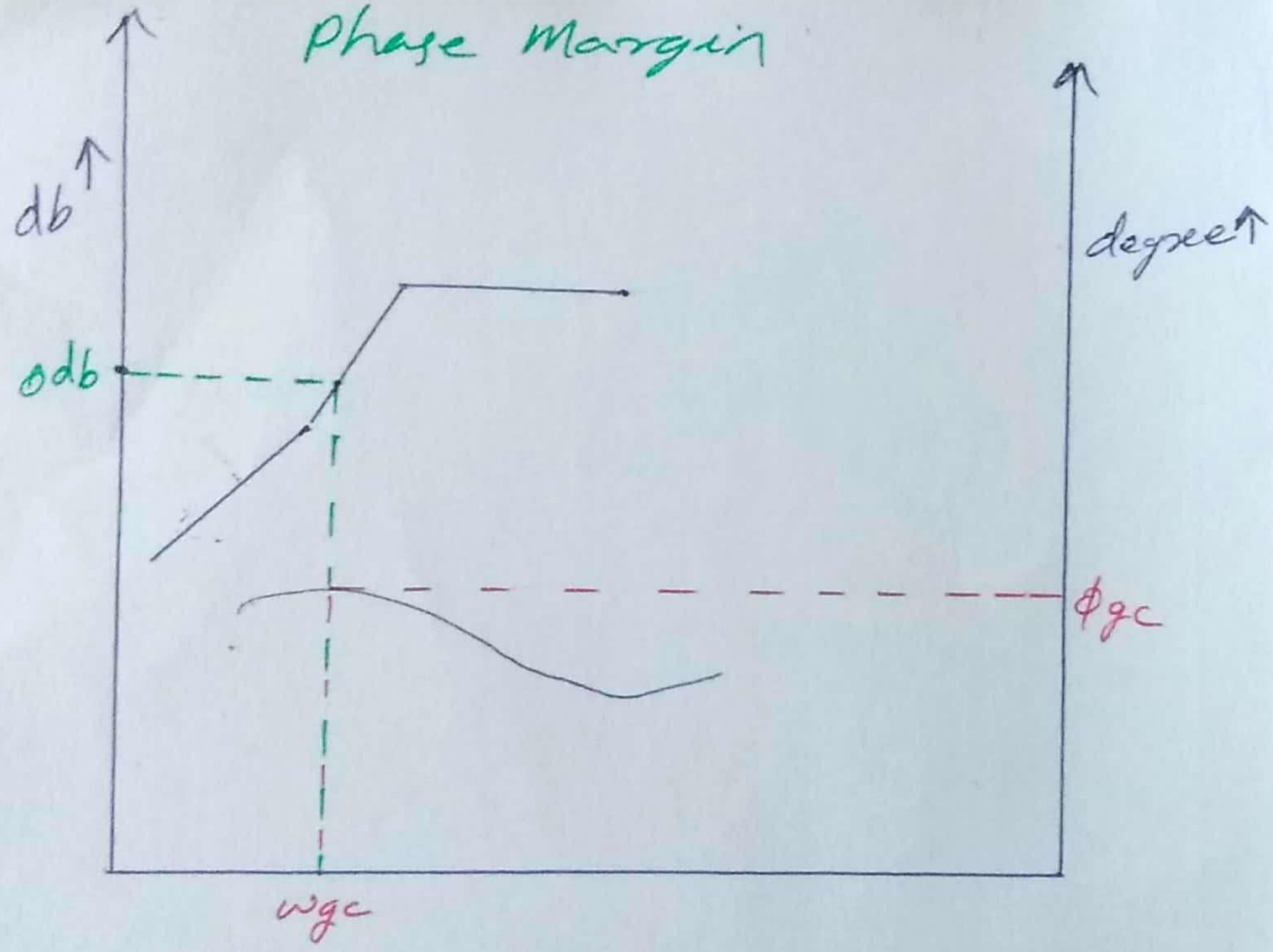
Now, $20 \log K = x$; $\log K = x/20$; $\therefore K = 10^{x/20}$

Gain Margin



$$K_g = \frac{1}{|G(j\omega_{pc})|}$$

Phase Margin



$\gamma = 180 + \phi_{gc}$

Sketch Bode plot for the following transfer function and determine the system gain K for the gain cross over frequency to be 5 rad/sec

$$G(s) = \frac{Ks^2}{(1+0.2s)\,(1+0.02s)}$$

$$G(j\omega) = \frac{K(j\omega)^2}{(1+0.2j\omega)(1+0.02j\omega)}$$

Let K = 1, $$G(j\omega) = \frac{(j\omega)^2}{(1+j0.2\omega)(1+j0.02\omega)}$$

MAGNITUDE PLOT

The corner frequencies are, $\omega_{c1} = \frac{1}{0.2} = 5$ rad / sec and $\omega_{c2} = \frac{1}{0.02} = 50$ rad / sec

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$(j\omega)^2$	–	+40	
$\frac{1}{1+j0.2\omega}$	$\omega_{c1} = \frac{1}{0.2} = 5$	–20	$40 - 20 = 20$
$\frac{1}{1+j0.02\omega}$	$\omega_{c2} = \frac{1}{0.02} = 50$	– 20	$20 - 20 = 0$

Choose a low frequency ω_l such that $\omega_l < \omega_{c1}$ and choose a high frequency ω_h such that $\omega_h > \omega_{c2}$.

Let, $\omega_l = 0.5$ rad/sec and $\omega_h = 100$ rad/sec.

Let, $A = \left|G(j\omega)\right|$ in db.

$$\text{At } \omega = \omega_l, \quad A = 20\log\left|(j\omega)^2\right| = 20\log(\omega)^2 = 20\log(0.5)^2 = -12 \text{ db}$$

$$\text{At } \omega = \omega_{c1}, \quad A = 20\log\left|(j\omega)^2\right| = 20\log(\omega)^2 = 20\log(5)^2 = 28 \text{ db}$$

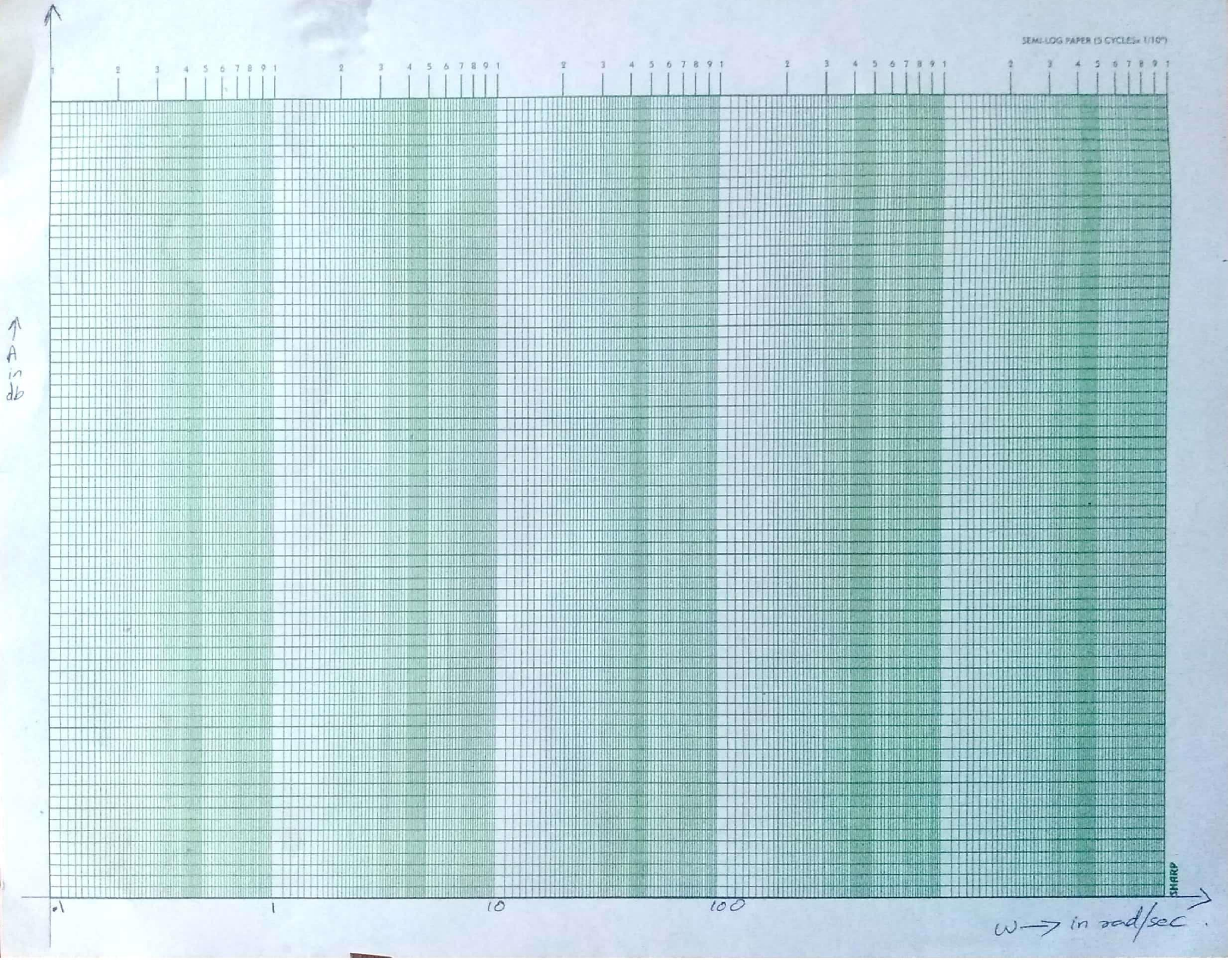
$$\text{At } \omega = \omega_{c2}, \quad A = \left[\text{slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c1}}\right] + A_{(\text{at } \omega = \omega_{c1})} = 20 \times \log\frac{50}{5} + 28 = 48 \text{ db}$$

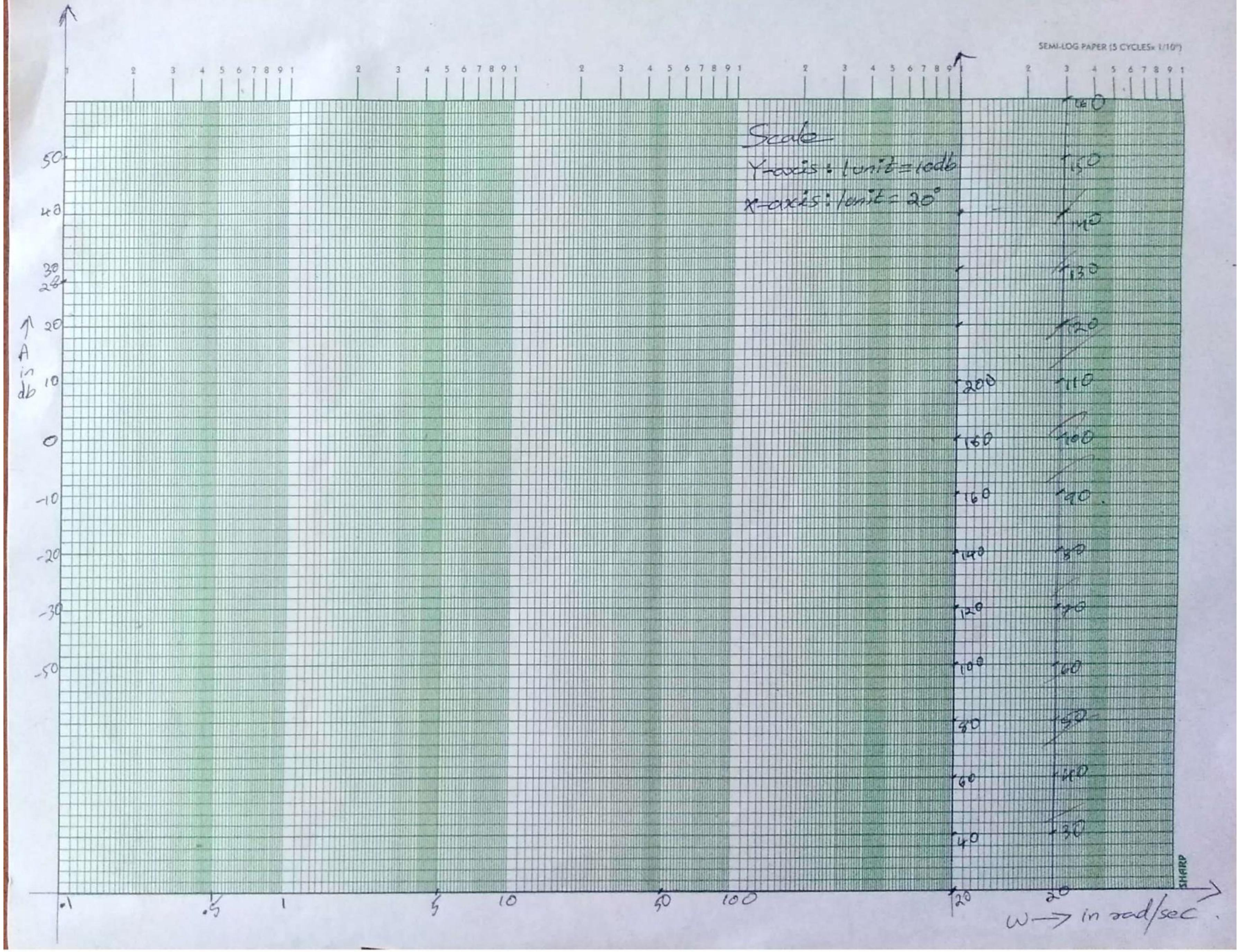
$$\text{At } \omega = \omega_h, \quad A = \left[\text{slope from } \omega_{c2} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c2}}\right] + A_{(\text{at } \omega = \omega_{c2})} = 0 \times \log\frac{100}{50} + 48 = 48 \text{ db}$$

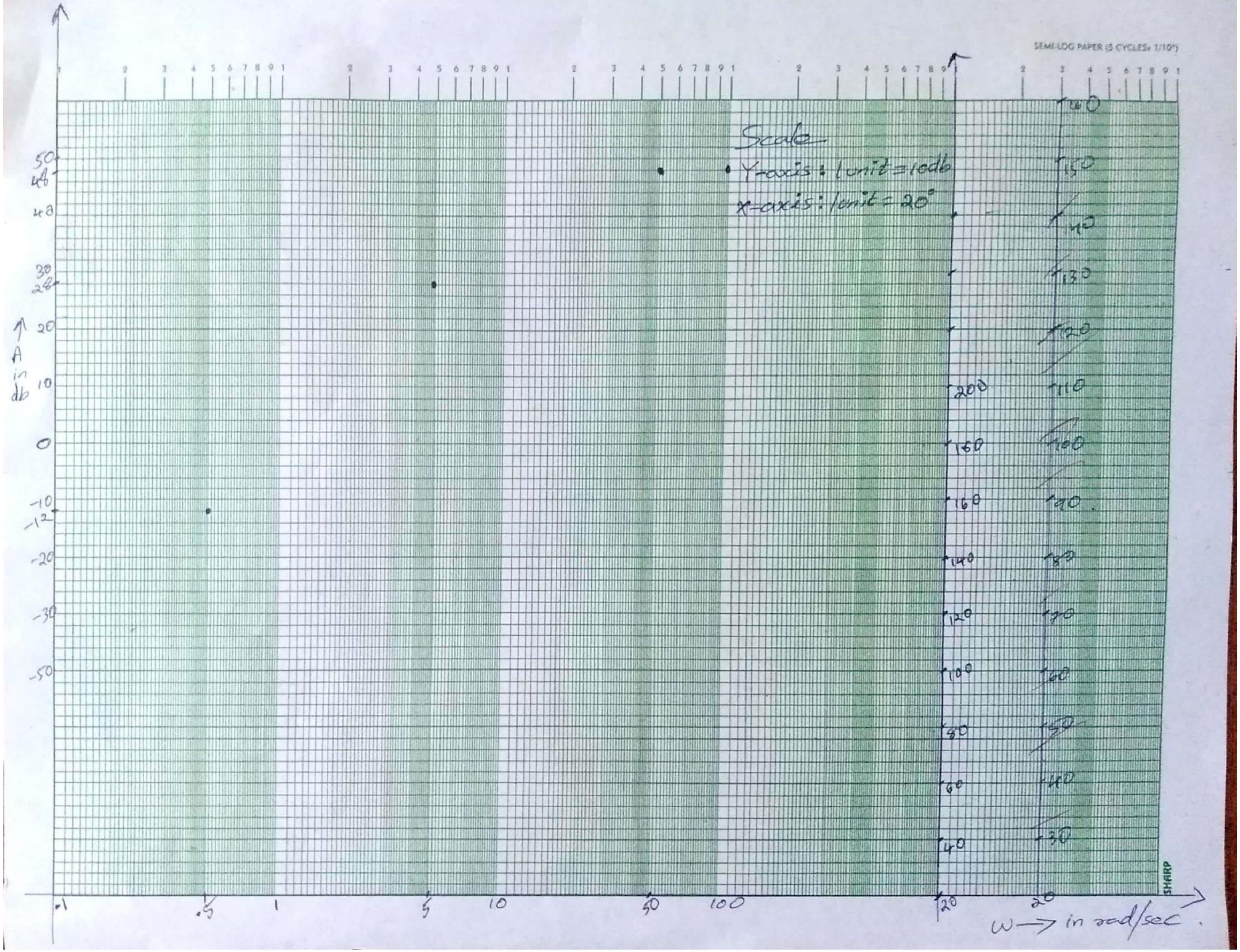
PHASE PLOT

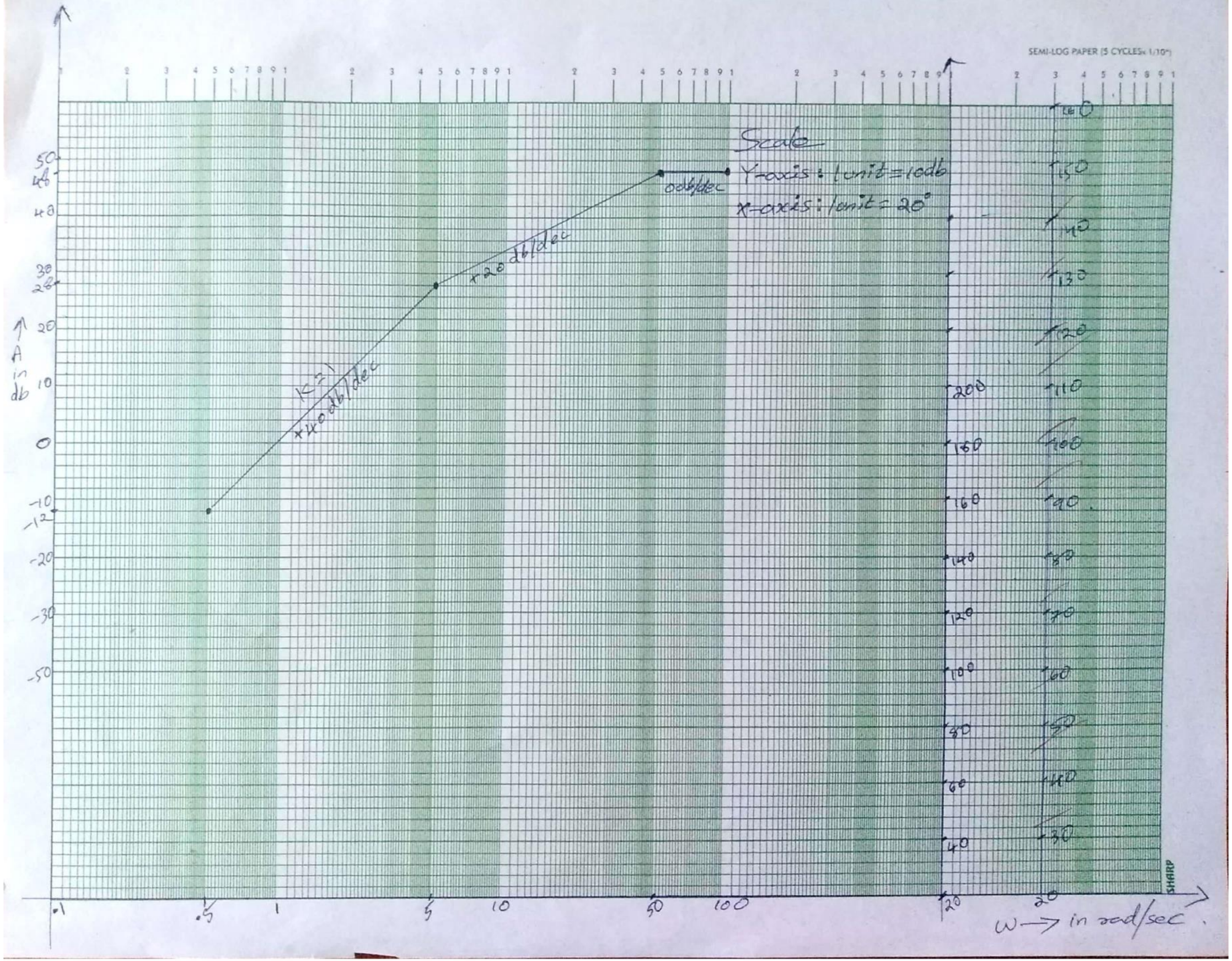
$$\phi = \angle G(j\omega) = 180^\circ - \tan^{-1}0.2\omega - \tan^{-1}0.02\omega$$

ω rad/sec	$\tan^{-1} 0.2\omega$ deg	$\tan^{-1} 0.02\omega$ deg	$\phi = \angle G(j\omega)$ deg	Point in phase plot
0.5	5.7	0.6	$173.7 \approx 174$	e
1	11.3	1.1	$167.6 \approx 168$	f
5	45	5.7	$129.3 \approx 130$	g
10	63.4	11.3	$105.3 \approx 106$	h
50	84.3	45	$50.7 \approx 50$	i
100	87.1	63.4	$29.5 \approx 30$	j

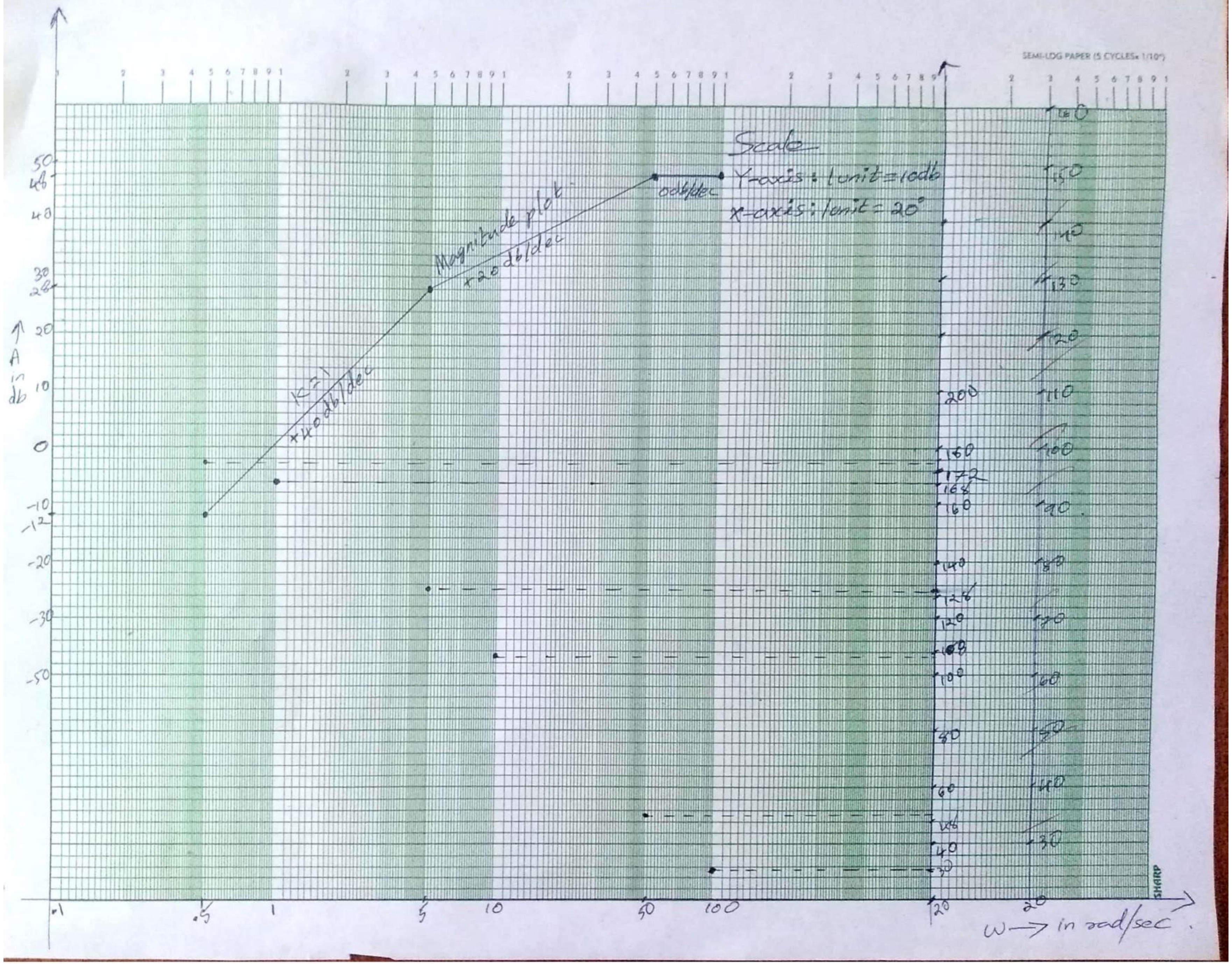

SEMI-LOG PAPER (5 CYCLES× 1/10")
A in db
·1
1
10
100
ω→ in rad/sec.
SHARP

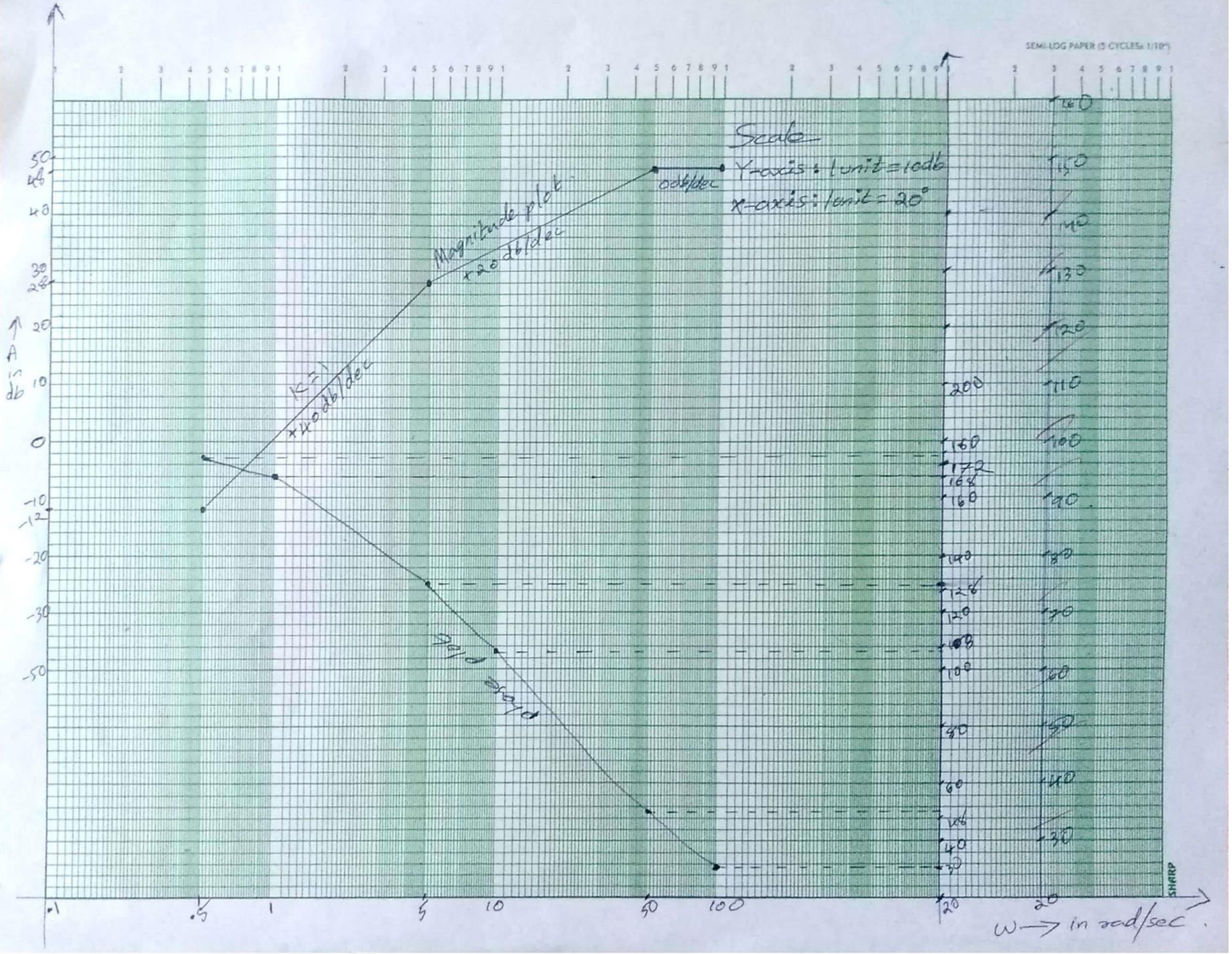

SEMI-LOG PAPER (5 CYCLES× 1/10")
Scale
Y-axis: 1unit = 10db
x-axis: 1unit = 20°
A in db
ω → in rad/sec
SHARP

