MODULE 5

COMPLEX VARIABLE - RESIDUE INTEGRATION

Laurent Series

Let $f(z)$ be analytic in the annular region between two concentric circles C_1 and C_2 with center z_0 and radii r_1 and r_2. Then for any z in the annular region $r_1 < |z - z_0| < r_2$, $f(z)$ can be uniquely represented by a Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

$$= a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots\cdots + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots$$

consisting of nonnegative and negative powers.

1) Find the Laurent series of $z^{-5}\sin z$ with center 0

Ans:

$$z^{-5}\sin z = \frac{1}{z^5}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right]$$

$$= \frac{1}{z^4} - \frac{1}{z^2 \cdot 3!} + \frac{1}{5!} - \frac{z^2}{7!} + \cdots\cdots \qquad (0 < |z| < \infty)$$

Here the annulus of convergence is the whole complex plane without the origin since the singular point is $z=0$.

2) Find the Laurents series of $z^2 e^{1/z}$ with centre 0

$$f(z) = z^2 e^{1/z}$$

$$= z^2\left[1 + \frac{1}{z\cdot 1!} + \frac{1}{z^2\cdot 2!} + \frac{1}{z^3\cdot 3!} + \cdots\right]$$

$$= z^2 + z + \frac{1}{2!} + \frac{1}{z\cdot 3!} + \cdots$$

$f(z)$ has singularity $z=0$ $\therefore$ $f(z)$ is analytic in $|z|>0$ which is the region of convergence.

3) Develop $\frac{1}{(1-z)}$ (a) in non negative powers of z.
(b) in negative powers of z.

(a) $$\frac{1}{(1-z)} = (1-z)^{-1} = 1+z+z^2+z^3+\cdots$$ which is valid if $|z|<1$

(b) $$f(z) = \frac{1}{1-z} = \frac{1}{-z\left[1-\frac{1}{z}\right]}$$

$$= \frac{-1}{z}\left[1-\frac{1}{z}\right]^{-1}$$

$$= \frac{-1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right]$$

$$= -\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{z^4} - \cdots$$ which is valid for $\left|\frac{1}{z}\right|<1$

4) Find all Taylor and Laurent series of $f(z) = \dfrac{-2z+3}{z^2-3z+2}$ with centre 0

Ans: By partial fraction

$$\frac{-2z+3}{z^2-3z+2} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$$

[Radius of convergence R is = to the distance from z_0 to the nearest singular point.]

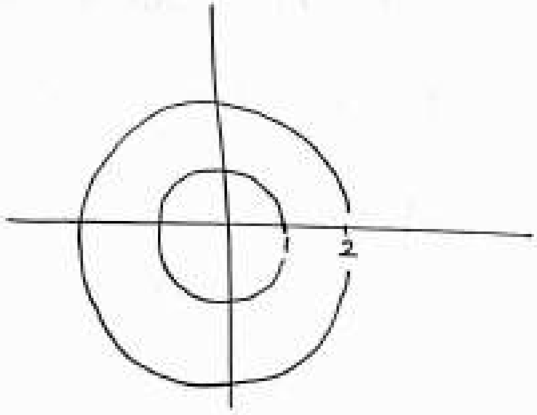
$$-2z+3 = A(z-2) + B(z-1)$$

$z=2 \Rightarrow -1 = \underline{\underline{B}}$

$z=+1 \Rightarrow 1 = -A \quad \therefore A = -1$

$$\frac{-2z+3}{z^2-3z+2} = \frac{-1}{(z-1)} + \frac{-1}{(z-2)}$$

Singular points are 1 & 2



$f(z)$ is analytic in the disc $|z|<1$

$$f(z) = \frac{-1}{z-1} + \frac{-1}{z-2}$$

$$= \frac{-1}{-(1-z)} + \frac{-1}{-2(1-z/2)}$$

$$= (1-z)^{-1} + \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= (1+z+z^2+z^3+\cdots) + \frac{1}{2}\left[1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots\right]$$

$$= \left(1+\frac{1}{2}\right) + \left(z + \frac{z}{4}\right) + \left(z^2 + \frac{z^2}{8}\right) + \left(z^3 + \frac{z^3}{16}\right) + \cdots$$

$$= \frac{3}{2} + \frac{5z}{4} + \frac{9z^2}{8} + \frac{17z^3}{16} + \cdots$$ is the taylor series expansion.