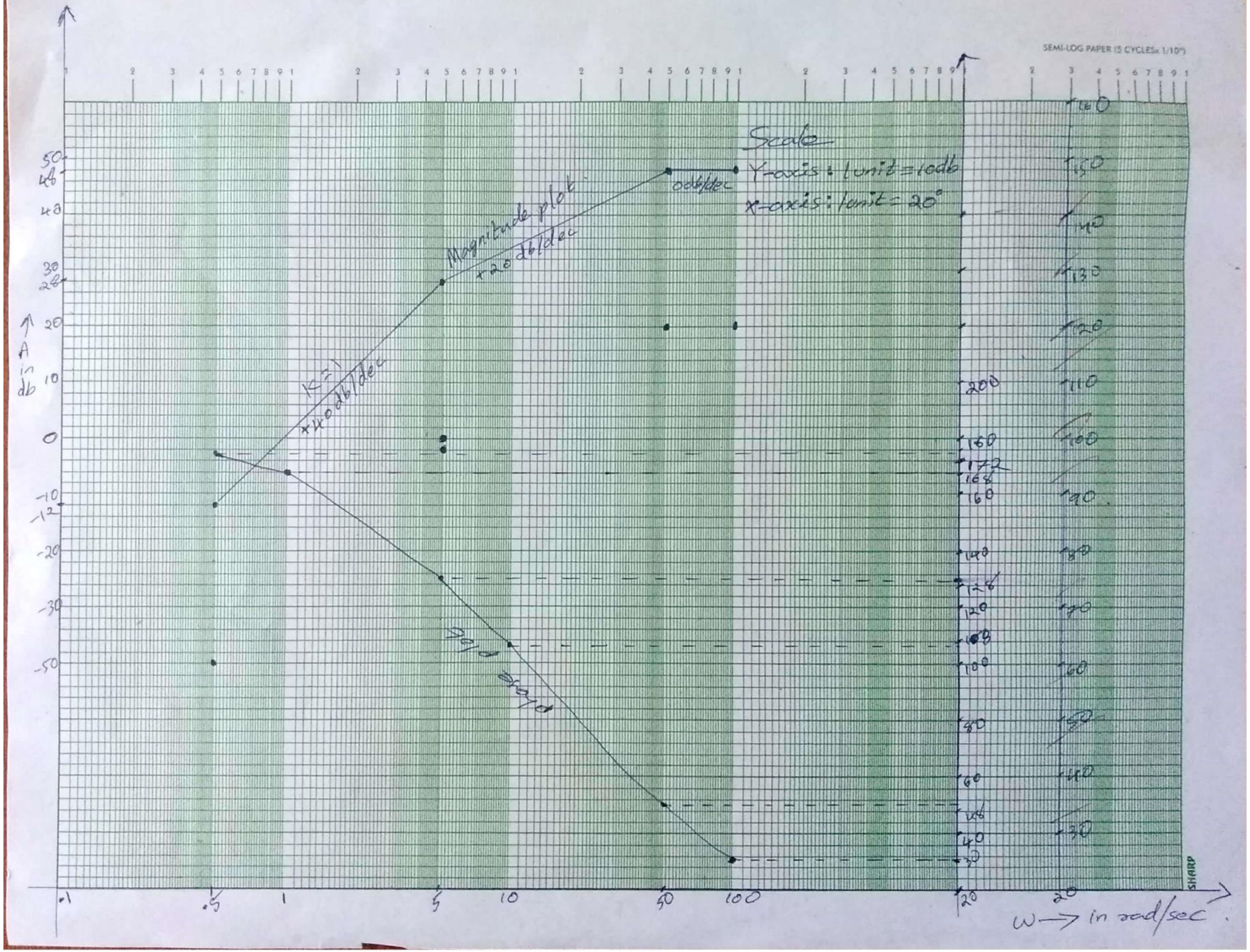

SEMI-LOG PAPER (5 CYCLES× 1/10")
Scale
Y-axis: 1 unit = 10db
x-axis: 1 unit = 20°
A in db
ω → in rad/sec
SHARP

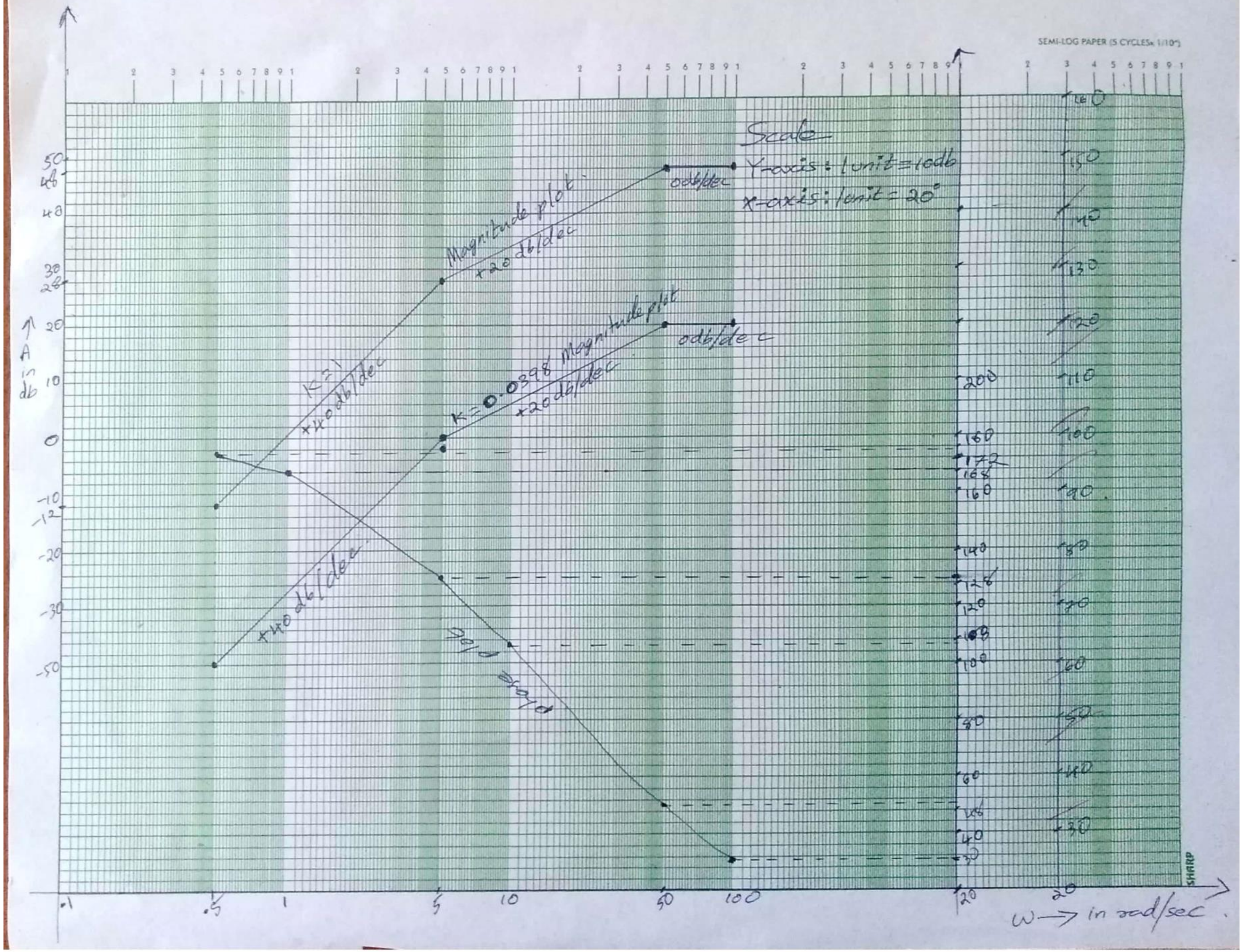


SEMI-LOG PAPER (5 CYCLES× 1/10")
Scale
Y-axis: 1unit = 10db
X-axis: 1unit = 20°
A in db
K=1
+40db/dec
+20db/dec
0db/dec
ω → in rad/sec
SHARP


SEMI-LOG PAPER (5 CYCLES× 1/10")
Scale
Y-axis: 1unit=10db
x-axis: 1unit=20°
Magnitude plot
K=1
+40db/dec
+20db/dec
0db/dec
A in db
ω → in rad/sec
SHARP


SEMI-LOG PAPER (5 CYCLES× 1/10")
Scale
Y-axis : 1unit = 10db
x-axis : 1unit = 20°
Magnitude plot
+20db/dec
0db/dec
K=1
+40db/dec
Phase plot
A in db
ω → in rad/sec


SEMI-LOG PAPER (5 CYCLES x 1/10")
Scale
Y-axis : 1unit = 10db
x-axis : 1unit = 20°
Magnitude plot
K=1
+40db/dec
+20db/dec
0db/dec
phase plot
A in db
ω → in rad/sec
SHARP


SEMI-LOG PAPER (5 CYCLES x 1/10")
Scale
Y-axis : 1 unit = 10db
x-axis : 1 unit = 20°
Magnitude plot
+20db/dec
0db/dec
K=1
+40db/dec
K = 0.0398 Magnitude plot
+20db/dec
0db/dec
+40db/dec
Phase plot
A in db
ω → in rad/sec
SHARP

CALCULATION OF K

Given that gain cross over frequency is 5rad/sec
At w=5rad/sec the gain is 28db.
If gain crossover frequency is 5rad/sec then at that frequency the db gain should be zero.
Hence to every point of magnitude plot a db gain of -28db should be added.
The addition of -28db shifts the plot downwards.
The magnitude correction is independent of frequency.

$$20 \log K = -28 \text{ db}$$

$$\log K = \frac{-28}{20}$$

$$K = 10^{-\left(\frac{28}{20}\right)} = 0.0398$$

Sketch the bode plot for the following transfer function and determine phase margin and gain margin

$$G(s) = \frac{75\,(1+0.2s)}{s\left(s^2+16s+100\right)}$$

comparing the quadratic factor in the denominator of G(s) with standard form of quadratic factor

$$s^2+16s+100 \;=\; s^2+2\zeta\omega_n s+\omega_n^2$$

On comparing

$$\omega_n^2 = 100 \quad \Rightarrow \quad \omega_n = 10$$

$$2\zeta\omega_n = 16 \quad \Rightarrow \quad \zeta = \frac{16}{2\omega_n} = \frac{16}{2\times 10} = 0.8$$

convert the given s-domain transfer function into bode form or time constant form.

convert the given s-domain transfer function into bode form or time constant form.

$$G(s) = \frac{75(1+0.2s)}{s\left(s^2+16s+100\right)} = \frac{75\,(1+0.2s)}{s \times 100\left(\dfrac{s^2}{100}+\dfrac{16s}{100}+1\right)} = \frac{0.75\,(1+0.2s)}{s\left(1+0.01s^2+0.16s\right)}$$

$$G(j\omega) = \frac{0.75\,(1+0.2\,j\omega)}{j\omega\left(1+0.01(j\omega)^2+0.16j\omega\right)} = \frac{0.75\,(1+j0.2\omega)}{j\omega\left(1-0.01\omega^2+j0.16\omega\right)}$$

MAGNITUDE PLOT

The corner frequencies are, $\omega_{c1} = \frac{1}{0.2} = 5$ rad / sec and $\omega_{c2} = \omega_n = 10$ rad / sec

For quadratic factor the frequency ω_n is the corner frequency ω_c.

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$\frac{0.75}{j\omega}$	–	–20	
$1 + j0.2\omega$	$\omega_{c1} = \frac{1}{0.2} = 5$	20	$-20 + 20 = 0$
$\frac{1}{1 - 0.01\omega^2 + j0.16\omega}$	$\omega_{c2} = \omega_n = 10$	– 40	$0 - 40 = -40$

find Lower frequency and higher frequencies $\omega_l < \omega_c$ and $\omega_h > \omega_c$

$\omega_l = 0.5$ rad/sec and $\omega_h = 20$ rad/sec. $A = |G(j\omega)|$ in db

Let us calculate A at ω_l, ω_{c1}, ω_{c2} and ω_h.

$$\text{At, } \quad \omega = \omega_l, \quad A = 20\log\left|\frac{0.75}{j\omega}\right| = 20\log\frac{0.75}{0.5} = 3.5 \text{ db}$$

$$\text{At, } \quad \omega = \omega_{c1}, \quad A = 20\log\left|\frac{0.75}{j\omega}\right| = 20\log\frac{0.75}{5} = -16.5 \text{ db}$$

$$\text{At, } \quad \omega = \omega_{c2}, \quad A = \left[\text{slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c1}}\right] + A_{(\text{at } \omega = \omega_{c1})}$$

$$= 0 \times \log\frac{10}{5} + (-16.5) = -16.5 \text{ db}$$

$$\text{At } \omega = \omega_h, \quad A = \left[\text{slope from } \omega_{c2} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c2}}\right] + A_{(\text{at } \omega = \omega_{c2})}$$

$$= -40 \times \log\frac{20}{10} + (-16.5) = -28.5 \text{ db}$$

PHASE PLOT

The phase angle of $G(j\omega)$ as a function of ω is given by,

$$\phi = \angle G(j\omega) = \tan^{-1}0.2\omega - 90^\circ - \tan^{-1}\frac{0.16\omega}{1-0.01\omega^2} \quad \text{for } \omega \leq \omega_n$$

$$\phi = \angle G(j\omega) = \tan^{-1}0.2\omega - 90^\circ - \left(\tan^{-1}\frac{0.16\omega}{1-0.01\omega^2} + 180^\circ\right) \text{ for } \omega > \omega_n$$

Note : *In quadratic factors the phase varies from 0° to 180°. But calculator calculates tan^{-1} only between 0° to 90°. Hence a correction of 180° should be added to phase after ω_n.*

ω rad/sec	$\tan^{-1}0.2\,\omega$ deg	$\tan^{-1}\dfrac{0.16\omega}{1-0.01\omega^2}$ deg	$\phi = \angle G(j\omega)$ deg	Points in phase plot
0.5	5.7	4.6	$-88.9 \approx -88$	e
1	11.3	9.2	$-87.9 \approx -88$	f
5	45	46.8	$-91.8 \approx -92$	g
10	63.4	90	$-116.6 \approx -116$	h
20	75.9	$-46.8+180=133.2$	$-147.3 \approx -148$	i
50	84.3	$-18.4+180=161.6$	$-167.3 \approx -168$	j
100	87.1	$-92+180=170.8$	$-173.7 \approx -174$	k

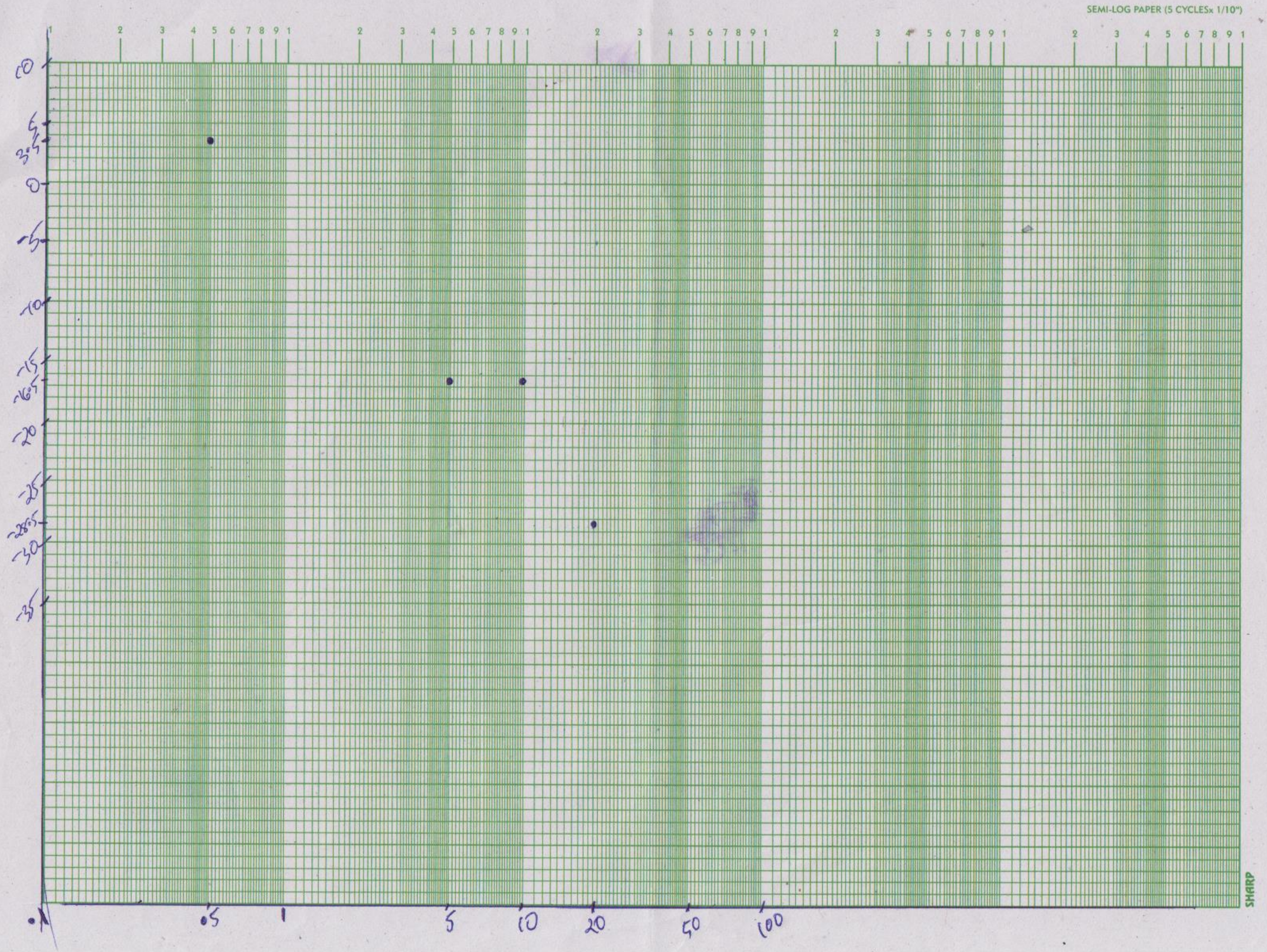
Let ϕ_{gc} be the phase of $G(j\omega)$ at gain cross-over frequency, ω_{gc}.

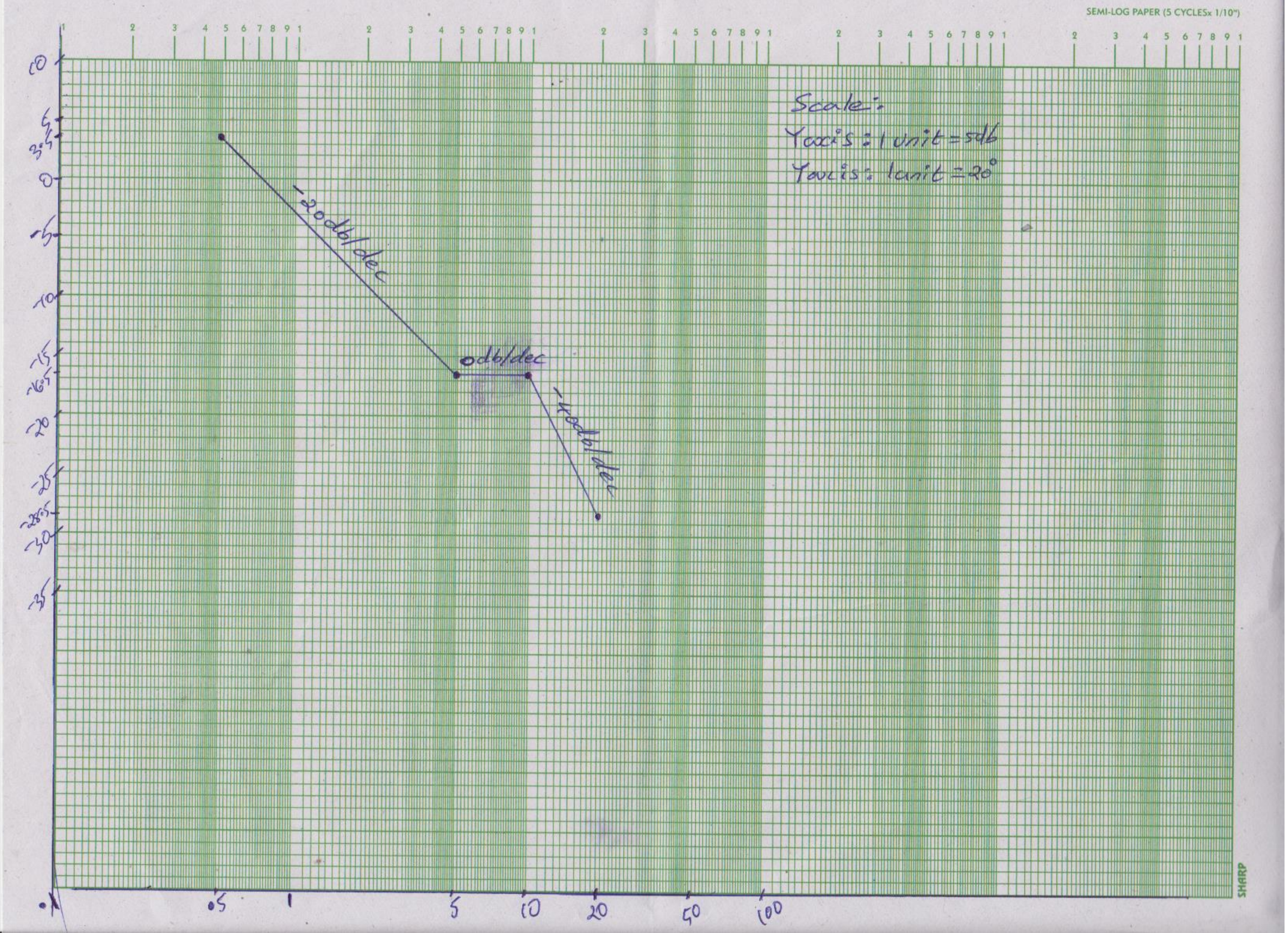
$$\phi_{gc} = -88°$$

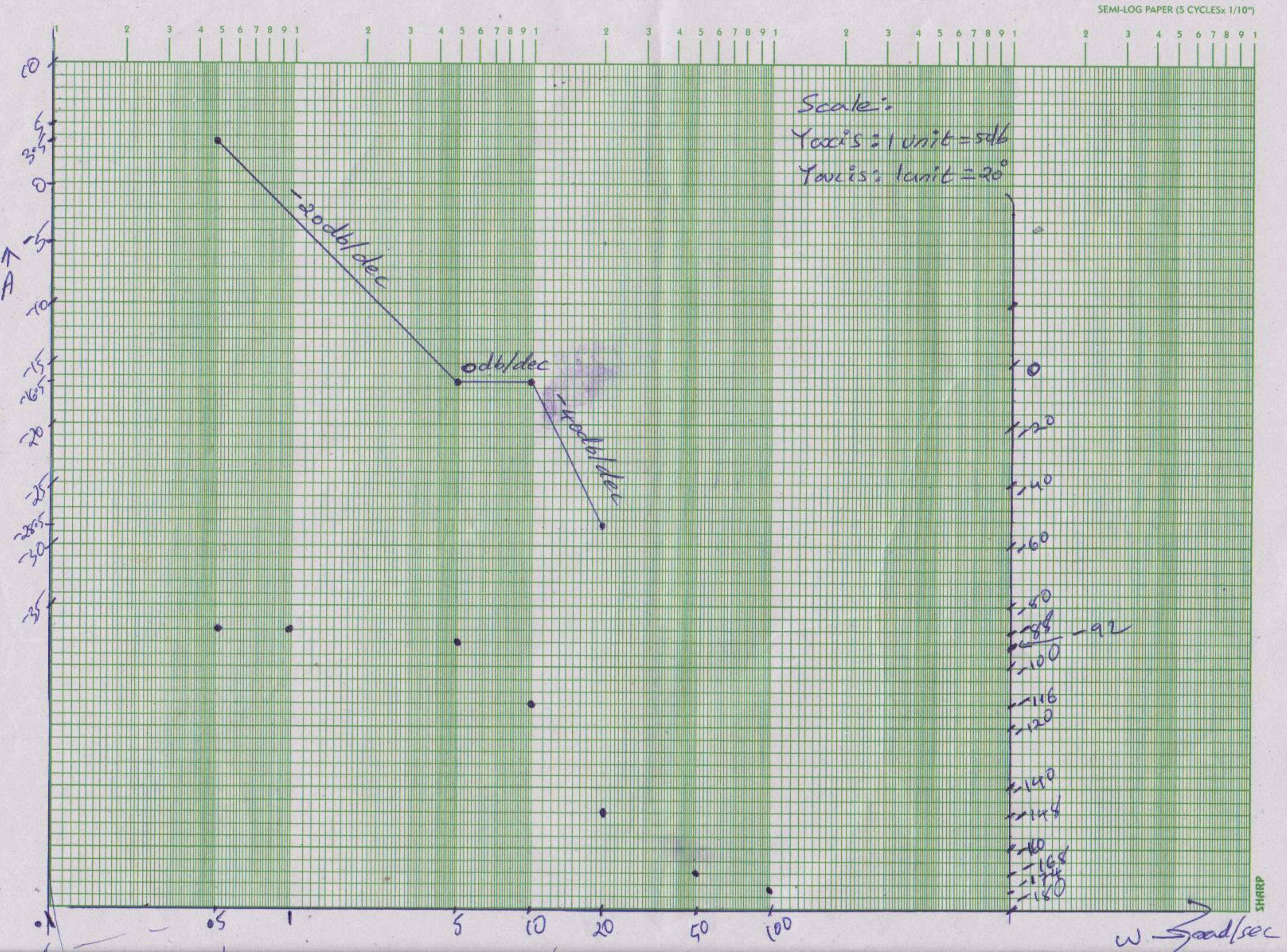
$$\therefore \text{Phase margin, } \gamma = 180° + \phi_{gc} = 180° - 88° = 92°$$

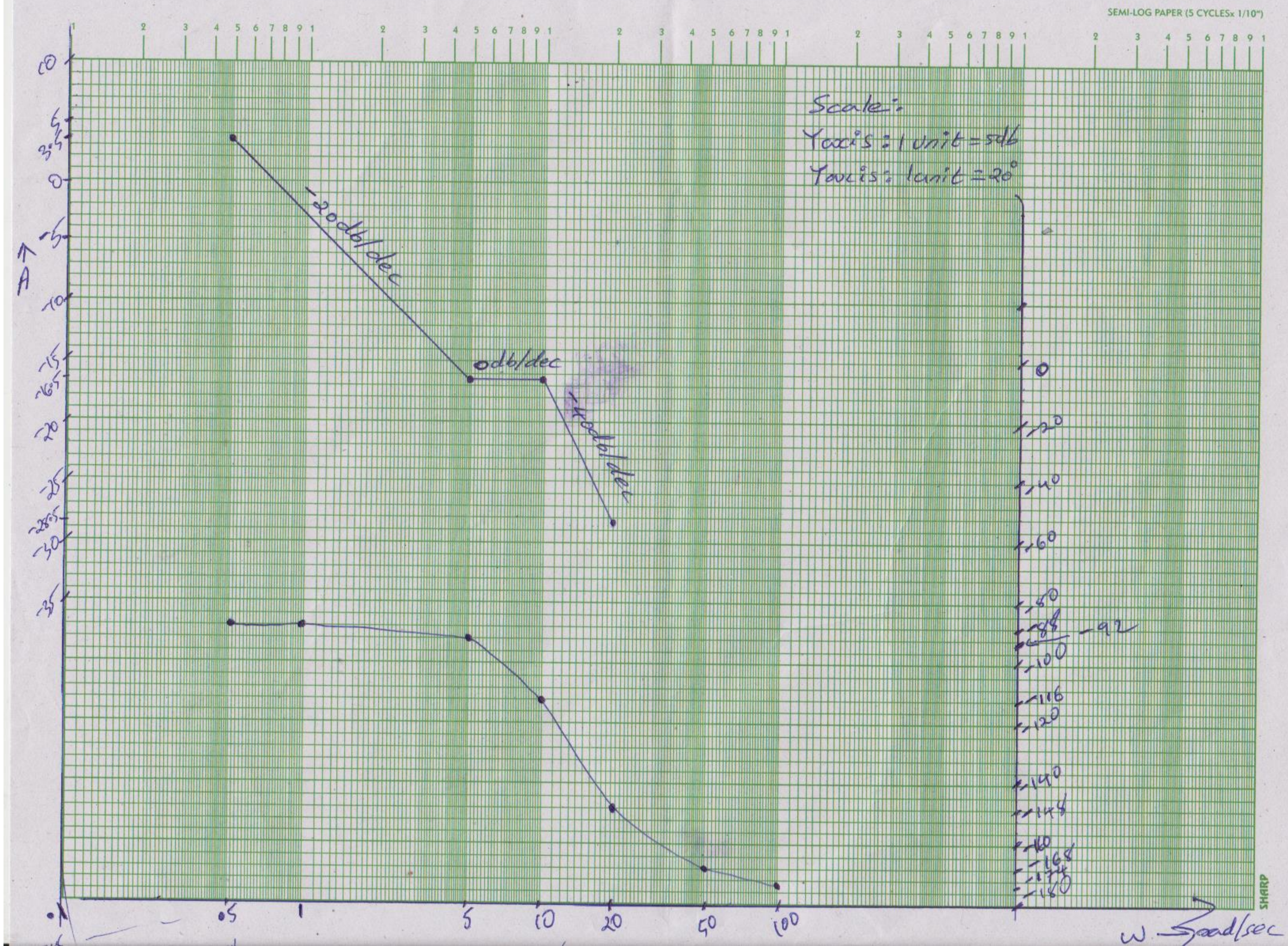
The phase plot crosses $-180°$ only at infinity. The $|G(j\omega)|$ at infinity is $-\infty$ db.

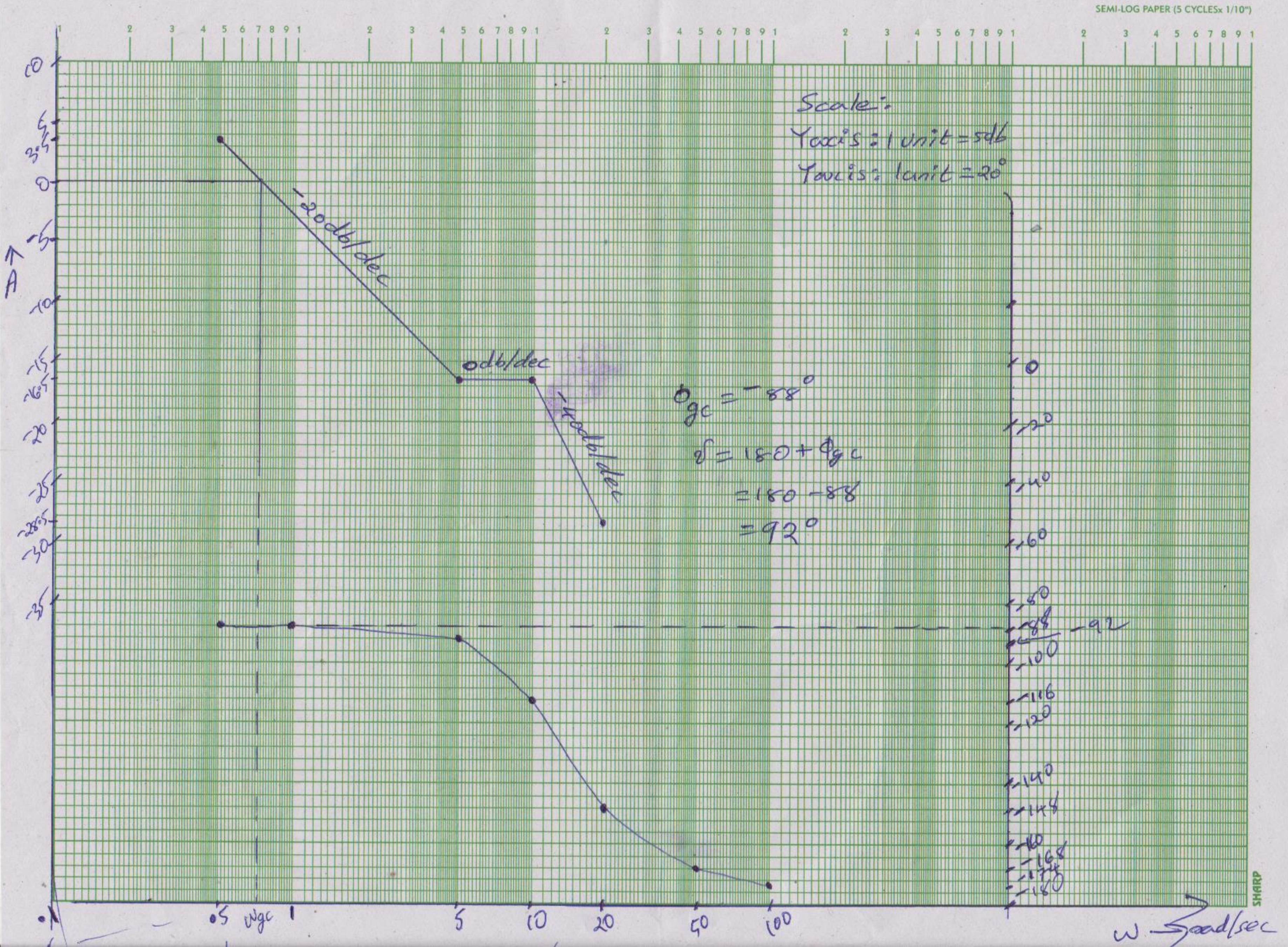
Hence gain margin is $+\infty$.


SEMI-LOG PAPER (5 CYCLESx 1/10")
1 2 3 4 5 6 7 8 9 1
10
5
3.5
0
-5
-10
-15
-16.5
-20
-25
-28.5
-30
-35
.1
.5
1
5
10
20
50
100
SHARP


SEMI-LOG PAPER (5 CYCLESx 1/10")
Scale:
Yaxis: 1 unit = 5db
Yaxis: 1unit = 20°
-20db/dec
0db/dec
-40db/dec
SHARP


SEMI-LOG PAPER (5 CYCLESx 1/10")
Scale:-
Yaxis : 1 unit = 5db
Yaxis: 1unit = 20°
-20db/dec
0db/dec
-40db/dec
A
-92
ω → rad/sec
SHARP


SEMI-LOG PAPER (5 CYCLESx 1/10")
Scale:
Yaxis: 1 unit = 5db
Yaxis: 1unit = 20°
-20db/dec
0db/dec
-40db/dec
-92
A
ω → rad/sec
SHARP


SEMI-LOG PAPER (5 CYCLESx 1/10")
Scale:-
Y axis : 1 unit = 5db
Y axis: 1 unit = 20°
-20db/dec
0db/dec
-40db/dec
Φgc = -88°
γ = 180 + Φgc
= 180 - 88
= 92°
A
ωgc
ω rad/sec
-92
SHARP

Given, $G(s) = \dfrac{K e^{-0.2s}}{s(s+2)(s+8)}$. Find K so that the system is stable with,

(a) gain margin equal to 2db, (b) phase margin equal to 45°.

$K = 1$

time constant form or bode form.

$$G(s) = \frac{e^{-0.2s}}{s(s+2)(s+8)} = \frac{e^{-0.2s}}{s \times 2\left(1+\frac{s}{2}\right) \times 8\left(1+\frac{s}{8}\right)} = \frac{0.0625\ e^{-0.2s}}{s(1+0.5s)(1+0.125s)}$$

$$G(j\omega) = \frac{0.0625\ e^{-j0.2\omega}}{j\omega\ (1+j0.5\omega)\ (1+j0.125\omega)}$$

$$\left|0.0625\, e^{-j0.2\omega}\right| = 0.0625 \quad \text{and} \quad \angle\left(0.0625 e^{-j0.2\omega}\right) = -0.2\omega \text{ radians.}$$

$$0.0625\, e^{-j0.2\omega} = 0.0625 \angle -0.2\omega$$

$$a + ib = r < \theta = re^{i\theta}$$

$$r = \sqrt{(a2 + b2)} \qquad \theta = \tan^{-1}\frac{b}{a}$$

$$a = r\cos\theta \qquad b = r\sin\theta$$

MAGNITUDE PLOT

The corner frequencies are, $\omega_{c1} = \dfrac{1}{0.5} = 2 \text{ rad / sec}$ and $\omega_{c2} = \dfrac{1}{0.125} = 8 \text{ rad / sec}$

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$\frac{0.0625}{j\omega}$	–	−20	
$\frac{1}{1+j0.5\omega}$	$\omega_{c1} = \frac{1}{0.5} = 2$	−20	$-20-20=-40$
$\frac{1}{1+j0.125\omega}$	$\omega_{c2} = \frac{1}{0.125} = 8$	−20	$-40-20=-60$

$\omega_l = 0.5$ rad / sec and $\omega_h = 50$ rad / sec.

calculate A at $\omega_l, \omega_{c1}, \omega_{c2}$ and ω_h.

$$\text{At } \omega = \omega_l, \quad A = 20\log\left|\frac{0.0625}{j\omega}\right| = 20\log\frac{0.0625}{0.5} = -18\text{db}$$

$$\text{At } \omega = \omega_{c1}, \quad A = 20\log\left|\frac{0.0625}{j\omega}\right| = 20\log\frac{0.0625}{2} = -30\text{db}$$

$$\text{At } \omega = \omega_{c2}, \quad A = \left[\text{Slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c1}}\right] + A_{(\text{at } \omega = \omega_{c1})} = -40 \times \log\frac{8}{2} + (-30) = -54\text{db}$$

$$\text{At } \omega = \omega_h, \quad A = \left[\text{Slope from } \omega_{c2} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c2}}\right] + A_{(\text{at } \omega = \omega_{c2})} = -60 \times \log\frac{50}{8} + (-54) = -102\text{db}$$

PHASE PLOT

$$\phi = -0.2\omega \times \frac{180^\circ}{\pi} - 90^\circ - \tan^{-1}0.5\omega - \tan^{-1}0.125\omega$$

ω rad/sec	$-0.2\ \omega\ (180^0/\pi)$ deg	$\tan^{-1} 0.5\ \omega$ deg	$\tan^{-1} 0.125\ \omega$ deg	$\phi = \angle G(j\omega)$ deg	Point in phase plot
0.01	−0.1145	0.2864	0.0716	$-90.4 \approx -90$	e
0.1	−1.145	2.862	0.716	$-94.7 \approx -94$	f
0.5	−5.7	14	3.6	$-113.3 \approx -114$	g
1	−11.4	26	7.12	$-134.4 \approx -134$	h
2	−22.9	45	14	$-171.9 \approx -172$	i
3	−34.37	56.30	20.56	$-201.2 \approx -202$	j
4	−45.84	63.43	26.57	$-225.8 \approx -226$	k

CALCULATION OF K

Phase margin, $\gamma = 180^0 + \phi_{gc}$, where ϕ_{gc} is the phase of $G(j\omega)$ at $\omega = \omega_{gc}$.

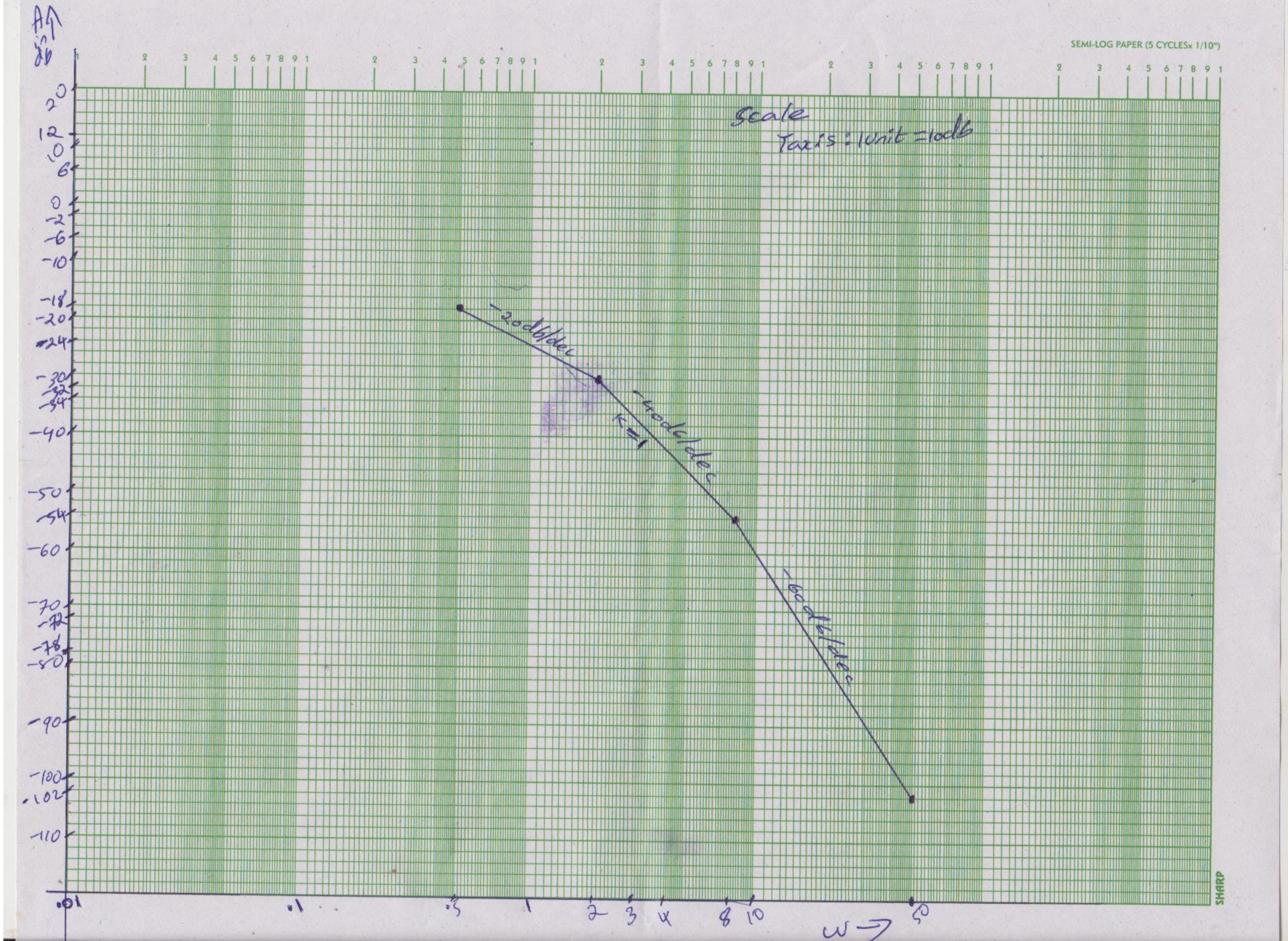
When $\gamma = 45°$, $\phi_{gc} = \gamma - 180^0 = 45^0 - 180^0 = -135°$.

With K =1, the db gain at =-135 degree is -24db.
This gain should be made zero to have to PM of 45degree.
Hence 24 degree should be added to every point.

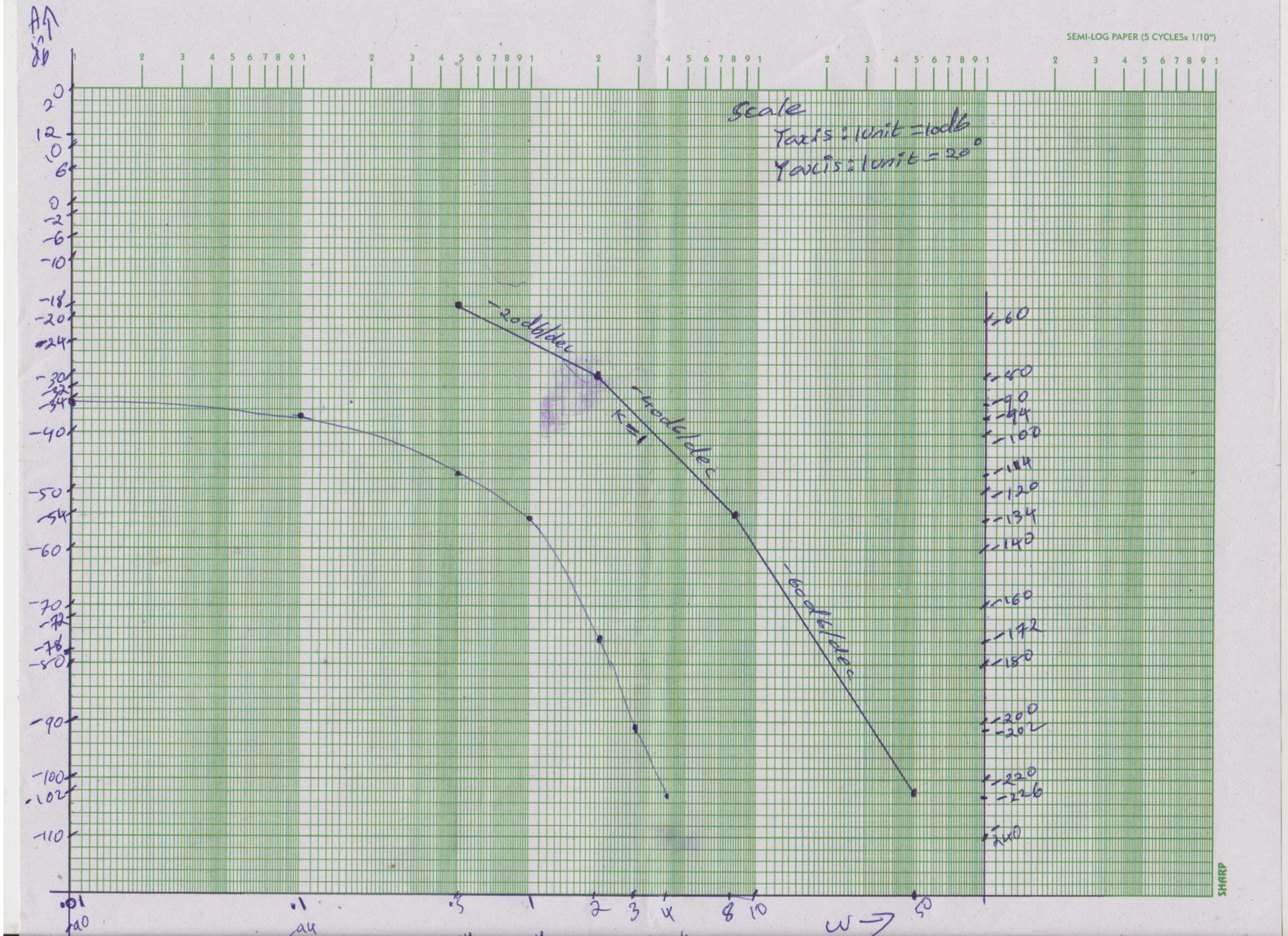
$$20 \log K = 24 \quad ; \quad K = 10^{24/20} \quad ; \quad K = 15.84$$

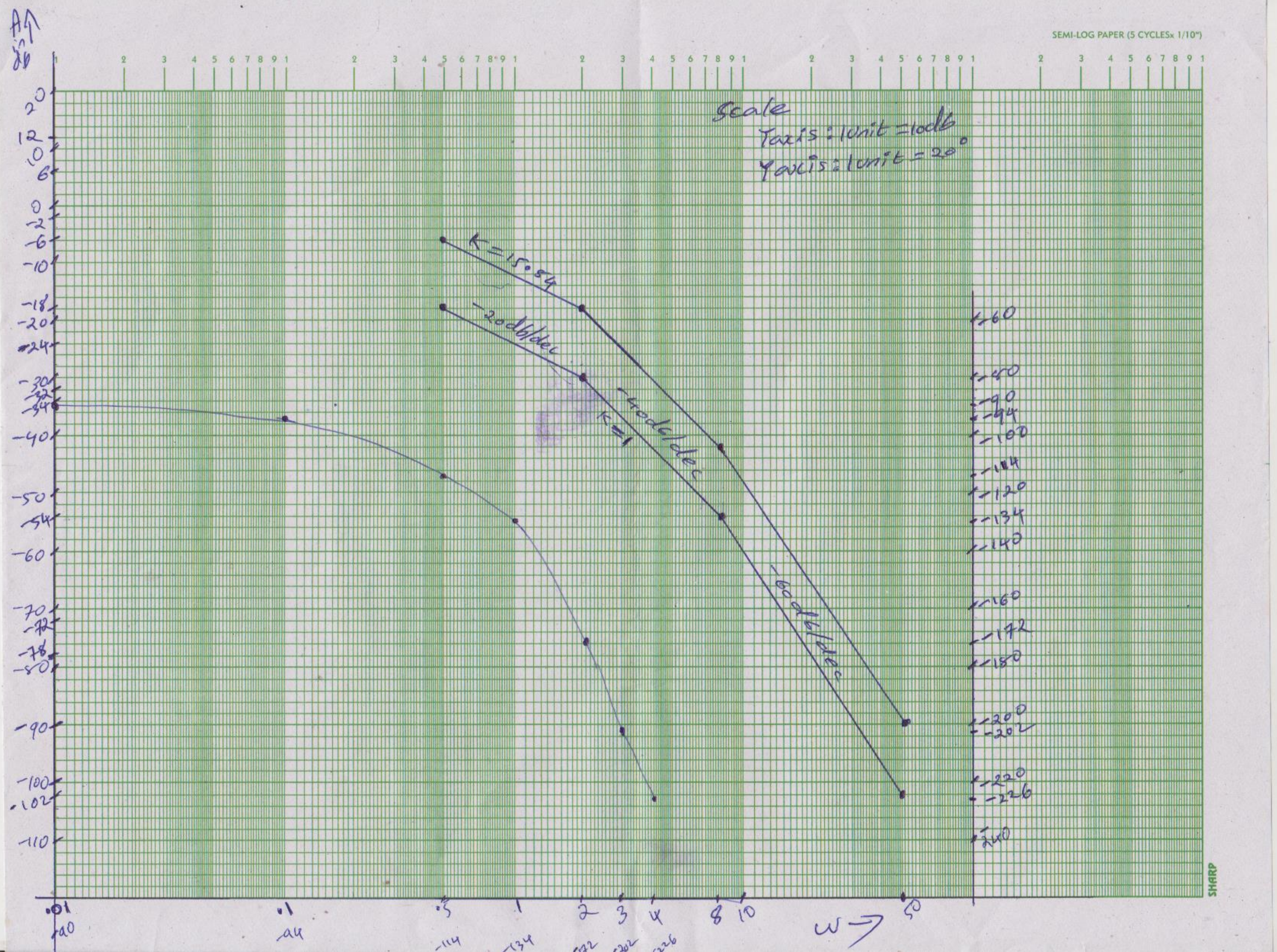
With K =1 the gain margin=-(-32)=32db.
Required gain margin is 2 db.
Hence 30 db should be added to every point of magnitude plot.

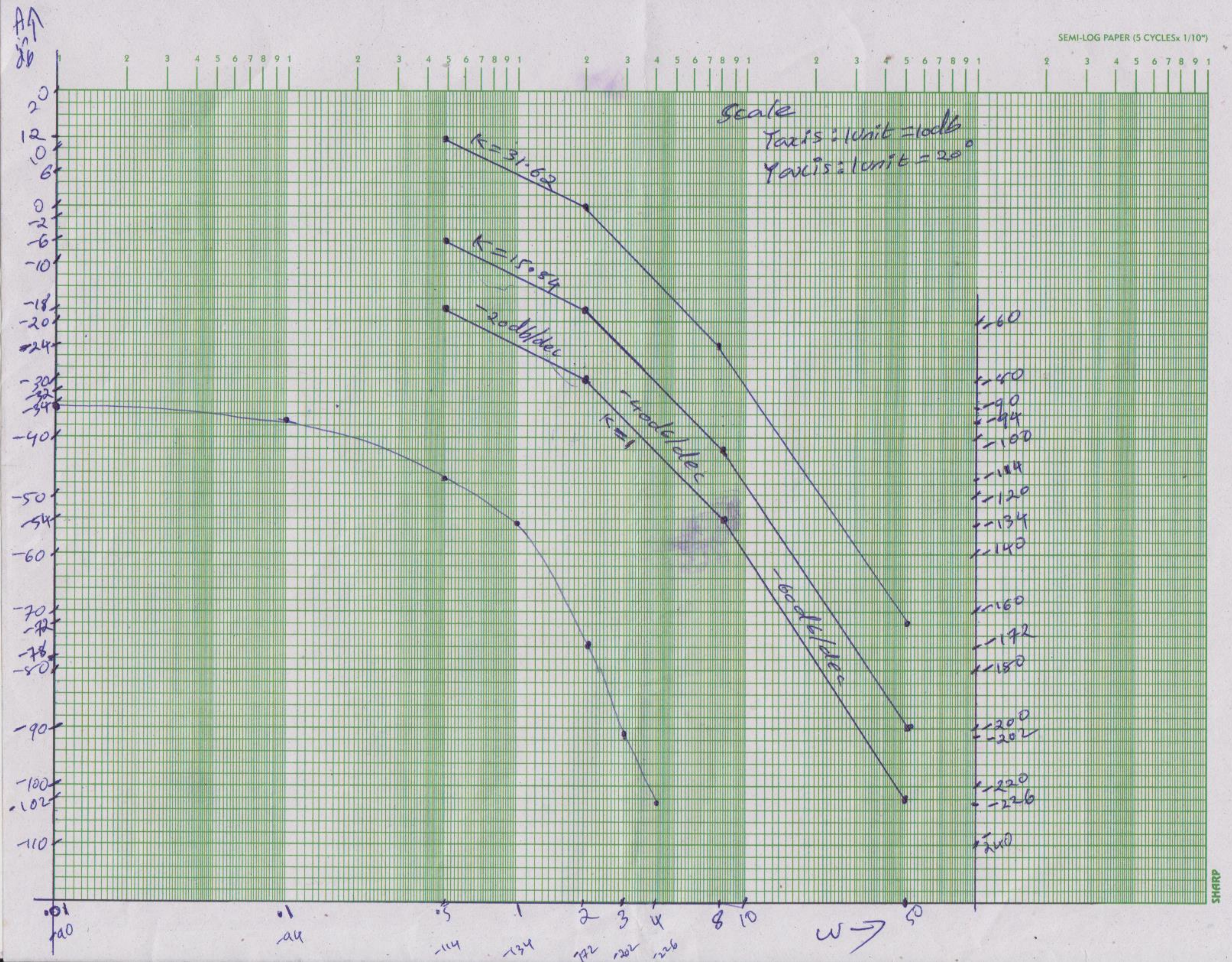
$$20 \log K = 30 \quad ; \quad K = 10^{30/20} \quad ; \quad K = 31.62$$



SEMI-LOG PAPER (5 CYCLESx 1/10")
Scale
Y axis : 1 unit = 10dB
-20dB/dec
-40dB/dec
K=1
-60dB/dec
ω →
SHARP


SEMI-LOG PAPER (5 CYCLES x 1/10")
SHARP
A in db
Scale
X axis : 1 unit = 10db
Y axis : 1 unit = 20°
-20db/dec
-40db/dec
-60db/dec
K=1
ω →


SEMI-LOG PAPER (5 CYCLESx 1/10")
A in db
Scale
X axis: 1unit = 10db
Y axis: 1unit = 20°
K=15.84
-20db/dec
-40db/dec
K=1
-60db/dec
ω →
SHARP



SEMI-LOG PAPER (5 CYCLES x 1/10")
Scale
X axis: 1 unit = 10 db
Y axis: 1 unit = 20°
K=31.62
K=15.84
-20db/dec
-40db/dec
K=1
-60db/dec
A in db
ω →
SHARP

Plot the Bode diagram for the following transfer function and obtain the gain and phase cross over frequencies.

$$G(s) = \frac{10}{s\,(1+0.4s)\,(1+0.1s)}$$

$$G(j\omega) = \frac{10}{j\omega\,(1+j0.4\omega)\,(1+j0.1\omega)}$$

MAGNITUDE PLOT

$$\omega_{c1} = \frac{1}{0.4} = 2.5 \text{ rad / sec} \quad \text{and} \quad \omega_{c2} = \frac{1}{0.1} = 10 \text{ rad / sec}$$

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$\frac{10}{j\omega}$	–	– 20	
$\frac{1}{1+j0.4\omega}$	$\omega_{c1} = \frac{1}{0.4} = 2.5$	– 20	–20 –20 = – 40
$\frac{1}{1+j0.1\omega}$	$\omega_{c2} = \frac{1}{0.1} = 10$	– 20	– 40 – 20 = – 60

$\omega_l = 0.1$ rad/sec, and $\omega_h = 50$ rad/sec.