$f(z)$ is also analytic in annulus $1<|z|<2$.

$$f(z) = \frac{-1}{z-1} - \frac{1}{z-2}$$

$$= \frac{-1}{z\left[1-\frac{1}{z}\right]} - \frac{1}{-2\left[1-\frac{z}{2}\right]}$$

$$= \frac{-1}{z}\left[1-\frac{1}{z}\right]^{-1} + \frac{1}{2}\left[1-\frac{z}{2}\right]^{-1}$$

$$= \frac{-1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right] + \frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right]$$

$$= \frac{1}{2}+\frac{z}{4}+\frac{z^2}{8}+\frac{z^3}{16}+\cdots - \frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}-\frac{1}{z^4}-\cdots$$

is the Laurents series expansion

$f(z)$ is also analytic in the annulus $|z|>2$

$$f(z) = \frac{-1}{(z-1)} - \frac{1}{(z-2)}$$

$$= \frac{-1}{z\left(1-\frac{1}{z}\right)} - \frac{1}{z\left(1-\frac{2}{z}\right)}$$

$$= \frac{-1}{z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$= \frac{-1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right] - \frac{1}{z}\left[1+\frac{2}{z}+\frac{4}{z^2}+\cdots\right]$$

$$= \frac{-1}{z}-\frac{1}{z^2}-\frac{1}{z^3}-\cdots - \frac{1}{z}-\frac{2}{z^2}-\frac{4}{z^3}-\cdots$$

$$= \frac{-2}{z}-\frac{3}{z^2}-\frac{5}{z^3}-\cdots$$ is the Laurents series expansion

Example 6.3.5. *Expand* $f(z) = \dfrac{z}{(z+1)(z+2)}$ *in Laurent's series about* $z = -2$ *in* $0 < |z+2| < 1$.

$$
\begin{aligned}
\text{Let} \quad \frac{z}{(z+1)(z+2)} &= \frac{A}{z+1} + \frac{B}{z+2} \\
&= \frac{A(z+2) + B(z+1)}{(z+1)(z+2)} \\
\implies z &= A(z+2) + B(z+1) \\
z = -1 &\implies A = -1 \\
z = -2 &\implies B = 1 \\
\therefore f(z) &= -\frac{1}{z+1} + \frac{1}{z+2}
\end{aligned}
$$

Laurent's series about $z = -2$

$$
\begin{aligned}
f(z) &= -\frac{1}{(z+2) - 2 + 1} + \frac{1}{z+2} \\
&= -\frac{1}{(z+2) - 1} + \frac{1}{z+2} \\
&= \frac{1}{1 - (z+2)} + \frac{1}{z+2} \\
&= \frac{1}{z+2} + [1 - (z+2)]^{-1} \\
&= \frac{1}{z+2} + 1 + (z+2) + (z+2)^2 + \cdots \text{ provided } 0 < |z+2| < 1.
\end{aligned}
$$

Example 6.3.6. *Find the Laurent's series expansion of* $\dfrac{1}{(z-1)(z-2)}$ *in the regions (i)* $|z| < 1$ *(ii)* $1 < |z| < 2$ *(iii)* $|z| > 2$ *and (iv)* $0 < |z-1| < 1$.

$$
\begin{aligned}
\text{Let} \quad \frac{1}{(z-1)(z-2)} &= \frac{A}{z-1} + \frac{B}{z-2} \\
&= \frac{A(z-2) + B(z-1)}{(z-1)(z-2)} \\
\implies 1 &= A(z-2) + B(z-1) \\
z = 1 &\implies A = -1 \\
z = 2 &\implies B = 1 \\
\therefore f(z) &= -\frac{1}{z-1} + \frac{1}{z-2}
\end{aligned}
$$