Let us calculate A at ω_l, ω_{c1}, ω_{c2} and ω_h.

$$\text{At } \omega = \omega_l, \quad A = 20\log\left|\frac{10}{j\omega}\right| = 20\log\frac{10}{0.1} = 40 \text{ db}$$

$$\text{At } \omega = \omega_{c1}, \quad A = 20\log\left|\frac{10}{j\omega}\right| = 20\log\frac{10}{2.5} = 12 \text{ db}$$

$$\text{At } \omega = \omega_{c2}, \quad A = \left[\text{Slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c1}}\right] + A_{(\text{at } \omega = \omega_{c1})} = -40 \times \log\frac{10}{2.5} + 12 = -12 \text{ db}$$

$$\text{At } \omega = \omega_h, \quad A = \left[\text{Slope from } \omega_{c2} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c2}}\right] + A_{(\text{at } \omega = \omega_{c2})} = -60 \times \log\frac{50}{10} + (-12) = -54 \text{ db}$$

PHASE PLOT

$$\phi = -90° - \tan^{-1} 0.4\omega - \tan^{-1} 0.1\omega$$

ω rad/sec	$\tan^{-1} 0.4\ \omega$ deg	$\tan^{-1} 0.1\ \omega$ deg	$\phi = \angle G(j\omega)$ deg	Points in phase plot
0.1	2.29	0.57	$-92.86 \approx -92$	e
1	21.80	5.71	$-117.5 \approx -118$	f
2.5	45.0	14.0	$-149 \approx -150$	g
4	57.99	21.8	$-169.79 \approx -170$	h
10	75.96	45.0	$-210.96 \approx -210$	i
20	82.87	63.43	$-236.3 \approx 236$	j

RESULT

Gain cross-over frequency = 5 rad/sec.

Phase cross-over frequency = 5 rad/sec.

For the function, $G(s) = \frac{5(1+2s)}{(1+4s)(1+0.25s)}$, draw the bode plot.

$$G(j\omega) = \frac{5(1+j2\omega)}{(1+j4\omega)(1+j0.25\omega)}$$

$$\omega_{c1} = \frac{1}{4} = 0.25 \text{ rad / sec}, \quad \omega_{c2} = \frac{1}{2} = 0.5 \text{ rad / sec}, \omega_{c3} = \frac{1}{0.25} = 4 \text{ rad / sec}$$

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/deg
5	–	0	–
$\frac{1}{1+j4\omega}$	$\omega_{c1} = \frac{1}{4} = 0.25$	–20	$0 - 20 = -20$
$1 + j2\omega$	$\omega_{c2} = \frac{1}{2} = 0.5$	20	$-20 + 20 = 0$
$\frac{1}{1+j0.25\omega}$	$\omega_{c3} = \frac{1}{0.25} = 4$	–20	$0 - 20 = -20$

At $\omega = \omega_l$, $A = |G(j\omega)| = 20 \log 5 = +14$ db

At $\omega = \omega_{c1}$, $A = |G(j\omega)| = 20 \log 5 = +14$ db

$$\text{At } \omega = \omega_{c2},\ A = \left[\text{Slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log \frac{\omega_{c2}}{\omega_{c1}}\right] + A_{(\text{at } \omega = \omega_{c1})} = -20 \times \log \frac{0.5}{0.25} + 14 = +8\text{db}$$

$$\text{At } \omega = \omega_{c3},\ A = \left[\text{Slope from } \omega_{c2} \text{ to } \omega_{c3} \times \log \frac{\omega_{c3}}{\omega_{c2}}\right] + A_{(\text{at } \omega = \omega_{c2})} = 0 \times \log \frac{4}{0.5} + 8 = +8\text{db}$$

$$\text{At } \omega = \omega_h,\ A = \left[\text{Slope from } \omega_{c3} \text{ to } \omega_h \times \log \frac{\omega_h}{\omega_{c3}}\right] + A_{(\text{at } \omega = \omega_{c3})} = -20 \log \frac{10}{4} + 8 = 0 \text{ db}$$

PHASE PLOT

$$\phi = \tan^{-1}(2\omega) - \tan^{-1}(4\omega) - \tan^{-1}(0.25\omega)$$

ω	$\tan^{-1} 2\omega$ deg	$\tan^{-1} 4\omega$ deg	$\tan^{-1} 0.25\omega$ deg	$\phi = \angle G(j\omega)$	Points in phase plot
0.1	11.3	21.8	1.43	$-11.93 \approx -12$	f
0.25	26.56	45.0	3.5	$-21.94 \approx -22$	g
0.5	45.0	63.43	7.1	$-25.53 \approx -26$	h
2	75.96	82.87	26.56	$-33.47 \approx -33$	i
4	82.87	86.42	45.0	$-48.55 \approx -49$	j
10	87.13	88.56	68.19	$-69.62 \approx -70$	k
50	89.42	89.71	85.42	$-85.71 \approx -86$	l

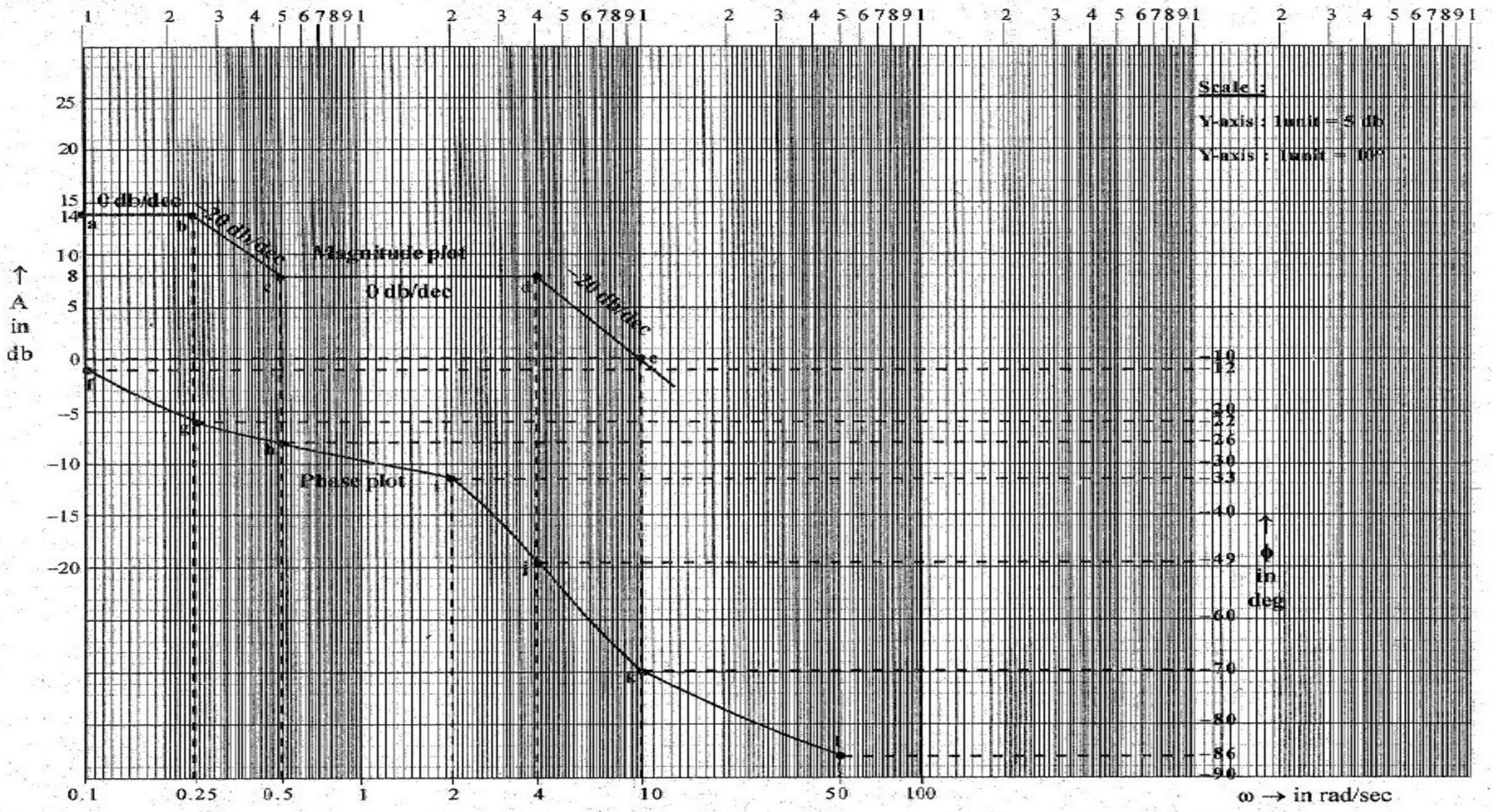


Fig 3.6.1 : Bode plot of transfer function, $G(j\omega) = \frac{5(1+j2\omega)}{(1+j4\omega)(1+j0.25\omega)}$.

NICHOLS PLOT

The Nichols plot is a frequency response plot of the open loop transfer function of a system. The Nichols plot is a graph between magnitude of $G(j\omega)$ in db and the phase of $G(j\omega)$ in degree, plotted on a ordinary graph sheet.

Steps

Consider open loop transfer function $G(j\omega)$ of the given system
Obtain the expression for $|G(j\omega)|$ in terms of ω
Obtain the expression for $\angle G(j\omega)$ in terms of ω
Tabulate the values of magnitude expressed in db and angle in degree for various values of ω
Select suitable scale on an ordinary graph paper with Y-axis representing magnitude in db and X-axis representing phase angle in degrees
Plot all the points tabulated on the graph paper.
The smooth curve obtained by joining all such plotted points represents magnitude-phase plot of a given system.

The gain margin in db is given by the negative of magnitude of $G(j\omega)$ at the phase cross over frequency ω_{pc}. The ω_{pc} is the frequency at at which phase of $G(j\omega)$ is -180 degree.

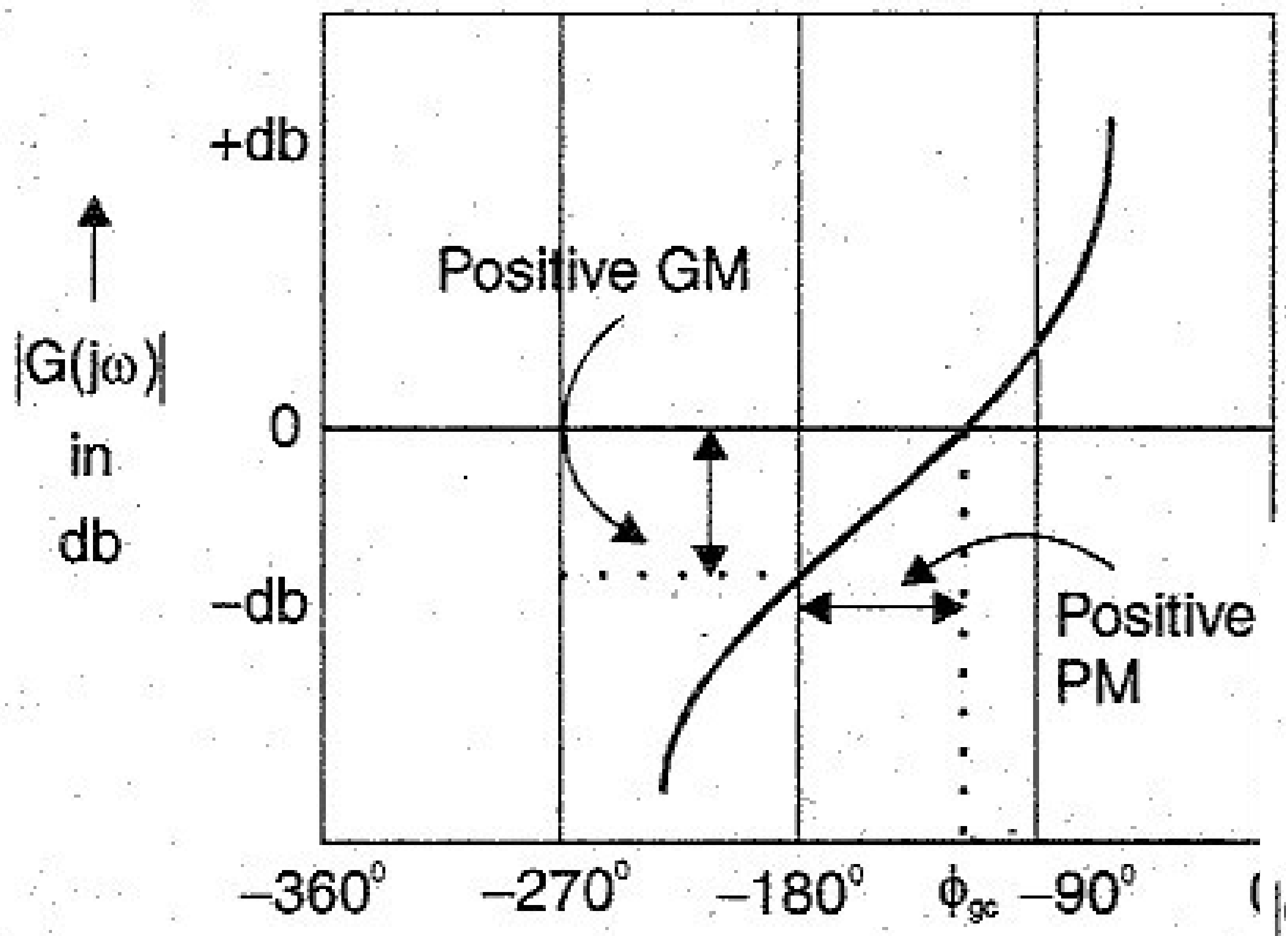
If the db magnitude of $G(j\omega)$ at ω_{pc} is negative then gain margin is positive and vice versa

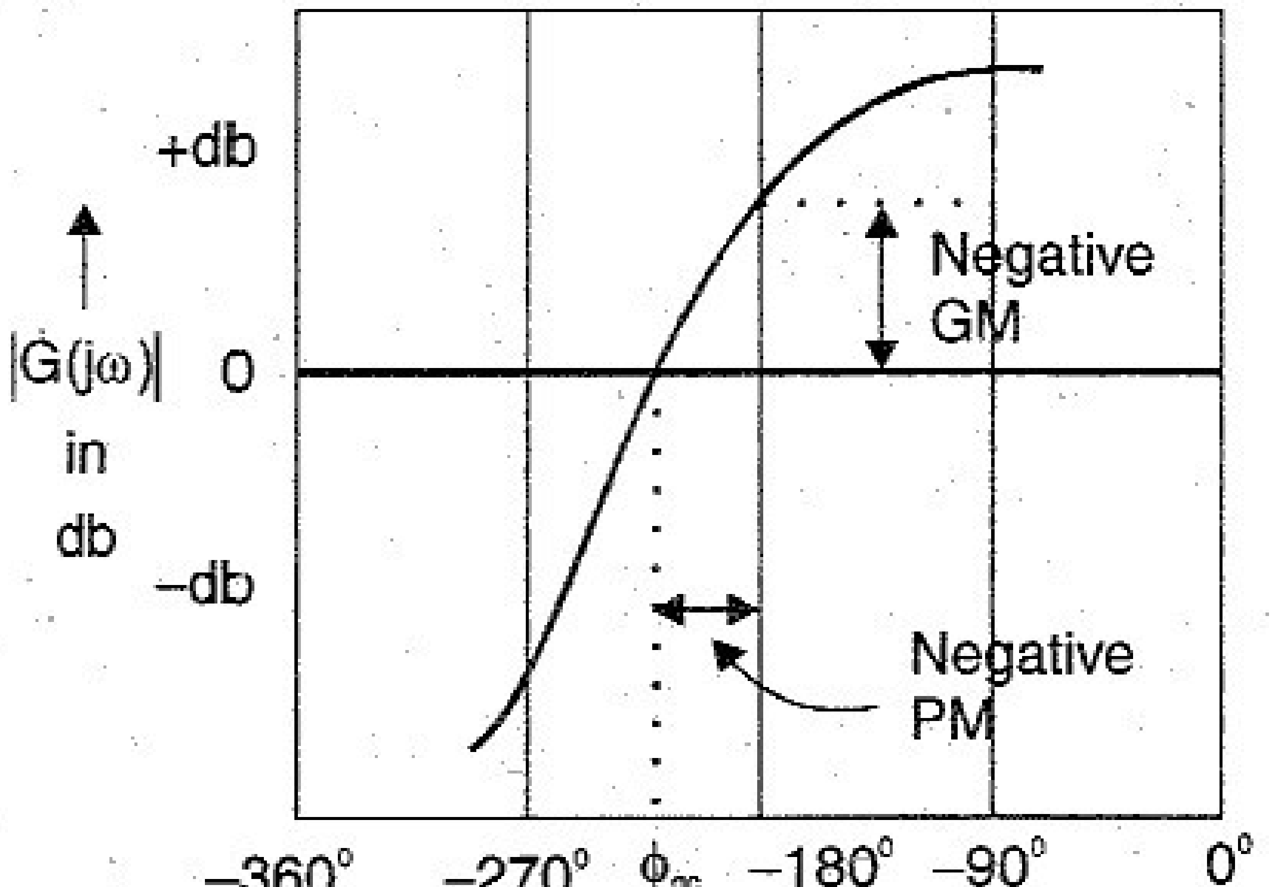
Let ϕ_{gc} be the phase angle of $G(j\omega)$ at gain cross over frequency ω_{gc}.

The ω_{gc} is the frequency at which the db magnitude of $G(j\omega)$ is zero.

Now the phase margin γ is given by $\gamma = 180° + \phi_{gc}$.

If ϕ_{gc} is less negative than -180 degree then phase margin is positive and vice versa.


+db
Positive GM
|G(jω)|
in
db
0
–db
Positive
PM
–360°
–270°
–180°
ϕ_{gc}
–90°


+db
Negative
GM
|G(jω)|
in
db
0
–db
Negative
PM
–360°
–270°
ϕ_{gc}
–180°
–90°
0°

Consider a unity feedback system having an open loop transfer function $G(s)=\dfrac{K(1+10s)}{s^2\,(1+s)\,(1+2s)}$

Sketch the Nichols plot and determine the value of K so that (i) Gain margin is 10db, (ii) Phase margin is 10°.

$$G(s)=\frac{K(1+10s)}{s^2(1+s)(1+2s)}$$

put K = 1. $s=j\omega$

$$G(j\omega)=\frac{(1+j10\omega)}{(j\omega)^2(1+j\omega)(1+j2\omega)}$$

$$G(j\omega)=\frac{\sqrt{1+(10\omega)^2}\ \angle\tan^{-1}10\omega}{\omega^2\angle180^0\sqrt{1+\omega^2}\ \angle\tan^{-1}\omega\sqrt{1+(2\omega)^2}\ \angle\tan^{-1}2\omega}$$

$$|G(j\omega)| = \frac{\sqrt{1+100\omega^2}}{\omega^2\sqrt{1+\omega^2}\ \sqrt{1+4\omega^2}}$$

$$|G(j\omega)|_{indb} = 20 \log \left[\frac{\sqrt{1+100\omega^2}}{\omega^2\sqrt{1+\omega^2}\ \sqrt{1+4\omega^2}} \right]$$

$$\angle G(j\omega) = \tan^{-1}10\omega - 180^0 - \tan^{-1}\omega - \tan^{-1}2\omega$$

ω rad/sec	0.2	0.4	0.6	0.8	1.0	1.5	2.0	3.0	4.0
$\lvert G(j\omega)\rvert$ db	34.1	25.4	19.3	14.3	10	1.4	–5.3	–15.2	–22.5
$\angle G(j\omega)$ deg	–150	–164	–181	–194	–204	–222	–232	–244	–250

when K=1

Gain margin $= -19.5$ db

Phase margin $= -45°$

Gain adjustment for required gain margin

The gain margin =10db which means magnitude of $G(j\omega)$ =-10db at ω_{pc}

When K=1 corresponding magnitude of $G(j\omega)$ is +19.5db at ω_{pc} .

Hence -29.5db should be added to every point of $G(j\omega)$

$$20 \log K_1 = -29.5 \text{ db} \quad \Rightarrow \quad \log K_1 = \frac{-29.5}{20} \quad \Rightarrow \quad K_1 = 10^{\frac{-29.5}{20}} = 0.0335$$

Gain adjustment for required phase margin

Phase margin, $\gamma_2 = 180° + \phi_{gc2}$

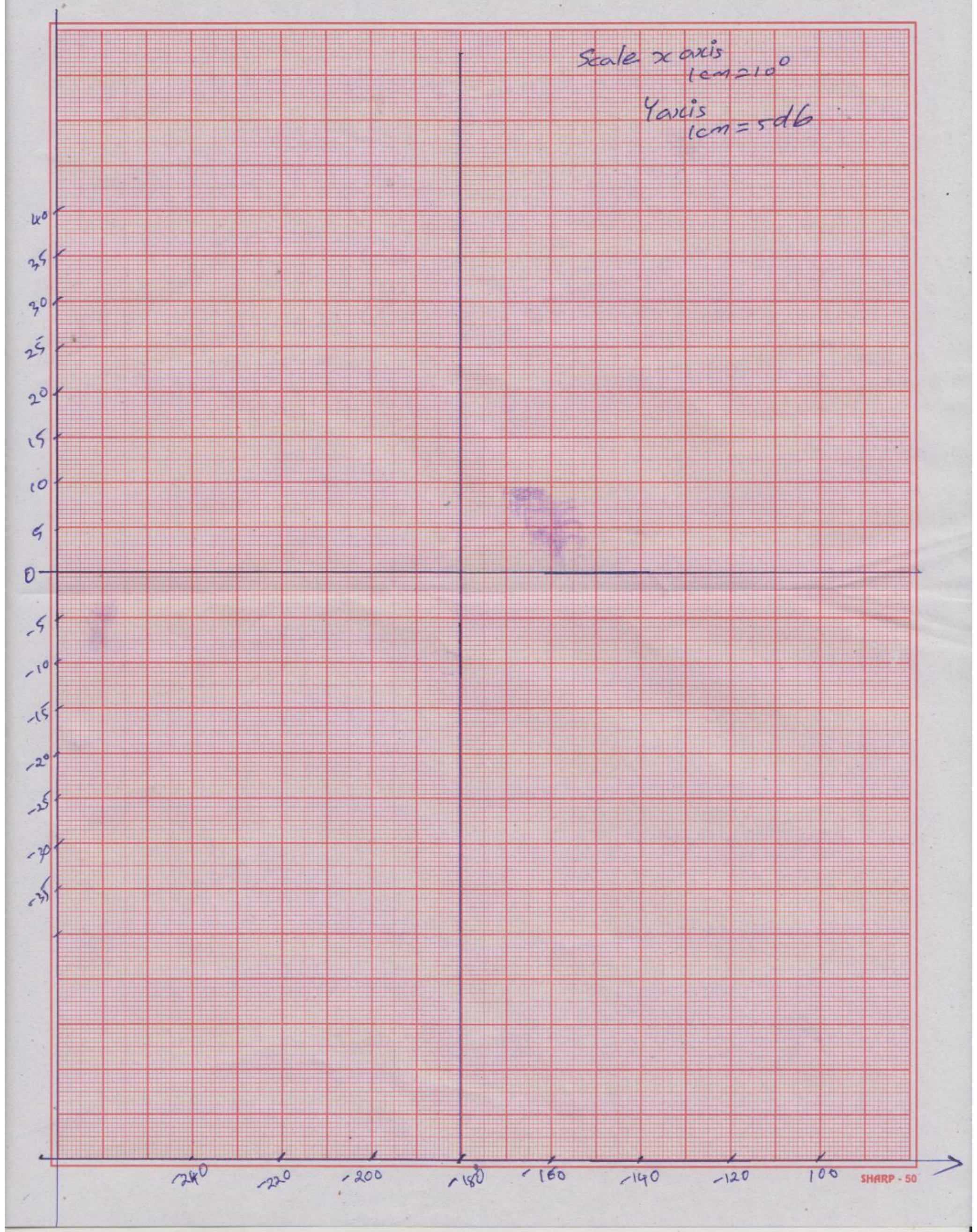
$\phi_{gc2} = \gamma_2 - 180° = 10° - 180° = -170°$

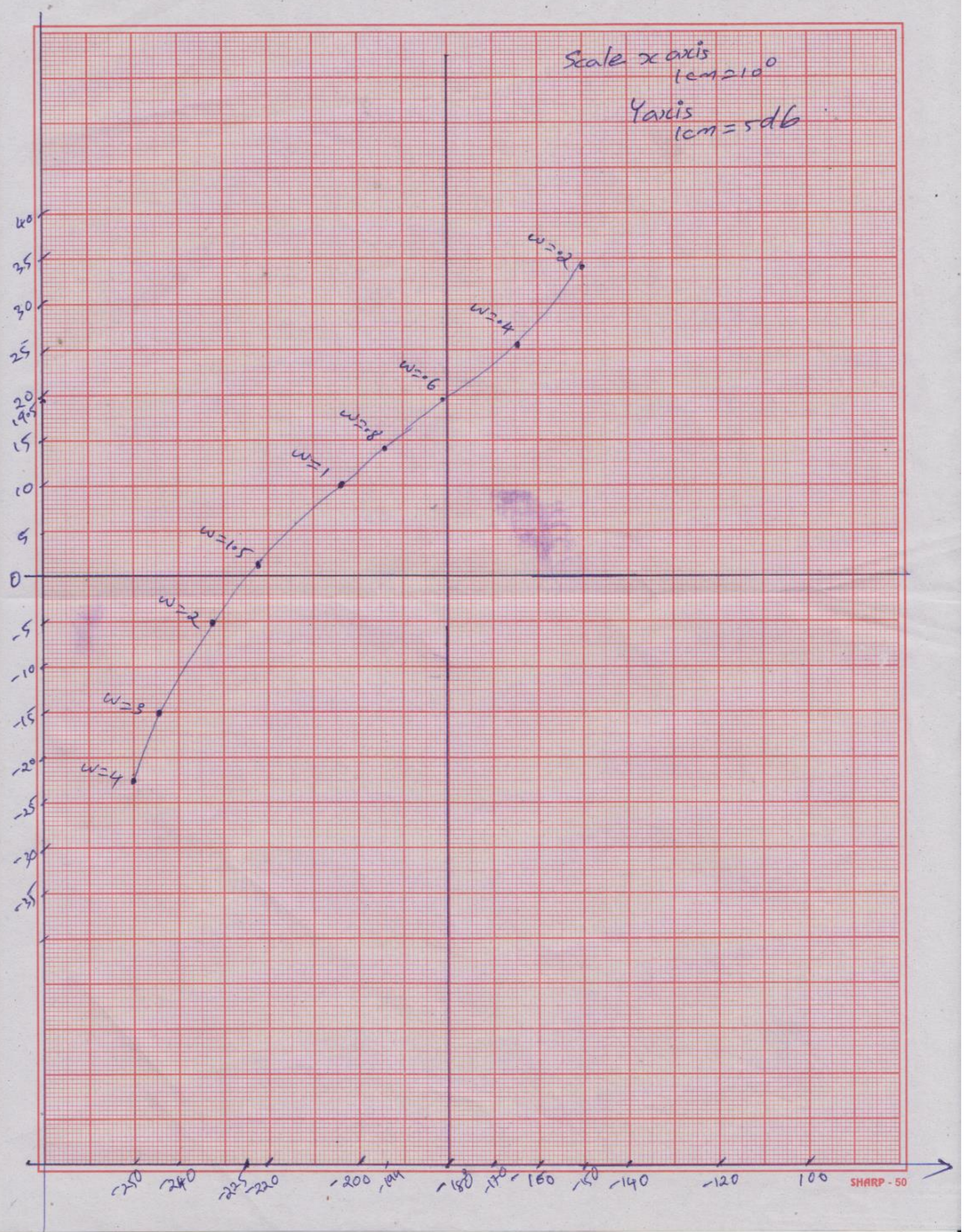
When K=1 corresponding magnitude of $G(j\omega)$ is +23db at -170 degree.

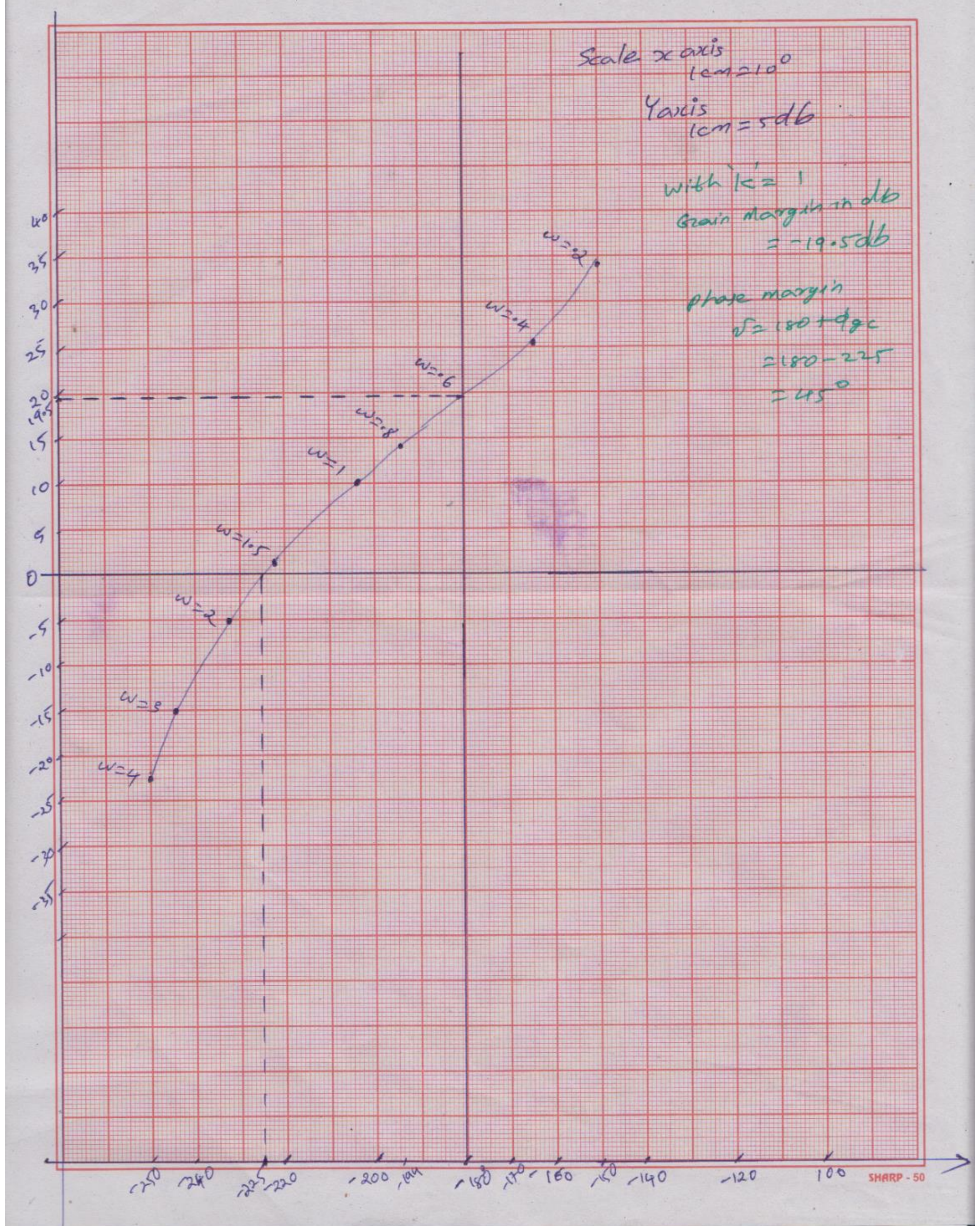
But for a phase margin of 10 degree, this gain should be made zero.

Hence -23db should be added to every point of $G(j\omega)$

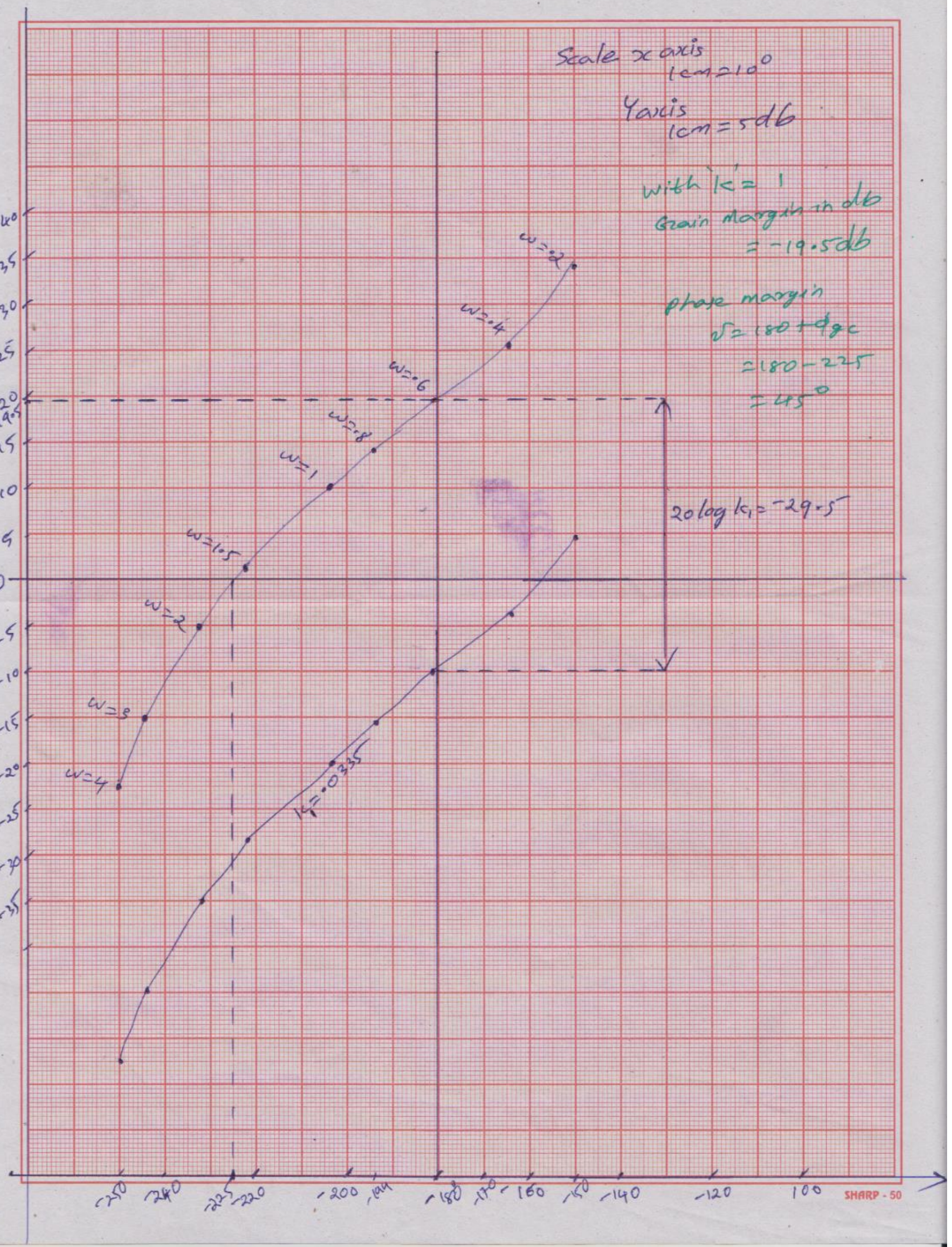
$$20 \log K_2 = -23 \quad \Rightarrow \quad \log K_2 = -23/20 \quad \Rightarrow \quad K_2 = 10^{\frac{-23}{20}} = 0.07$$

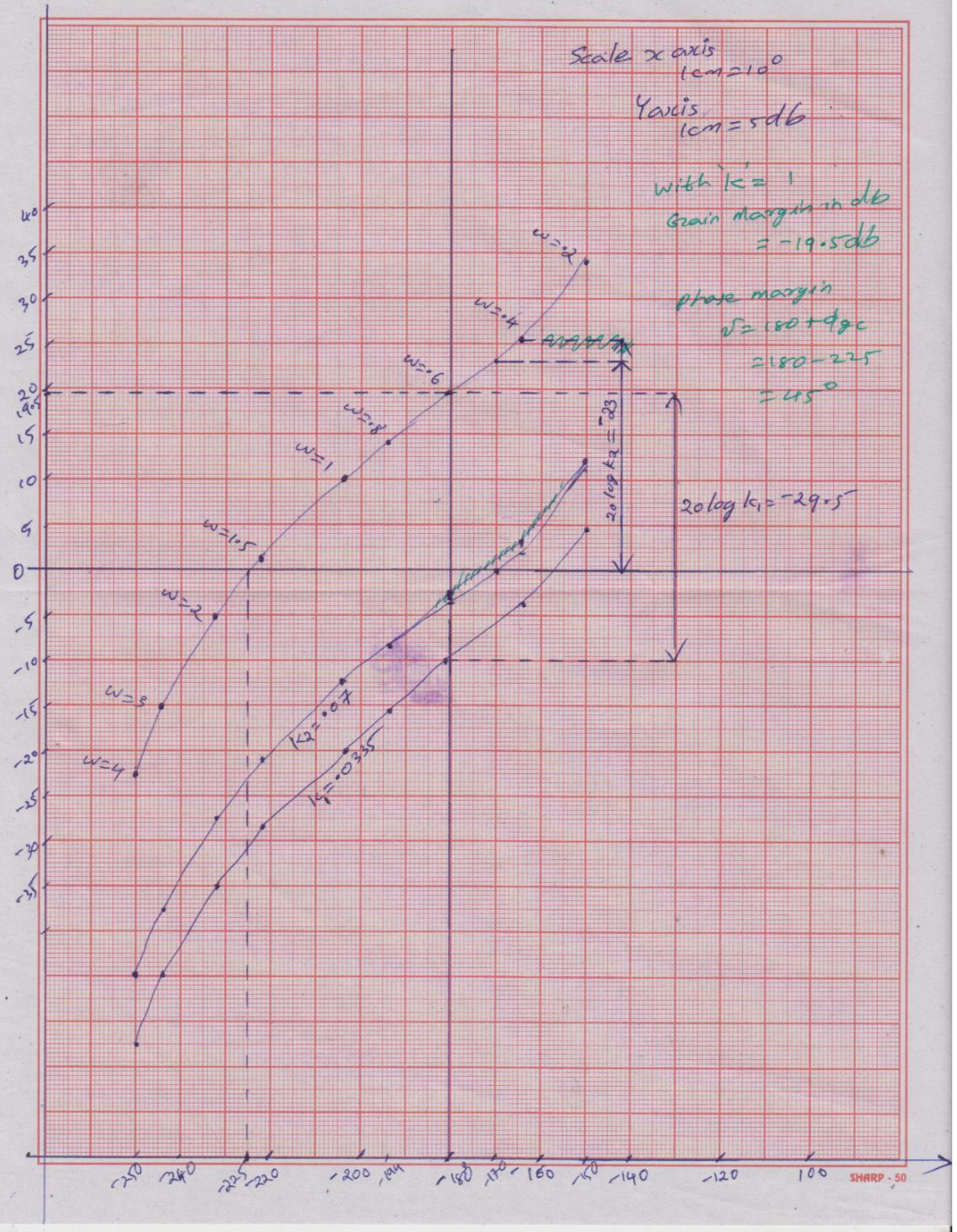

Scale x axis
1cm=10°
Y axis
1cm=5dB
40
35
30
25
20
15
10
5
0
-5
-10
-15
-20
-25
-30
-35
-240
-220
-200
-180
-160
-140
-120
100
SHARP - 50


Scale x axis
1cm = 10°
Y axis
1cm = 5 dB
ω=.2
ω=.4
ω=.6
ω=.8
ω=1
ω=1.5
ω=2
ω=3
ω=4
40
35
30
25
20
15
10
5
0
-5
-10
-15
-20
-25
-30
-35
-250
-240
-225
-220
-200
-180
-170
-160
-150
-140
-120
100
SHARP - 50



Scale x axis
1cm=10°
Y axis
1cm=5db
with 'k'= 1
Gain margin in db
= -19.5db
phase margin
γ= 180+φgc
=180-225
=45°
ω=.2
ω=.4
ω=.6
ω=.8
ω=1
ω=1.5
ω=2
ω=3
ω=4
40
35
30
25
20
19.5
15
10
5
0
-5
-10
-15
-20
-25
-30
-35
-250
-240
-225
-220
-200
-180
-170
-160
-150
-140
-120
100
SHARP - 50


Scale x axis
1cm = 10°
Y axis
1cm = 5db
with k = 1
Gain margin in db
= -19.5db
phase margin
γ = 180 + φgc
= 180 - 225
= 45°
20 log k1 = -29.5
k1 = .0335
SHARP - 50



Scale x axis
1cm = 10°
Y axis
1cm = 5db
with 'k' = 1
Gain margin in db
= -19.5db
phase margin
γ = 180 + φgc
= 180 - 225
= 45°
20 log k₁ = -29.5
20 log k₂ = -23
ω = .2
ω = .4
ω = .6
ω = .8
ω = 1
ω = 1.5
ω = 2
ω = 3
ω = 4
k₂ = .07
k₁ = .0335
SHARP - 50

MODULE VI

Polar plot

Nyquist stability criterion

Nichols chart

Non-minimum phase system

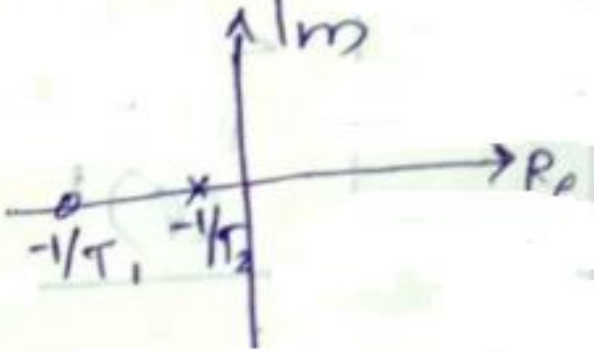
transportation lag

Minimum phase system and non minimum phase system

T.F having no poles and zeros in right half of ‘S’ plane is called minimum phase T.F.

System with minimum phase T.F are called minimum phase system.

$$G(j\omega) = \frac{1+j\omega T_1}{1+j\omega T_2}$$



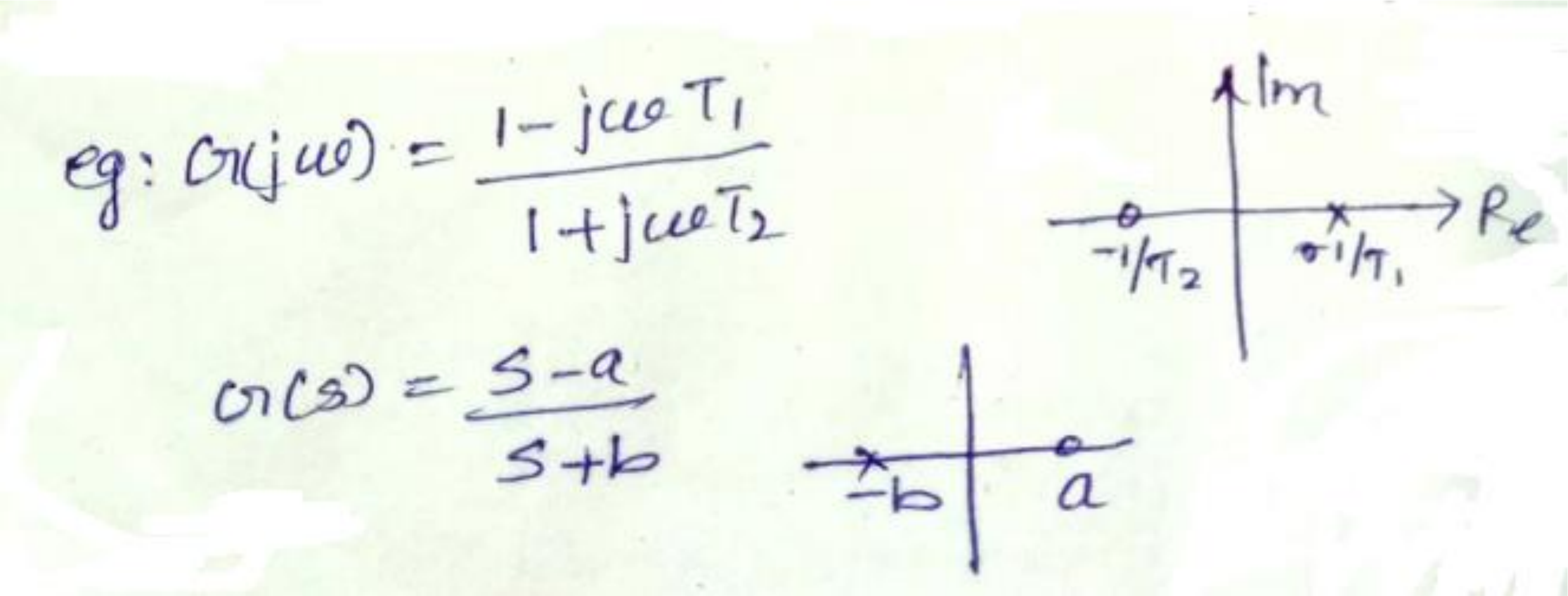
$$G(s) = \frac{s+a}{s+b}$$

zero = $-a$

pole = $-b$

The T.F having poles and/or zeros in the right half of ‘S’ plane are called non minimum phase T.F.

System with non minimum phase T.F are called non minimum phase system



POLAR PLOT

The polar plot of a sinusoidal transfer function G(jω) is a plot of the magnitude of G(jω) versus the phase angle of G(jω) on polar coordinates as ω is varied from zero to infinity.

the polar plot is the locus of vectors |G(jω)| ∠G(jω) as ω is varied from zero to infinity.

The polar plot is also called ***Nyquist plot.***

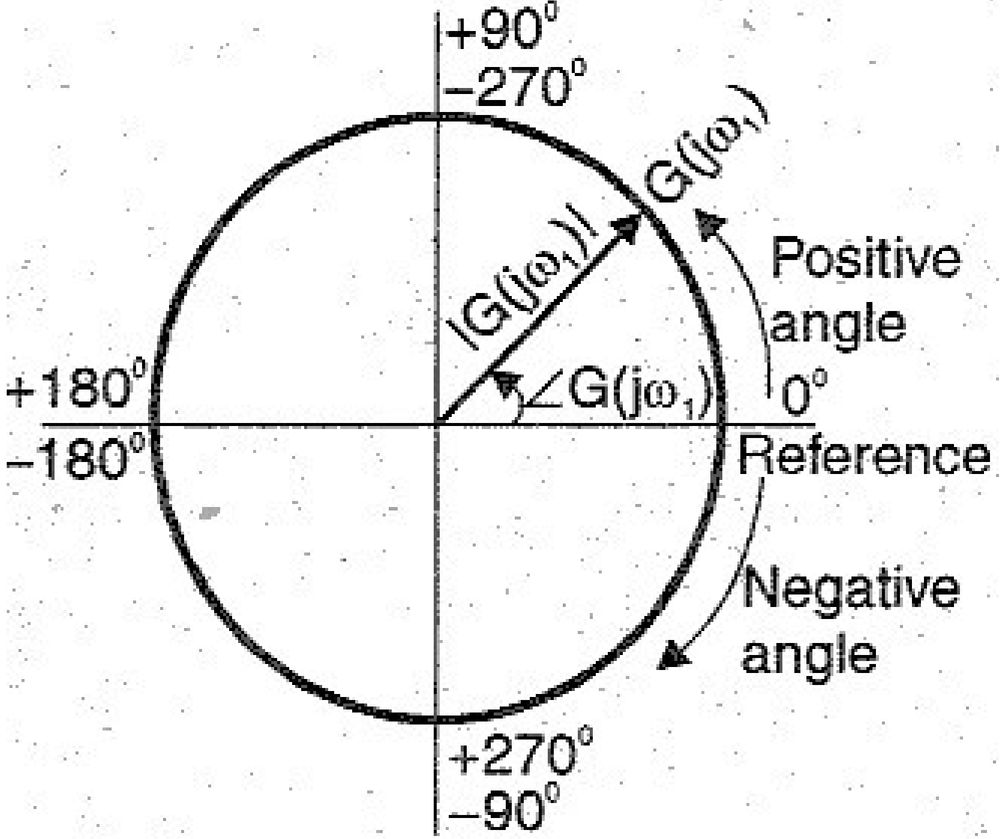
The polar plot is usually plotted on a polar graph sheet.

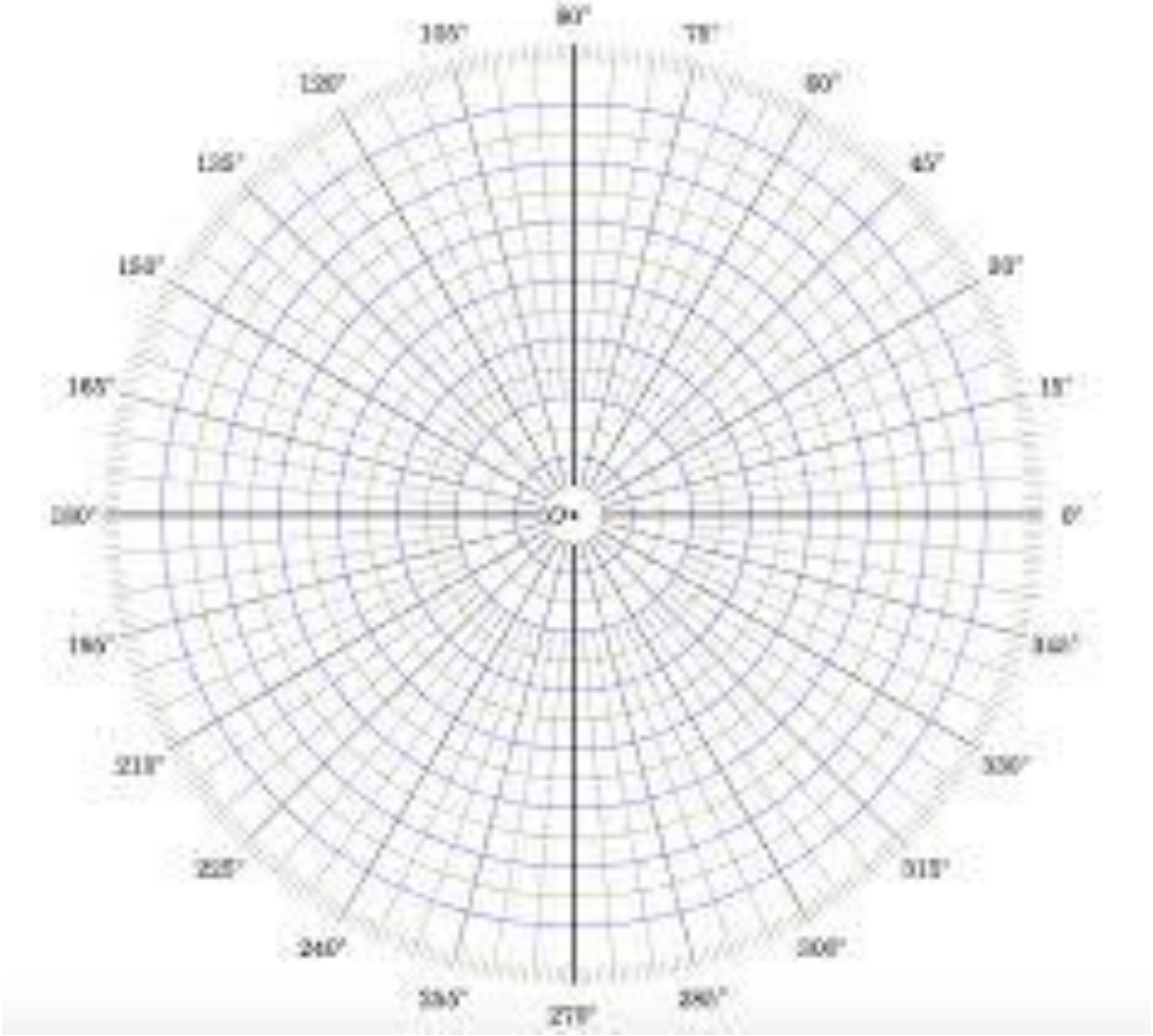
polar graph sheet has concentric circles and radial lines.

The circles represent the magnitude and the radial lines represent the phase angles.

Each point on the polar graph has a magnitude and phase angle.

magnitude of a point is given by the value of the circle passing through that point and the phase angle is given by the radial line passing through that point.


+90°
−270°
G(jω1)
|G(jω1)|
Positive
angle
∠G(jω1)
0°
Reference
+180°
−180°
Negative
angle
+270°
−90°


90°
105°
75°
120°
60°
135°
45°
150°
30°
165°
15°
180°
0°
195°
345°
210°
330°
225°
315°
240°
300°
255°
285°
270°

In order to plot the polar plot, magnitude and phase of $G(j\omega)$ are computed for various values of ω and tabulated.

Usually the choice of frequencies are corner frequencies and frequencies around corner frequencies.

Alternatively, if $G(j\omega)$ can be expressed in rectangular coordinates as,

$$G(j\omega) = G_R(j\omega) + jG_I(j\omega)$$

where, $G_R(j\omega)$ = Real part of $G(j\omega)$; $G_I(j\omega)$ = Imaginary part of $G(j\omega)$.

then the polar plot can be plotted in ordinary graph sheet between $G_R(j\omega)$ and $G_I(j\omega)$ by varying ω from 0 to ∞.

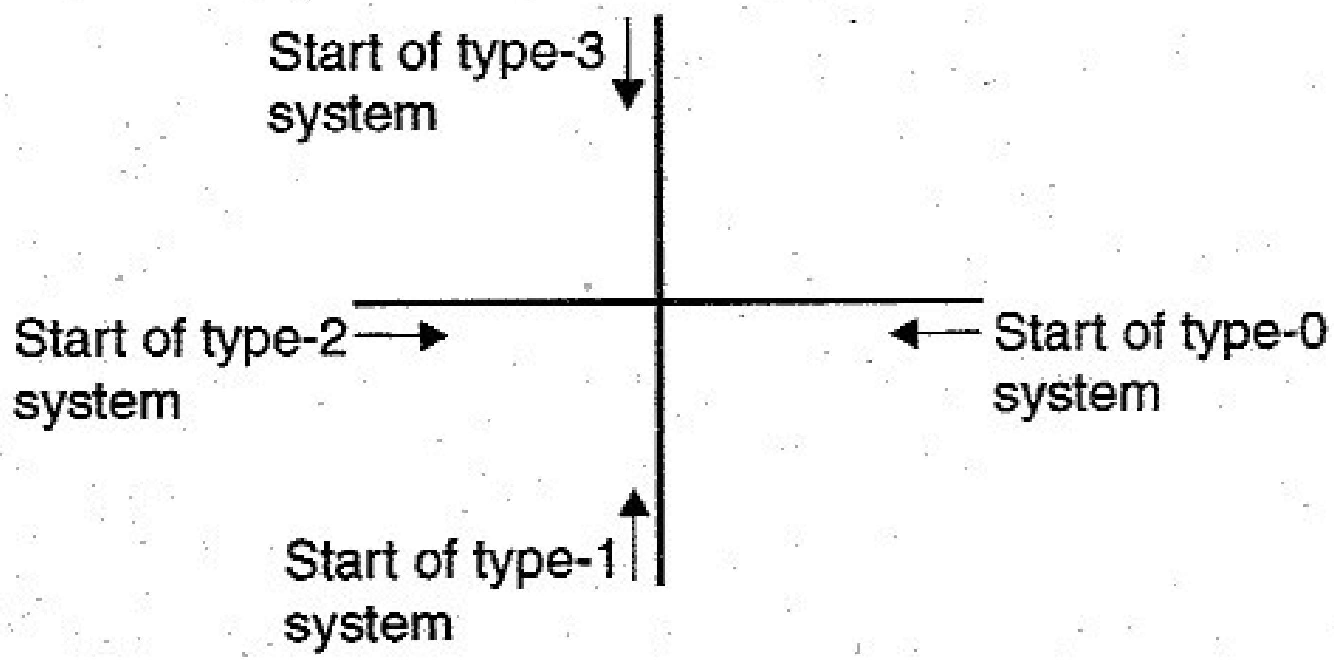
to plot the polar plot on ordinary graph sheet, the magnitude and phase of $G(j\omega)$ are computed for various values of ω.

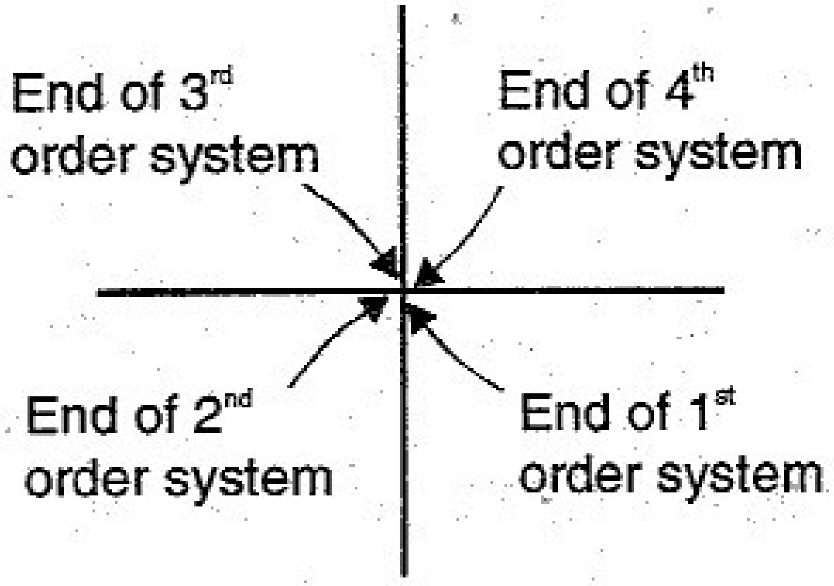
Then convert the polar coordinates to rectangular coordinates using

P $\rightarrow$ R conversion (polar to rectangular conversion) in the calculator.

The change in shape of polar plot can be predicted due to addition of a pole or zero.

1. When a pole is added to a system, the polar plot end point will shift by -90^{0}.
2. When a zero is added to a system the polar plot end point will shift by $+90^{0}$.

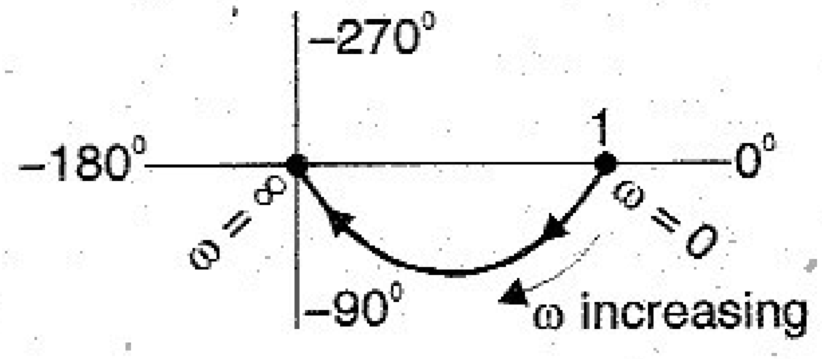




Typical Sketches of Polar Plot

Type : 0, Order : 1 $G(s)=\dfrac{1}{1+sT}$

$$G(j\omega)=\frac{1}{1+j\omega T}=\frac{1}{\sqrt{1+\omega^2T^2}\,\angle\tan^{-1}\omega T}=\frac{1}{\sqrt{1+\omega^2T^2}}\angle-\tan^{-1}\omega T$$

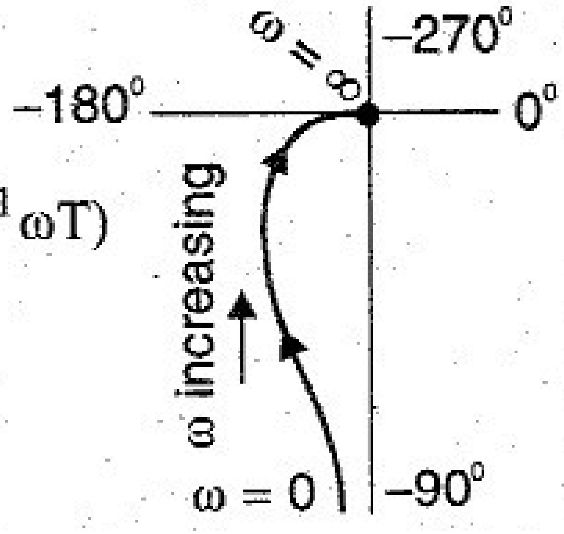


As $\omega\to 0$, $G(j\omega)\to 1\angle 0^\circ$

As $\omega\to\infty$, $G(j\omega)\to 0\angle -90^\circ$

Type : 1, Order : 2 $G(s)=\dfrac{1}{s(1+sT)}$

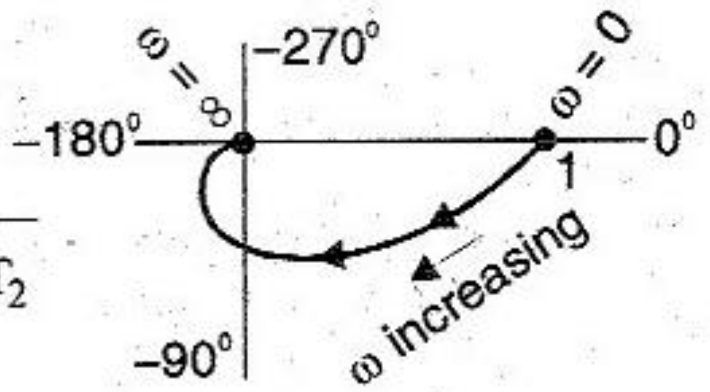
$$G(j\omega)=\frac{1}{j\omega(1+j\omega T)}=\frac{1}{\omega\,\angle 90^\circ\,\sqrt{1+\omega^2T^2}\,\angle\tan^{-1}\omega T}=\frac{1}{\omega\sqrt{1+\omega^2T^2}}\angle(-90^\circ-\tan^{-1}\omega T)$$



As $\omega\to 0$, $G(j\omega)\to \infty\angle -90^\circ$

As $\omega\to\infty$, $G(j\omega)\to 0\angle -180^\circ$

Type : 0, Order : 2 $G(s) = \frac{1}{(1+sT_1)(1+sT_2)}$



$$G(j\omega) = \frac{1}{(1+j\omega T_1)\ (1+j\omega T_2)} = \frac{1}{\sqrt{1+\omega^2 T_1^2}\angle \tan^{-1}\omega T_1 \sqrt{1+\omega^2 T_2^2}\ \angle \tan^{-1}\omega T_2}$$

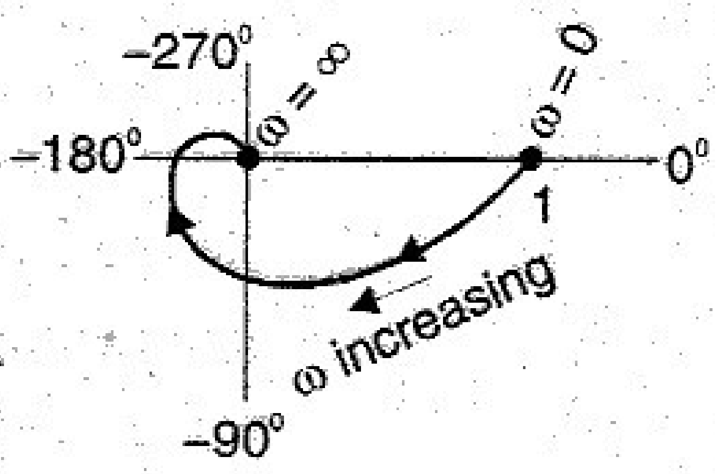
$$= \frac{1}{\sqrt{\left(1+\omega^2 T_1^2\right)\left(1+\omega^2 T_2^2\right)}}\angle(-\tan^{-1}\omega T_1 - \tan^{-1}\omega T_2)$$

As $\omega \to 0$, $G(j\omega) \to 1\angle 0^\circ$

As $\omega \to \infty$, $G(j\omega) \to 0\angle -180^\circ$

Type : 0, Order : 3

$$G(s) = \frac{1}{(1+sT_1)(1+sT_2)(1+sT_3)}$$



$$G(j\omega) = \frac{1}{(1+j\omega T_1)\ (1+j\omega T_2)\ (1+j\omega T_3)}$$

$$= \frac{1}{\sqrt{1+\omega^2T_1^2}\angle\tan^{-1}\omega T_1\ \sqrt{1+\omega^2T_2^2}\angle\tan^{-1}\omega T_2\ \sqrt{1+\omega^2T_3^2}\angle\tan^{-1}\omega T_3}$$

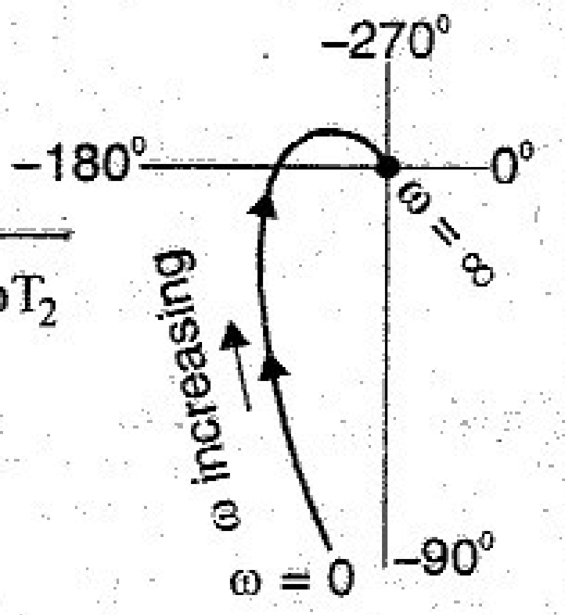
$$= \frac{1}{\sqrt{\left(1+\omega^2T_1^2\right)\left(1+\omega^2T_2^2\right)\left(1+\omega^2T_3^2\right)}}\angle(-\tan^{-1}\omega T_1 - \tan^{-1}\omega T_2 - \tan^{-1}\omega T_3)$$

As $\omega \to 0$, $\quad G(j\omega) \to 1\angle 0°$

As $\omega \to \infty$, $\quad G(j\omega) \to 0\angle -270°$

Type : 1, Order : 3

$$G(s) = \frac{1}{s(1+sT_1)(1+sT_2)}$$



$$G(j\omega) = \frac{1}{j\omega(1+j\omega T_1)\ (1+j\omega T_2)} = \frac{1}{\omega\angle 90^\circ\ \sqrt{1+\omega^2T_1^2}\,\angle\tan^{-1}\omega T_1\ \sqrt{1+\omega^2T_2^2}\ \angle\tan^{-1}\omega T_2}$$

$$= \frac{1}{\omega\sqrt{\left(1+\omega^2T_1^2\right)\left(1+\omega^2T_2^2\right)}}\angle(-90^\circ - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2)$$

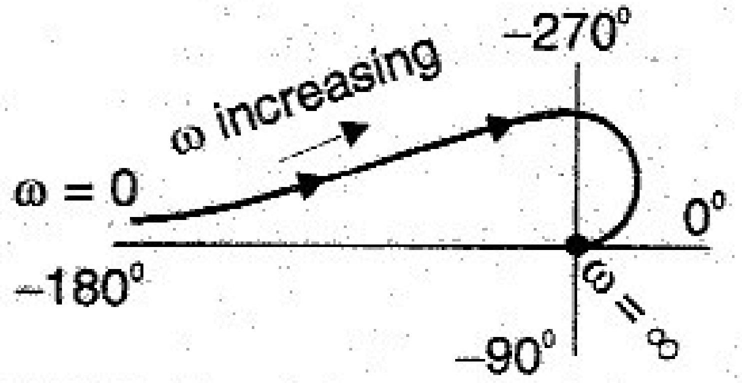
As $\omega \to 0$, $\quad G(j\omega) \to \infty\angle -90^\circ$

As $\omega \to \infty$, $\quad G(j\omega) \to 0\angle -270^\circ$

Type : 2, Order : 4 $G(s) = \dfrac{1}{s^2(1+sT_1)(1+sT_2)}$

$$G(j\omega) = \frac{1}{(j\omega)^2\ (1+j\omega T_1)\ (1+j\omega T_2)} = \frac{1}{\omega^2\angle -180^\circ\ \sqrt{1+\omega^2T_1^2}\angle\tan^{-1}\omega T_1\ \sqrt{1+\omega^2T_2^2}\angle\tan^{-1}\omega T_2}$$

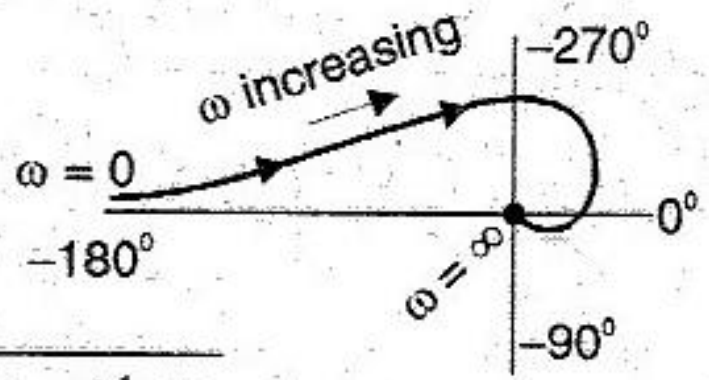
$$= \frac{1}{\omega^2\sqrt{\left(1+\omega^2T_1^2\right)\left(1+\omega^2T_2^2\right)}}\angle(-180^\circ - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2)$$



As $\omega \to 0$, $G(j\omega) \to \infty\angle -180^\circ$

As $\omega \to \infty$, $G(j\omega) \to 0\angle -360^\circ$

Type : 2, Order : 5 $G(s)=\dfrac{1}{s^2(1+sT_1)(1+sT_2)(1+sT_3)}$



$$G(j\omega)=\frac{1}{(j\omega)^2\left(1+j\omega T_1\right)\left(1+j\omega T_2\right)\left(1+j\omega T_3\right)}$$

$$=\frac{1}{\omega^2\angle-180^\circ\sqrt{1+\omega^2T_1^2}\angle\tan^{-1}\omega T_1\sqrt{1+\omega^2T_2^2}\angle\tan^{-1}\omega T_2\sqrt{1+\omega^2T_3^2}\angle\tan^{-1}\omega T_3}$$

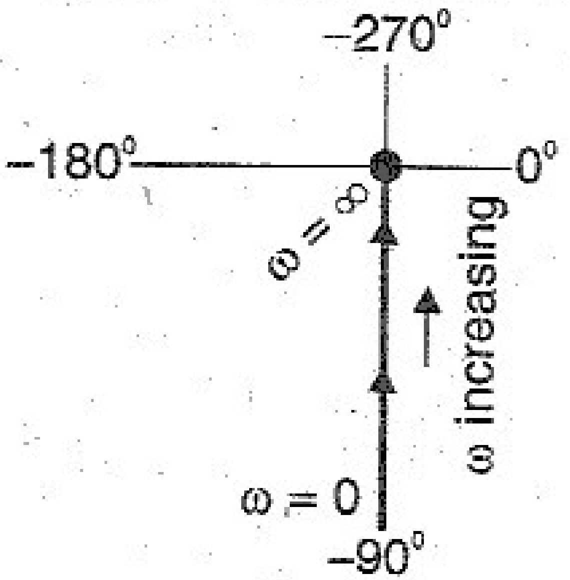
$$=\frac{1}{\omega^2\sqrt{\left(1+\omega^2T_1^2\right)\left(1+\omega^2T_2^2\right)\left(1+\omega^2T_3^2\right)}}\angle(-180^\circ-\tan^{-1}\omega T_1-\tan^{-1}\omega T_2-\tan^{-1}\omega T_3)$$

As $\omega\to 0$, $\quad G(j\omega)\to\infty\angle-180^\circ$

As $\omega\to\infty$, $\quad G(j\omega)\to 0\angle-450^\circ=0\angle-90^\circ$

Type : 1, Order : 1

$$G(s) = \frac{1}{s}$$

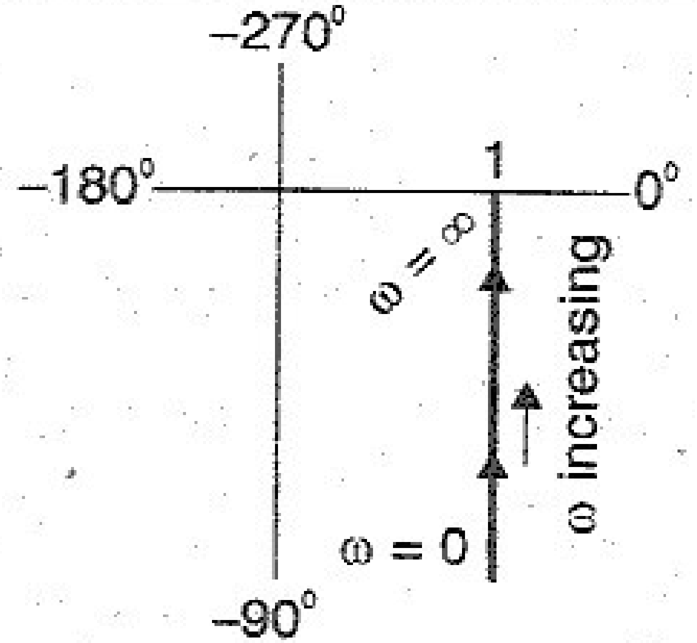


$$G(j\omega) = \frac{1}{j\omega} = \frac{1}{\omega\angle 90^\circ} = \frac{1}{\omega}\angle -90^\circ$$

As $\omega \to 0$, $G(j\omega) \to \infty\angle -90^\circ$

As $\omega \to \infty$, $G(j\omega) \to 0\angle -90^\circ$

$$G(s) = \frac{1+sT}{sT}$$



$$G(j\omega) = \frac{1+j\omega T}{j\omega T} = \frac{1}{j\omega T} + 1 = \frac{1}{\omega T\angle 90^\circ} + 1 = \frac{1}{\omega T}\angle -90^\circ + 1$$

As $\omega \to 0$, $G(j\omega) \to \infty\angle -90^\circ + 1$

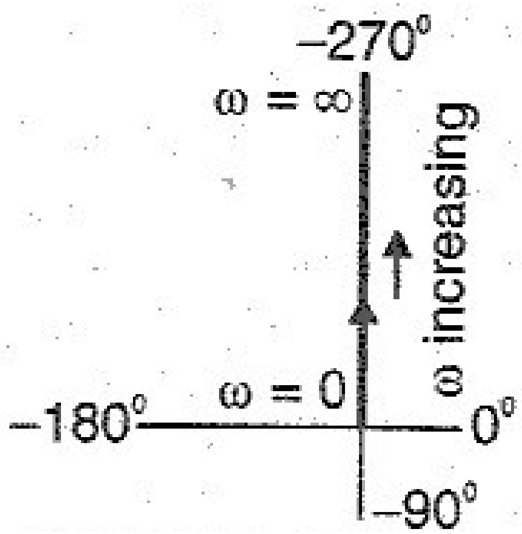
As $\omega \to \infty$, $G(j\omega) \to 0\angle -90^\circ + 1$

$G(s) = s$

$G(j\omega) = j\omega = \omega\angle 90^\circ$

As $\omega \to 0$, $\quad G(j\omega) \to 0\angle 90^\circ$

As $\omega \to \infty$, $\quad G(j\omega) \to \infty\angle 90^\circ$

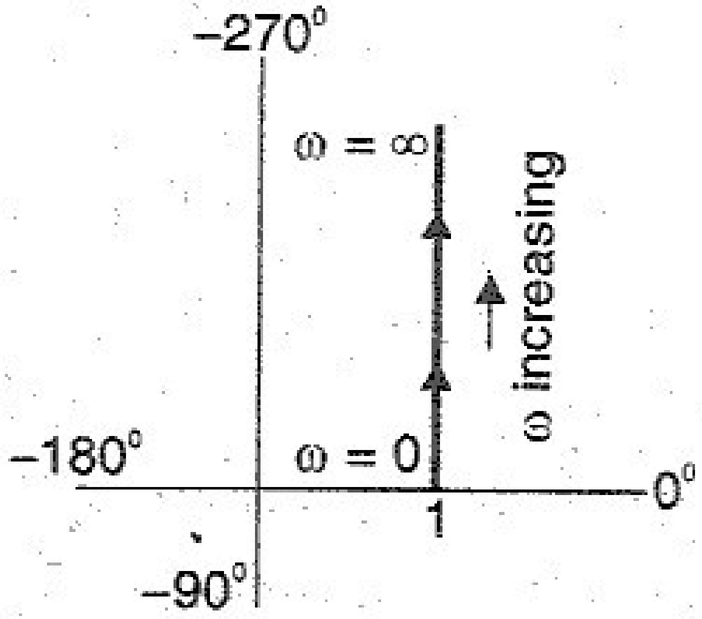


$G(s) = 1 + sT$

$G(j\omega) = 1 + j\omega T = 1 + \omega T\angle 90^\circ$

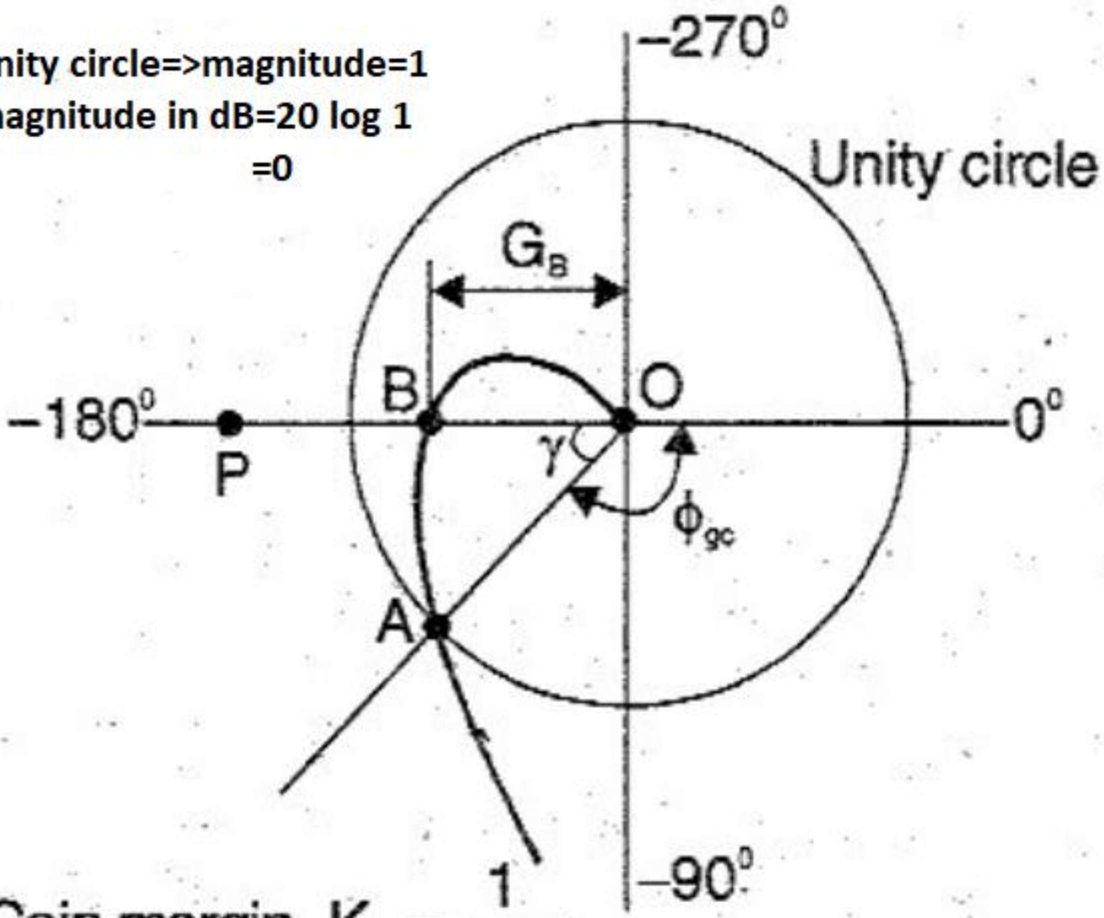
As $\omega \to 0$, $\quad G(j\omega) \to 1 + 0\angle 90^\circ$

As $\omega \to \infty$, $\quad G(j\omega) \to 1 + \infty\angle 90^\circ$



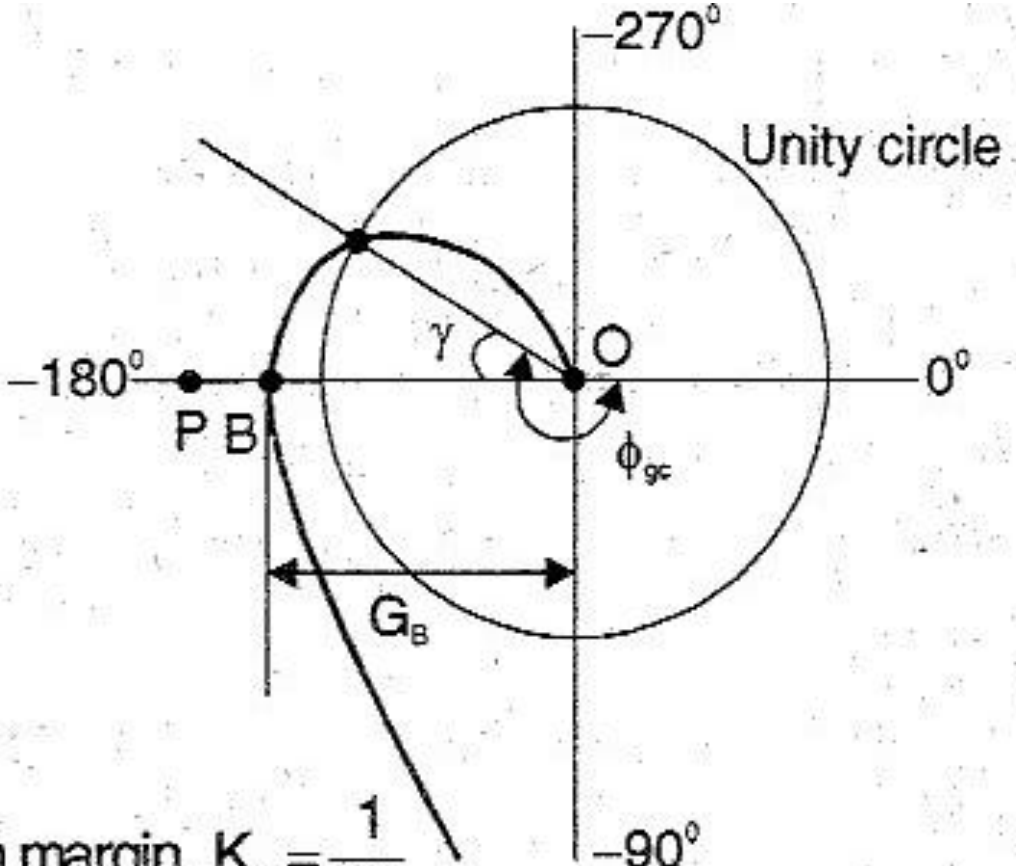
DETERMINATION OF GAIN MARGIN AND PHASE MARGIN FROM POLAR PLOT

unity circle=>magnitude=1
magnitude in dB=20 log 1
=0



Gain margin, $K_g = \frac{1}{G_B}$

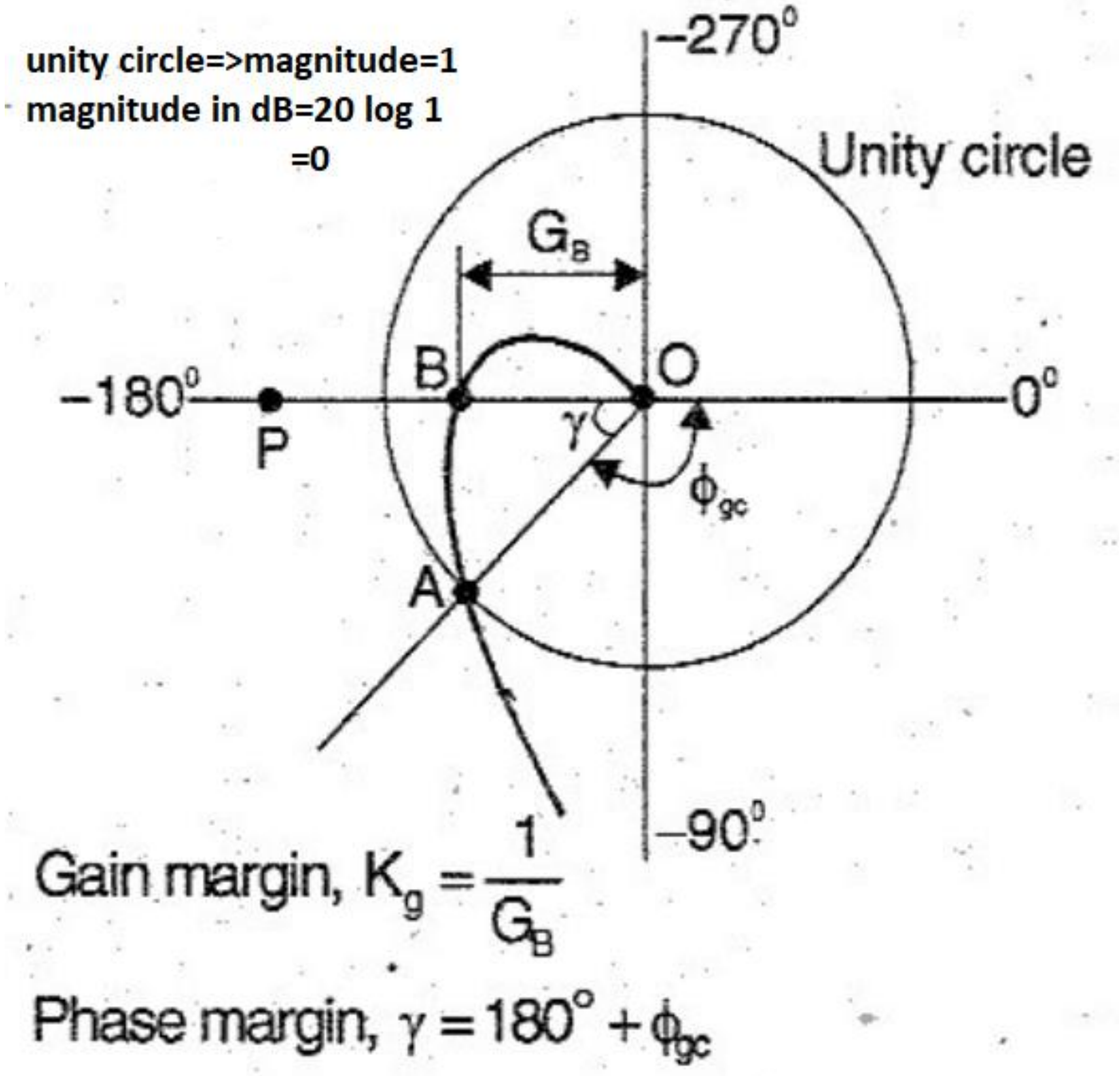
Phase margin, $\gamma = 180^\circ + \phi_{gc}$

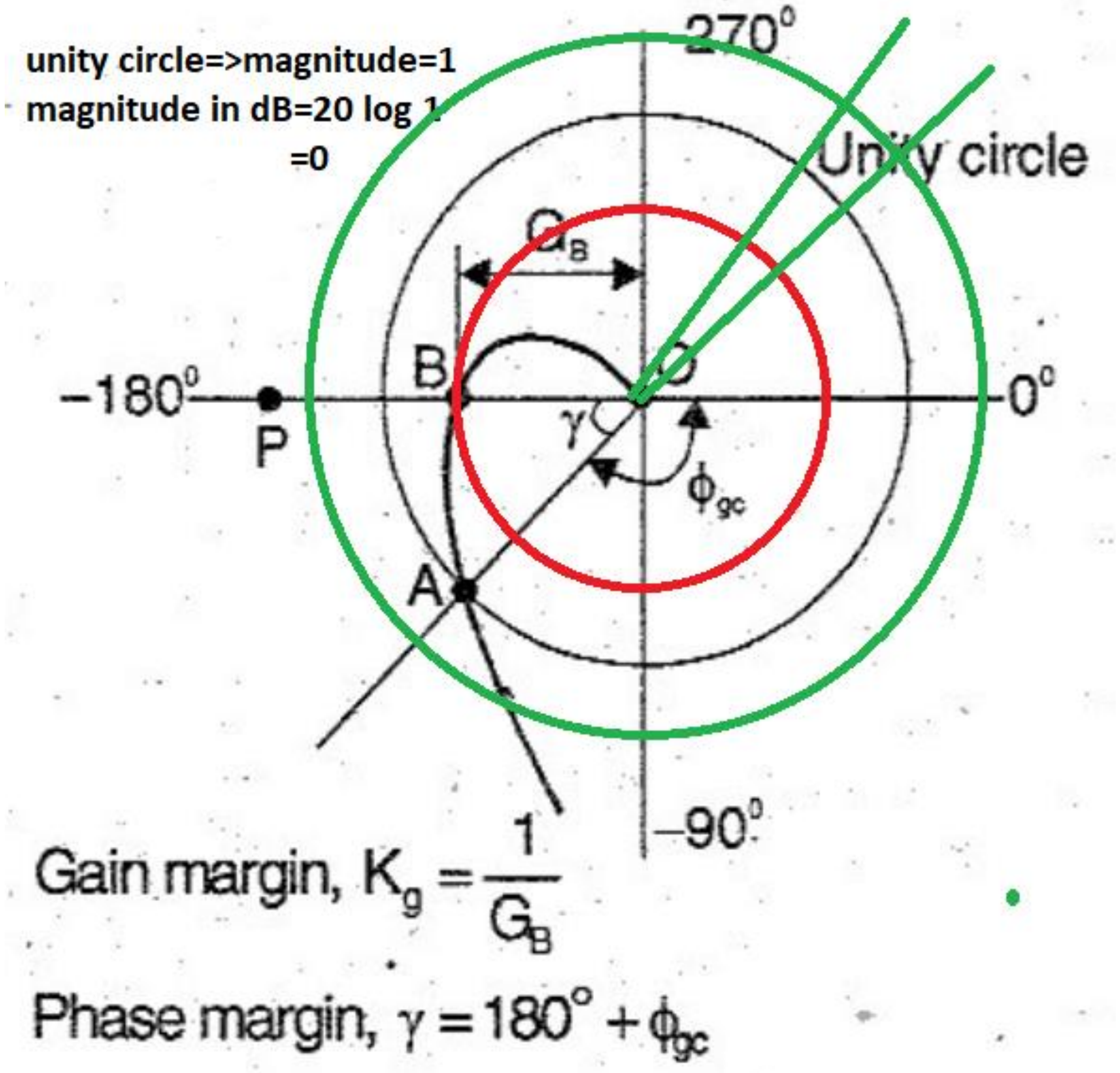


Gain margin, $K_g = \frac{1}{G_B}$

Phase margin, $\gamma = 180^\circ + \phi_{gc}$

DETERMINATION OF GAIN MARGIN AND PHASE MARGIN FROM POLAR PLOT





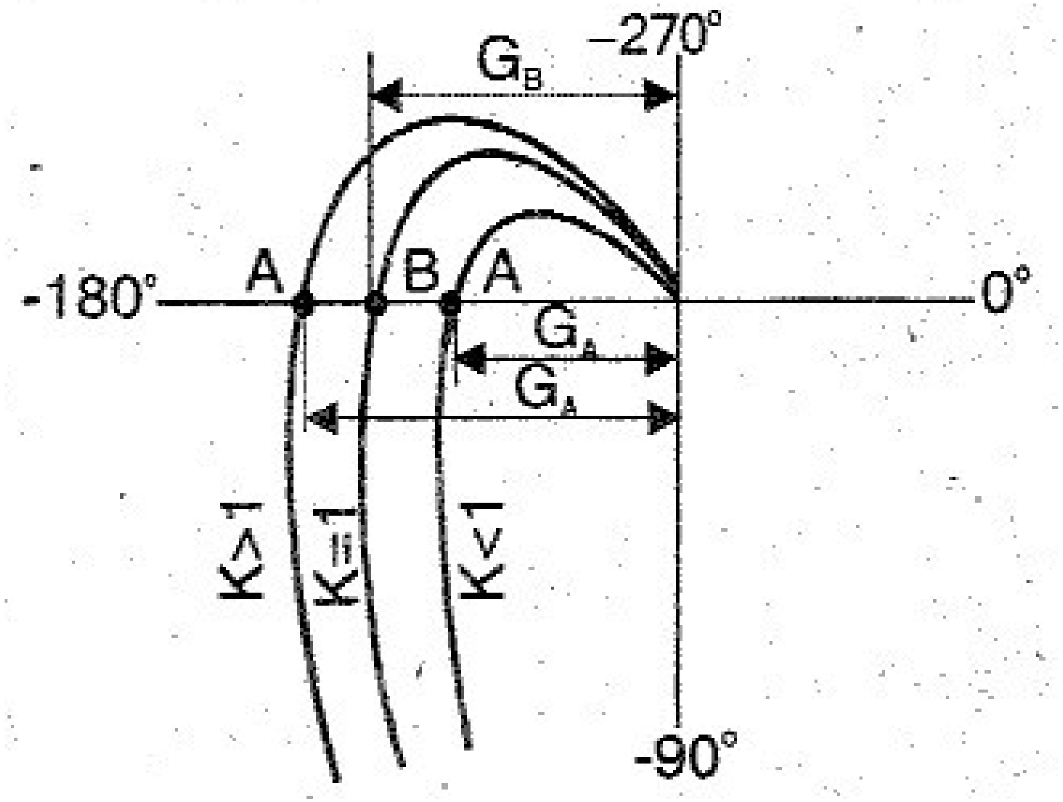
GAIN ADJUSTMENT USING POLAR PLOT

To Determine K for Specified GM

Draw $G(j\omega)$ locus with K =1.

Let it cut the $-180°$ axis at point B corresponding to a gain of G_B.

Let the specified gainmargin be x db.



For this gain margin, the $G(j\omega)$ locus will cut $-180°$ at point A whose magnitude is G_A.

$$\text{Now, } 20\log\frac{1}{G_A} = x \Rightarrow \log\frac{1}{G_A} = \frac{x}{20} \Rightarrow \frac{1}{G_A} = 10^{x/20}$$

$$\therefore G_A = \frac{1}{10^{x/20}}$$

Now the value of K is given by, $K = \frac{G_A}{G_B}$.

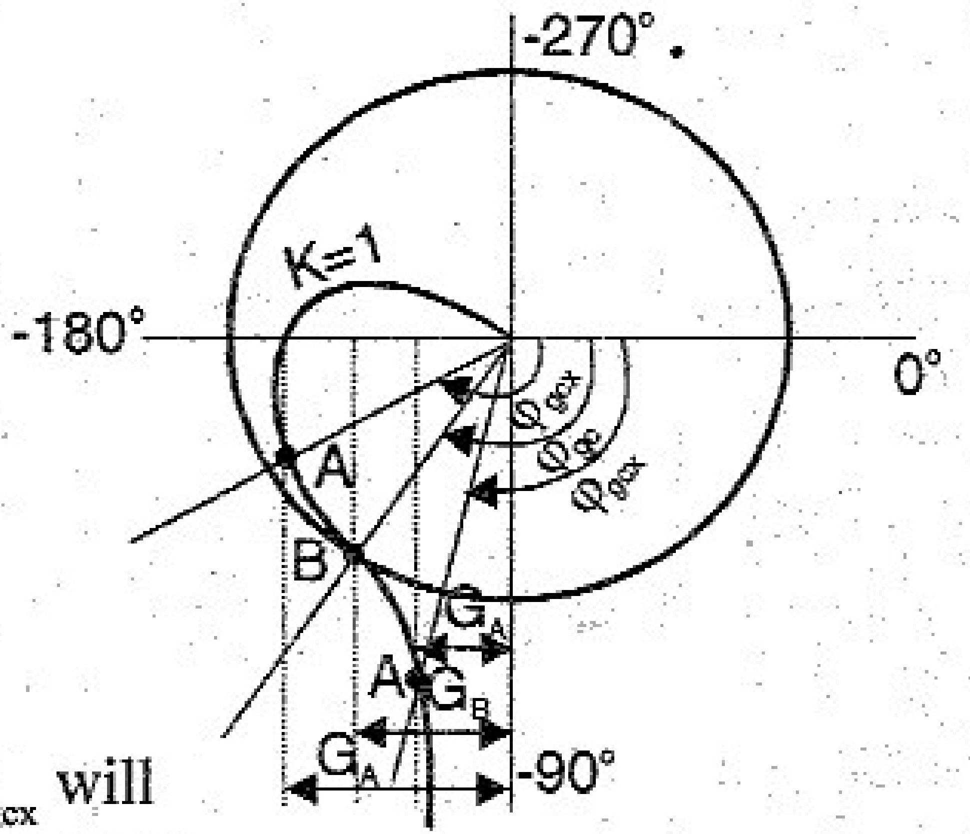
If, K >1, then the system gain should be increased.

If, K < 1, then the system gain should be reduced.

To Determine K for Specified PM

Draw $G(j\omega)$ locus with K = 1. Let it cut the unity circle at point B. (The gain at point B is G_B and equal to unity).

Let the specified phase margin be x°



For a phase margin of x°, let ϕ_{gcx} be the phase angle of $G(j\omega)$ at gain crossover frequency.

$$x° = 180° + \phi_{gcx} \quad \Rightarrow \quad \phi_{gcx} = x° - 180°$$

In the polar plot, the radial line corresponding to ϕ_{gcx} will cut the locus of $G(j\omega)$ with K = 1 at point A and the magnitude corresponding to that point be G_A

$$K = \frac{G_B}{G_A} = \frac{1}{G_A} \qquad (\because G_B = 1)$$

The open loop transfer function of a unity feedback system is given by $G(s) = 1/s(1+s)(1+2s)$.
Sketch the polar plot and determine the gain margin and phase margin.

$$G(s) = 1/s(1+s)(1+2s)$$

$$G(j\omega) = \frac{1}{j\omega\,(1+j\omega)\,(1+j2\omega)}$$

The corner frequencies are $\omega_{c1} = 1/2 = 0.5$ rad/sec and $\omega_{c2} = 1$ rad/sec.

$$G(j\omega) = \frac{1}{(j\omega)\,(1+j\omega)\,(1+j2\omega)} = \frac{1}{\omega\angle 90^\circ\ \sqrt{1+\omega^2}\ \angle\tan^{-1}\omega\ \sqrt{1+4\omega^2}\ \angle\tan^{-1}2\omega}$$

$$= \frac{1}{\omega\sqrt{(1+\omega^2)\,(1+4\omega^2)}}\ \angle -90^\circ - \tan^{-1}\omega - \tan^{-1}2\omega$$

$$|G(j\omega)| = \frac{1}{\omega\sqrt{(1+\omega^2)(1+4\omega^2)}} = \frac{1}{\omega\sqrt{1+4\omega^2+\omega^2+4\omega^4}} = \frac{1}{\omega\sqrt{1+5\omega^2+4\omega^4}}$$

TABLE-1 : Magnitude and phase of G(jω) at various frequencies

ω rad/sec	0.35	0.4	0.45	0.5	0.6	0.7	1.0
|G(jω)|	2.2	1.8	1.5	1.2	0.9	0.7	0.3
∠G(jω) deg	-144	-150	-156	-162	-171	-179.5 ≈ -180	-198

$$\angle G(j\omega) = -90° - \tan^{-1}\omega - \tan^{-1}2\omega$$

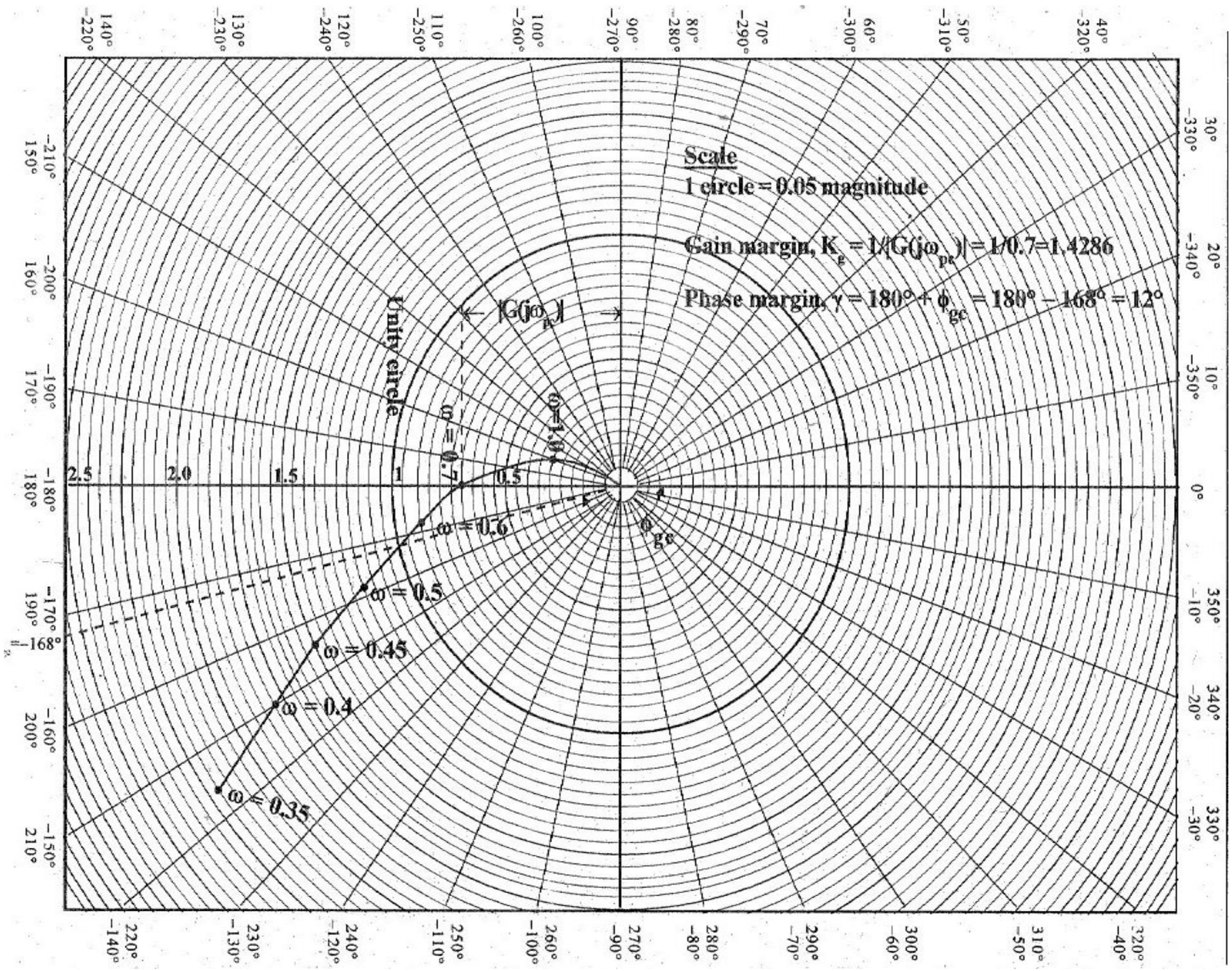
TABLE-2 : Real and imaginary part of G(jω) at various frequencies

ω rad/sec	0.35	0.4	0.45	0.5	0.6	0.7	1.0
$G_R(j\omega)$	-1.78	-1.56	-1.37	-1.14	-0.89	-0.7	-0.29
$G_I(j\omega)$	-1.29	-0.9	-0.61	-0.37	-0.14	0	0.09

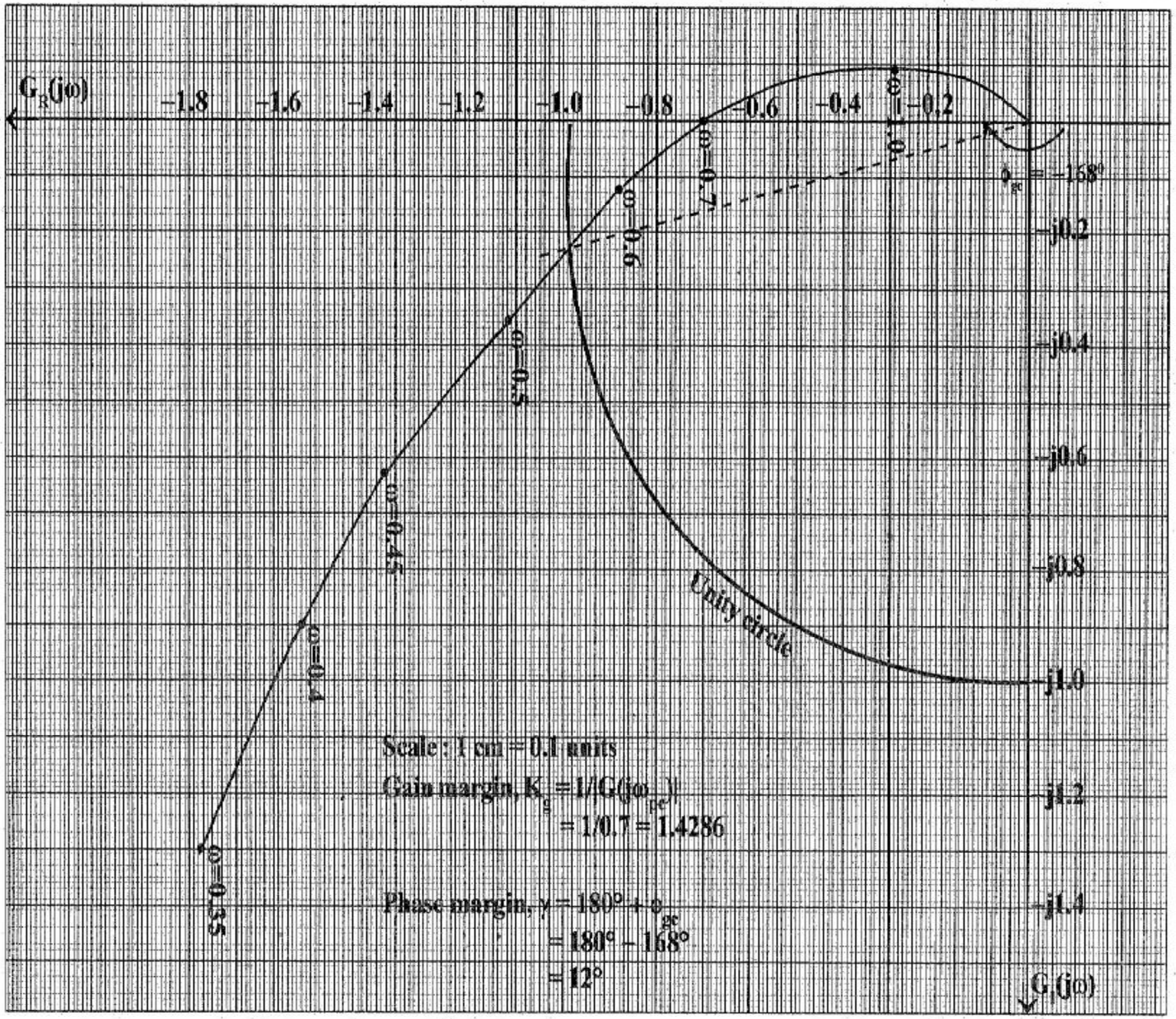
RESULT

Gain margin, $K_g = 1.4286$

Phase margin, $\gamma = +12°$



Scale
1 circle = 0.05 magnitude
Gain margin, $K_g = 1/|G(j\omega_{pc})| = 1/0.7 = 1.4286$
Phase margin, $\gamma = 180° + \phi_{gc} = 180° - 168° = 12°$
Unity circle
$|G(j\omega_{pc})|$
$\omega = 1.0$
$\omega = 0.7$
$\omega = 0.6$
$\omega = 0.5$
$\omega = 0.45$
$\omega = 0.4$
$\omega = 0.35$
ϕ_{gc}
-168°
2.5
2.0
1.5
1
0.5



G_R(jω)
−1.8
−1.6
−1.4
−1.2
−1.0
−0.8
−0.6
−0.4
−0.2
ω=1.0
ω=0.7
ω=0.6
ω=0.5
ω=0.45
ω=0.4
ω=0.35
φgc = −168°
−j0.2
−j0.4
−j0.6
−j0.8
−j1.0
−j1.2
−j1.4
Unity circle
G_I(jω)
Scale : 1 cm = 0.1 units
Gain margin, Kg = 1/G(jωpc)
= 1/0.7 = 1.4286
Phase margin, γ = 180° + φgc
= 180° − 168°
= 12°

The open loop transfer function of a unity feedback system is given by G(s) = 1/ $s^2(1+s)(1+2s)$. Sketch the polar plot and determine the gain margin and phase margin.

$G(s) = 1/\, s^2(1+s)(1+2s)$ $\quad\quad$ $G(j\omega) = \dfrac{1}{(j\omega)^2\,(1+j\omega)\,(1+j2\omega)}$

$\omega_{c1} = 0.5$ rad/sec and $\omega_{c2} = 1$ rad/sec.

$$G(j\omega) = \frac{1}{(j\omega)^2\,(1+j\omega)\,(1+j2\omega)}$$

$$= \frac{1}{\omega^2\angle 180^\circ\ \sqrt{1+\omega^2}\ \angle\tan^{-1}\omega\ \sqrt{1+4\omega^2}\ \angle\tan^{-1}2\omega}$$

$$G(j\omega) = \frac{1}{\omega^2\sqrt{1+\omega^2}\ \sqrt{1+4\omega^2}}\ \angle(-180 - \tan^{-1}\omega - \tan^{-1}2\omega)$$

$$|G(j\omega)| = \frac{1}{\omega^2\ \sqrt{1+\omega^2}\ \sqrt{1+4\omega^2}} = \frac{1}{\omega^2\sqrt{(1+\omega^2)\,(1+4\omega^2)}}$$

$$= \frac{1}{\omega^2\ \sqrt{1+5\omega^2+4\omega^4}}$$

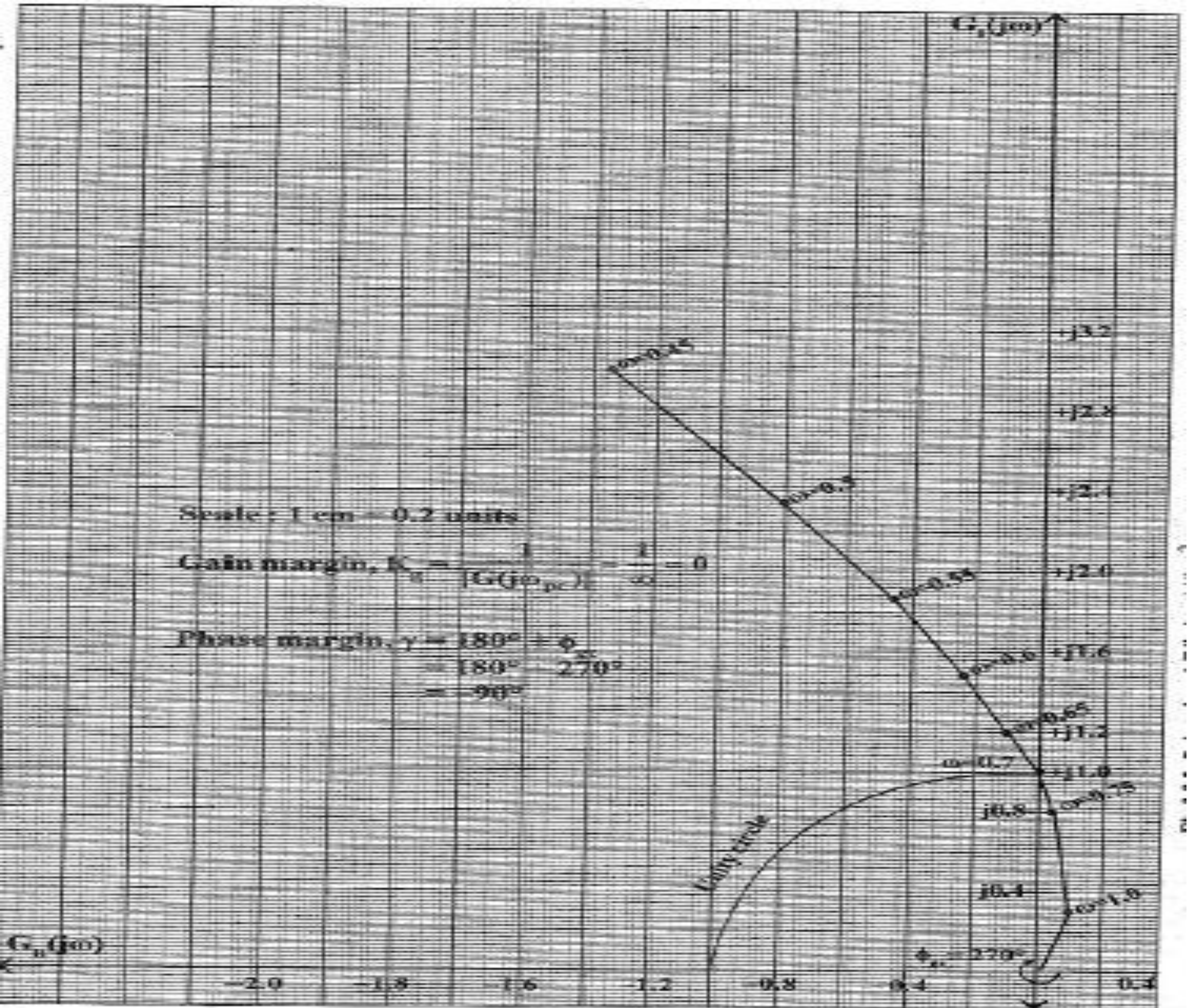
$$\angle G(j\omega) = -180^\circ - \tan^{-1}\omega - \tan^{-1}2\omega.$$

TABLE-1 : Magnitude and phase plot of $G(j\omega)$ at various frequencies

ω rad/sec	0.45	0.5	0.55	0.6	0.65	0.7	0.75	1.0
$\lvert G(j\omega)\rvert$	3.3	2.5	1.9	1.5	1.2	$0.97 \approx 1$	0.8	0.3
$\angle G(j\omega)$ deg	–246	–251	–256	–261	–265	–269	–273	–288

TABLE-2 : Real and imaginary parts of $G(j\omega)$

ω rad/sec	0.45	0.5	0.55	0.6	0.65	0.7	0.75	1.0
$G_R(j\omega)$	–1.34	–0.81	–0.46	–0.23	–0.1	–0.02	0.04	0.09
$G_I(j\omega)$	3.01	2.36	1.84	1.48	1.2	1.0	0.8	0.29



RESULT

Gain margin, $K_g = 0$

Phase margin, $\gamma = -90°$

Consider a unity feedback system having an open loop transfer function $G(s) = \frac{K}{s(1+0.2s)(1+0.05s)}$

Sketch the polar plot and determine the value of K so that (i) Gain margin is 18 db (ii) Phase margin is 60°.

$$G(s) = \frac{K}{s(1+0.2s)(1+0.05s)}$$

Put K = 1 and s = jω

$$G(j\omega) = \frac{1}{j\omega(1+j0.2\omega)(1+j0.05\omega)}$$

$\omega_{c1} = 1/0.2 = 5$ rad/sec and $\omega_{c2} = 1/0.05 = 20$ rad/sec.

$$G(j\omega) = \frac{1}{j\omega(1+j0.2\omega)(1+j0.05\omega)}$$

$$= \frac{1}{\omega\angle 90^\circ \sqrt{1+(0.2\omega)^2}\angle \tan^{-1}0.2\omega \sqrt{1+(0.05\omega)^2}\angle \tan^{-1}0.05\omega}$$

$$= \frac{1}{\omega\sqrt{1+(0.2\omega)^2}\sqrt{1+(0.05\omega)^2}} \angle(-90^\circ - \tan^{-1}0.2\omega - \tan^{-1}0.05\omega)$$

$$|G(j\omega)| = \frac{1}{\omega\sqrt{1+(0.2\omega)^2}\sqrt{1+(0.05\omega)^2}} \quad \text{and} \quad \angle G(j\omega) = -90^\circ - \tan^{-1}0.2\omega - \tan^{-1}0.05\omega$$

TABLE-1 : Magnitude and Phase of G(jω) at Various Frequencies

ω rad/sec	0.6	0.8	1	2	3	4
|G(jω)|	1.65	1.23	1.0	0.5	0.3	0.2
∠G(jω) deg	–98	–101	–104	–117.5	–129.4	–140

ω rad/sec	5	6	7	9	10	11	14
|G(jω)|	0.14	0.1	0.07	0.05	0.04	0.03	0.02
∠G(jω) deg	–149	–157	–164	–176	–180	–184	–195

TABLE-2 : Real and Imaginary Parts of G(jω) at Various Frequencies

ω rad/sec	0.6	0.8	1	2	3	4
$G_R(j\omega)$	–0.23	–0.23	–0.24	–0.23	–0.19	–0.15
$G_I(j\omega)$	–1.63	–1.21	–0.97	–0.44	–0.23	–0.13

ω rad/sec	5	6	7	9	10	11	14
$G_R(j\omega)$	–0.120	–0.092	–0.067	–0.050	–0.04	–0.030	–0.019
$G_I(j\omega)$	–0.072	–0.039	–0.019	–0.0034	0	0.002	0.005

there are two plots, marked as curve-I and curve-II. These two loci are sketched with different scales to clearly determine the gain margin and phase margin.

From the polar plot, with K = 1,

Gain margin, $K_g = 1/0.04 = 25$.

Gain margin in db = 20 log 25 = 28 db.

Phase margin, $\gamma = 76°$.

Case (i)

With K = 1, let G(jω) cut the −180° axis at point B and gain corresponding to that point be G_B. From the polar plot $G_B = 0.04$. The gain margin of 28 db with K = 1 has to be reduced to 18 db and so K has to be increased to a value greater than one.

Let G_A be the gain at −180° for a gain margin of 18 db.

$$\text{Now, } 20 \log \frac{1}{G_A} = 18 \quad \Rightarrow \quad \log \frac{1}{G_A} = \frac{18}{20} \quad \Rightarrow \quad \frac{1}{G_A} = 10^{18/20}$$

$$\therefore G_A = \frac{1}{10^{18/20}} = 0.125$$

The value of K is given by, $K = \dfrac{G_A}{G_B} = \dfrac{0.125}{0.04} = 3.125$

Case (ii)

With K = 1, the phase margin is 76°. This has to be reduced to 60°. Hence gain has to be increased.

Let ϕ_{gc2} be the phase of G(jω) for a phase margin of 60°

$$\therefore\ 60° = 180° + \phi_{gc2}$$

$$\phi_{gc2} = 60° - 180° = -120°$$

In the polar plot the –120° line cut the locus of G(jω) at point C and cut the unity circle at point D.

Let, G_C = Magnitude of G(jω) at point C.

G_D = Magnitude of G(jω) at point D.

From the polar plot, $G_C = 0.425$ and $G_D = 1$.

$$\text{Now, } K = \frac{G_D}{G_C} = \frac{1}{0.425} = 2.353$$

RESULT

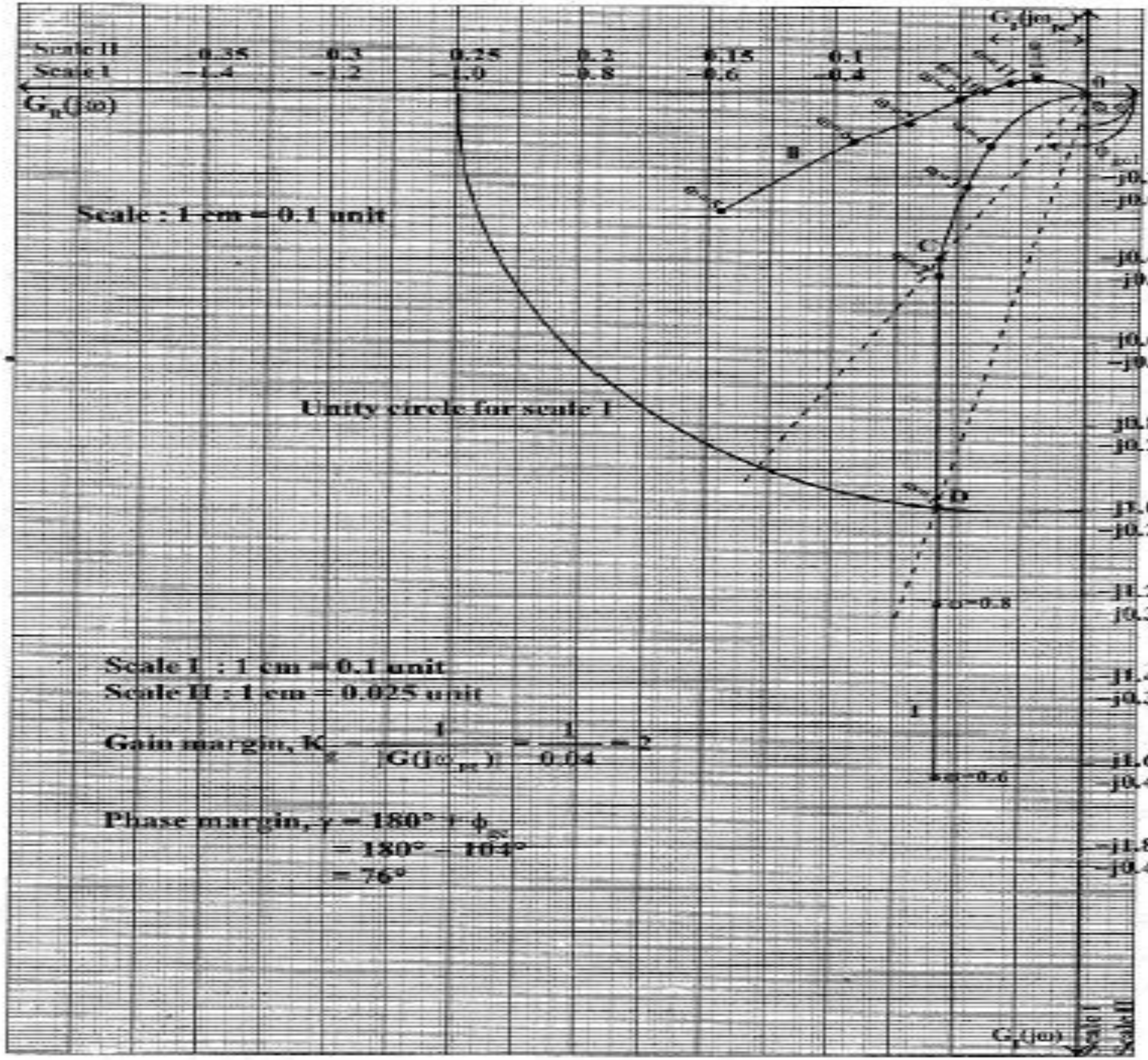
(a) When K = 1, Gain margin, K_g = 25

Gain margin in db = 28db

(b) When K = 1, Phase margin, γ = 76°

(c) For a gain margin of 18 db, K = 3.125

(d) For a phase margin of 60°, K = 2.353



Scale II
Scale I
-0.35
-1.4
0.3
-1.2
0.25
-1.0
0.2
-0.8
0.15
-0.6
0.1
-0.4
G_R(jω)
G_I(jω)
Scale : 1 cm = 0.1 unit
B
C
D
-j0.2
-j0.05
-j0.4
-j0.1
-j0.6
-j0.15
-j0.8
-j0.2
-j1.0
-j0.25
-j1.2
-j0.3
-j1.4
-j0.35
-j1.6
-j0.4
-j1.8
-j0.45
Unity circle for scale I
ω=0.8
ω=0.6
Scale I : 1 cm = 0.1 unit
Scale II : 1 cm = 0.025 unit
Gain margin, K_g = 1/|G(jω_pc)| = 1/0.04 = 2
Phase margin, γ = 180° + φ_gc
= 180° – 104°
= 76°

Transportation Lag

It is also called dead time or time delay.

In practical systems due to several reasons, it is necessary to stop certain action in a system for some time.

Such time delay is called transportation lag.

For example in modern systems using micro controllers, it is difficult to match the speed of peripherals with micro controller.

In such a case it is necessary to provide purposely a time delay to micro controllers to adjust with the speed of other supporting peripherals.

The transportation lag is given by the expression e^{-Ts} in Laplace domain

$$e^{-Ts}$$

$$G(S) = e^{-Ts}$$

$$e(j\omega) = e^{-j\omega T}$$

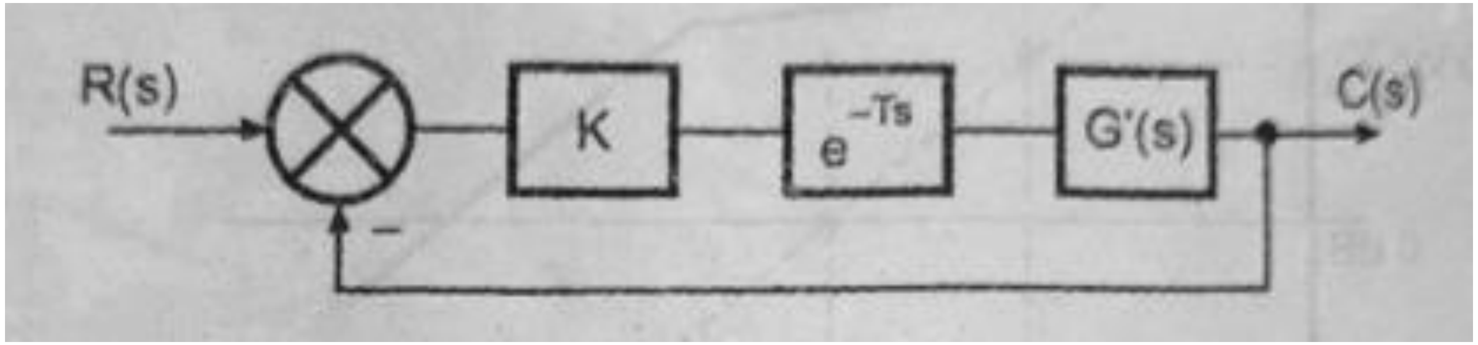
$$e(j\omega) = e^{-j\omega T}$$

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$e^{-j\omega} = \cos\theta - j\sin\omega T$$

$$|G(j\omega)| = \sqrt{\cos^2\omega T + \sin^2\omega T} = 1$$

$$\text{in dB} = 20\log 1 = 0\,\text{dB}$$



Introducing time delay in system has no effect on the magnitude plot.

But $\angle G(j\omega)$ of transportation lag is

$$= \tan^{-1}\left[\frac{\sin \omega T}{\cos \omega T}\right] = -\tan^{-1}(\tan \omega T) = -\omega T \text{ radians}$$

$$= -57.3\,\omega T \text{ degree}$$

The phase angle is linearly vary with ω

ω	$\angle G(j\omega) = -57.3\,\omega T$ degree $= -57.3\,\omega$ degree (for T = 1)
0.1	-5.73
0.5	-28.65
1	-114.6
5	-286.5
10	-573

<u>Nichols chart</u>

Nichols transformed the constant M and N circle contours on the polar plots to log magnitude versus phase angle plot.

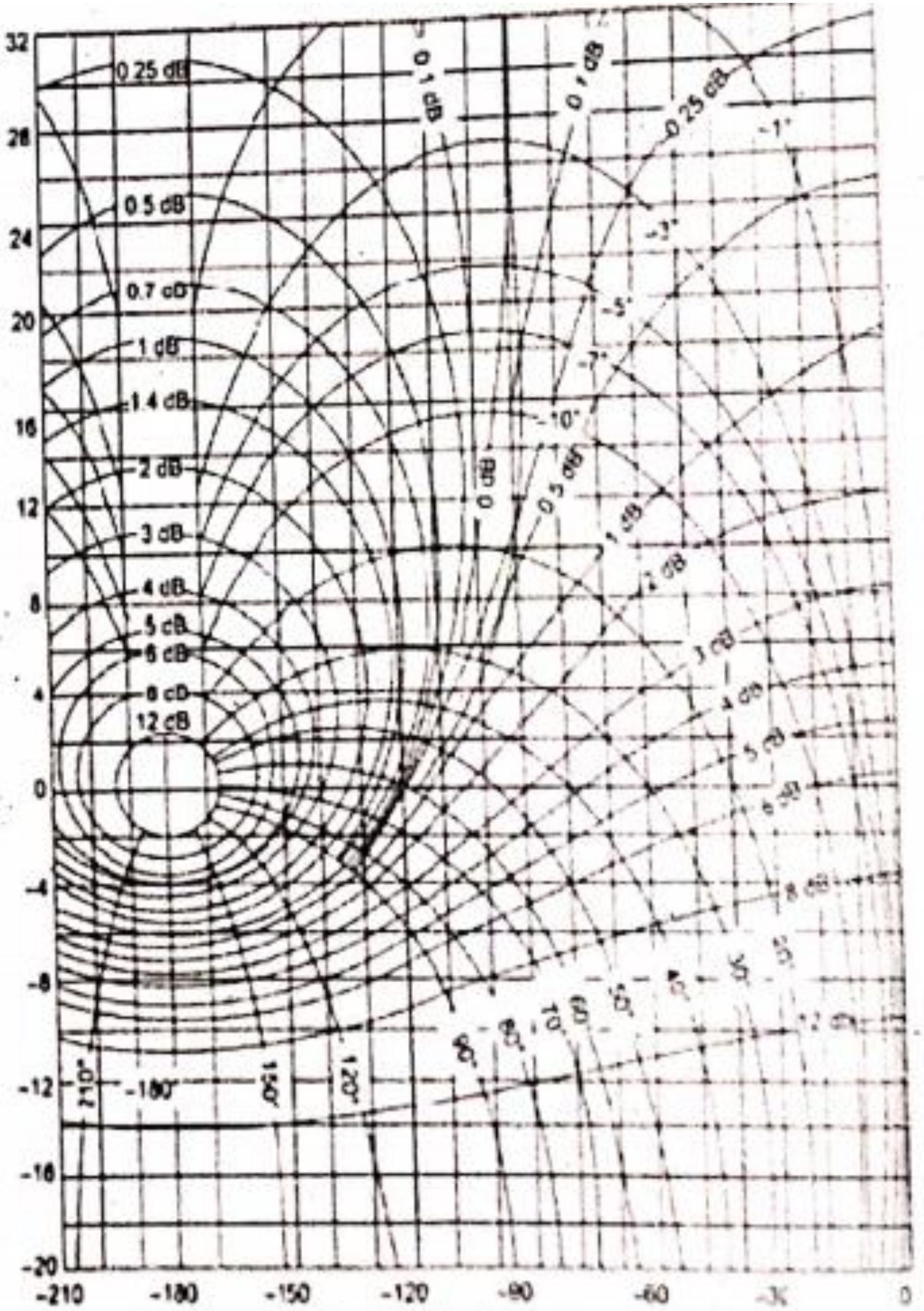
M circles are called constant magnitude loci while **N circles** are called as constant phase angle loci

The frequency response characteristics of a system can be studied by plotting the log magnitude in dB versus the phase angle for various frequencies.

When the open loop gain in dB versus loop phase angle in degree is plotted for different frequencies and M and N circles are superimposed on it, the resultant plot thus obtained is called Nichols chart

With the help of Nichols chart the following can be evaluated:

1. Complete closed loop frequency response.
2. Parameters M, bandwidth, gain and ω can be calculated for the closed loop system


32
28
24
20
16
12
8
4
0
-4
-8
-12
-16
-20
0.25 dB
0.5 dB
0.7 dB
1 dB
1.4 dB
2 dB
3 dB
4 dB
5 dB
6 dB
8 dB
12 dB
0.1 dB
0 dB
0.5 dB
1 dB
2 dB
3 dB
4 dB
5 dB
6 dB
8 dB
12 dB
-0.1 dB
-0.25 dB
-1°
-2°
-5°
-7°
-10°
210°
-180°
150°
120°
90°
80°
70°
60°
50°
40°
30°
20°
-210
-180
-150
-120
-90
-60
0
Loop Phase < (G) in degrees

The Nichols chart may be thought of as a Nyquist plot on a log scale.
A Nyquist plot is a plot in the complex plane of

$$G(j\omega) = \underbrace{\text{Re}(G(j\omega))}_{\text{x-coordinate}} + \underbrace{j\text{Im}(G(j\omega))}_{\text{y-coordinate}}$$

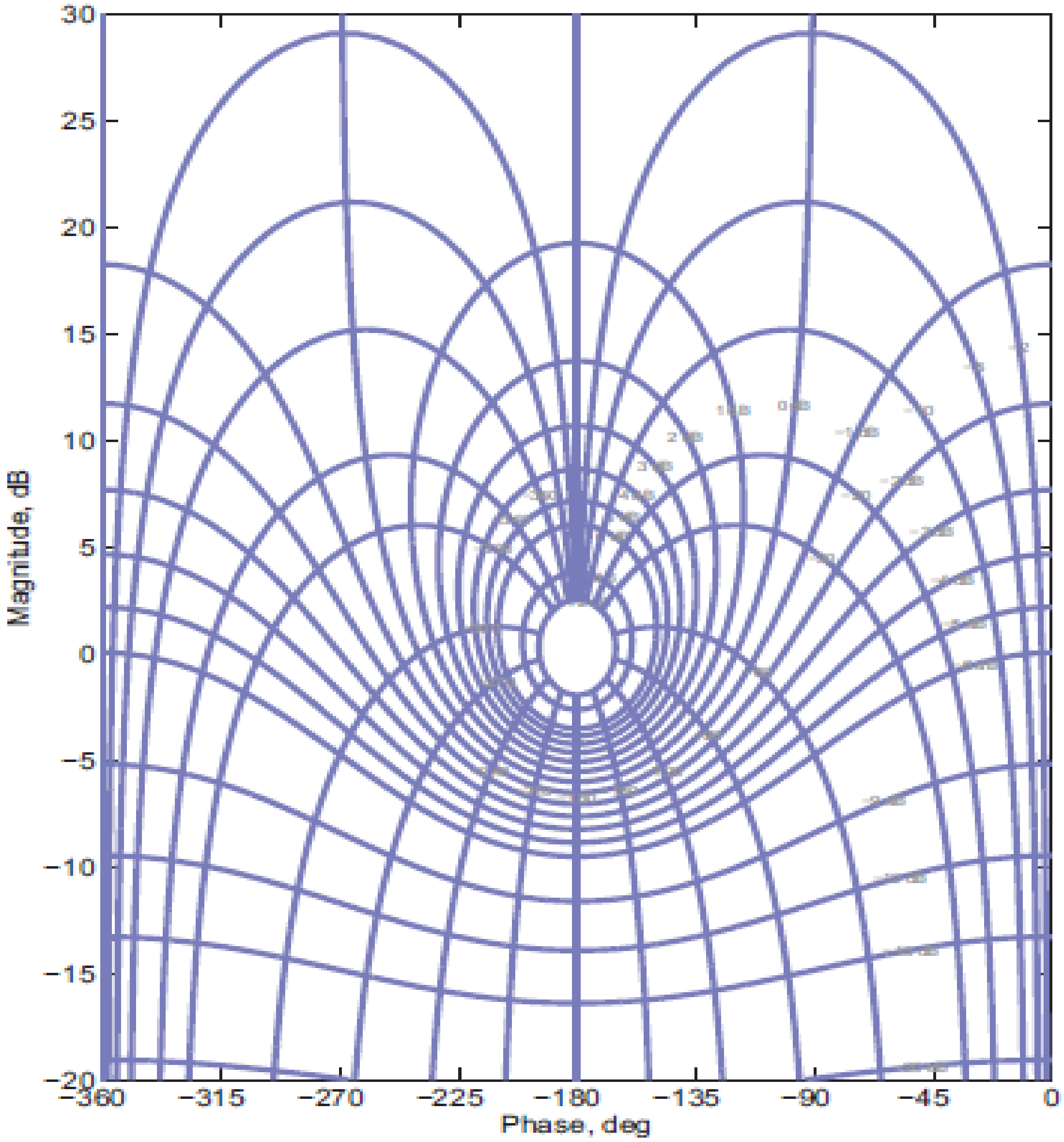
Instead, on a Nichols chart, we plot

$$\log\ G(j\omega) = \underbrace{\log|G(j\omega)|}_{\text{y-coordinate}} + j\underbrace{\angle(G(j\omega))}_{\text{x-coordinate}}$$

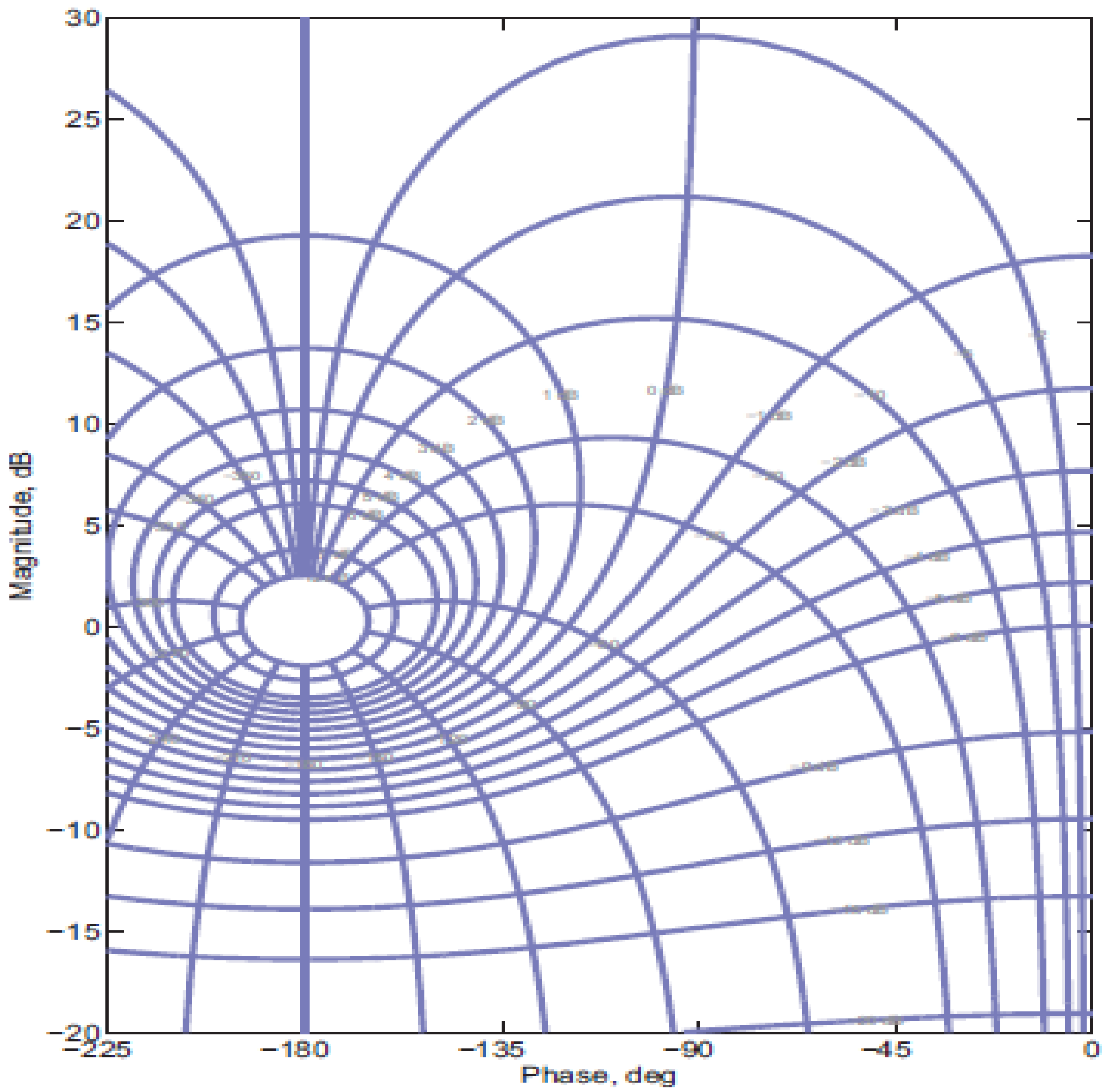
Notice that we reverse the coordinates -the real part is plotted on the vertical, and the imaginary part is plotted on the horizontal.
In addition, the chart has contours of constant closed-loop magnitude and phase,

$$M = \left|\frac{G}{1+G}\right|$$

$$N = \angle\frac{G}{1+G}$$



Magnitude, dB
30
25
20
15
10
5
0
−5
−10
−15
−20
−360
−315
−270
−225
−180
−135
−90
−45
0
Phase, deg



Magnitude, dB
30
25
20
15
10
5
0
−5
−10
−15
−20
−225
−180
−135
−90
−45
0
Phase, deg