(i) When $|z| < 1$

$$
\begin{aligned}
f(z) &= -\frac{1}{z-1} + \frac{1}{z-2} \\
&= -\frac{1}{(-1)(1-z)} + \frac{1}{(-2)\left[1-\left(\frac{z}{2}\right)\right]} \\
&= (1-z)^{-1} - \frac{1}{2}\left[1-\left(\frac{z}{2}\right)\right]^{-1} \\
&= 1 + z + z^2 + \cdots - \frac{1}{2}\left[1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \cdots\right]
\end{aligned}
$$

provided $|z| < 1$ and $\left|\frac{z}{2}\right| < 1 \implies |z| < 1$ and $|z| < 2 \implies |z| < 1$

$$
= \frac{1}{2} + \frac{3z}{2} - \frac{7z^2}{8} \cdots
$$

(ii) $1 < |z| < 2$

$$
\begin{aligned}
f(z) &= -\frac{1}{z-1} + \frac{1}{z-2} \\
&= -\frac{1}{z\left[1-\left(\frac{1}{z}\right)\right]} + \frac{1}{(-2)\left[1-\left(\frac{z}{2}\right)\right]} \\
&= -\frac{1}{z}\left[1-\left(\frac{1}{z}\right)\right]^{-1} - \frac{1}{2}\left[1-\left(\frac{z}{2}\right)\right]^{-1} \\
&= -\frac{1}{z}\left[1 + \left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 + \cdots\right] - \frac{1}{2}\left[1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \cdots\right]
\end{aligned}
$$

provided $\left|\frac{1}{z}\right| < 1$ and $\left|\frac{z}{2}\right| < 1 \implies 1 < |z|$ and $|z| < 2 \implies 1 < |z| < 2$

(iii) $|z| > 2$

$$
\begin{aligned}
f(z) &= -\frac{1}{z-1} + \frac{1}{z-2} \\
&= -\frac{1}{z\left[1-\left(\frac{1}{z}\right)\right]} + \frac{1}{z\left[1-\left(\frac{2}{z}\right)\right]} \\
&= -\frac{1}{z}\left[1-\left(\frac{1}{z}\right)\right]^{-1} + \frac{1}{z}\left[1-\left(\frac{2}{z}\right)\right]^{-1} \\
&= -\frac{1}{z}\left[1 + \left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 + \cdots\right] + \frac{1}{z}\left[1 + \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 + \cdots\right]
\end{aligned}
$$

provided $\left|\frac{1}{z}\right| < 1$ and $\left|\frac{2}{z}\right| < 1 \implies 1 < |z|$ and $2 < |z| \implies |z| > 2$

(iv) $0 < |z-1| < 1$

$$\begin{aligned}
f(z) &= -\frac{1}{z-1} + \frac{1}{z-2} \\
&= -\frac{1}{z-1} + \frac{1}{(z-1)-1} \\
&= -\frac{1}{z-1} + \frac{1}{(-1)[1-(z-1)]} \\
&= -\frac{1}{z-1} - [1-(z-1)]^{-1} \\
&= -\frac{1}{z-1} - [1+(z-1)+(z-1)^2+\cdots] \qquad \text{provided } |z-1|<1
\end{aligned}$$

Example 6.3.7. *Find the Laurent's series expansion of* $\frac{1}{z-z^3}$ *in* $1 < |z+1| < 2$.

$$\begin{aligned}
\frac{1}{z-z^3} &= \frac{1}{z(1-z^2)} \\
&= \frac{1}{z(1+z)(1-z)}
\end{aligned}$$

Let
$$\begin{aligned}
\frac{1}{z-z^3} &= \frac{A}{z} + \frac{B}{z+1} + \frac{C}{1-z} \\
&= \frac{A(z+1)(1-z)+Bz(1-z)+Cz(z+1)}{(z+1)(z+2)} \\
\implies 1 &= A(z+1)(1-z)+Bz(1-z)+Cz(z+1)
\end{aligned}$$

$$z=0 \implies A=1,\ z=-1 \implies B=-\frac{1}{2},\ z=1 \implies B=\frac{1}{2}$$

$$\begin{aligned}
\therefore f(z) &= \frac{1}{z} + \frac{\left(-\frac{1}{2}\right)}{z+1} + \frac{\left(\frac{1}{2}\right)}{1-z} \\
&= \frac{1}{z} - \frac{1}{2(z+1)} + \frac{1}{2(1-z)} \\
&= \frac{1}{(z+1)-1} - \frac{1}{2(z+1)} + \frac{1}{2[1-((z+1)-1)]} \\
&= \frac{1}{(z+1)-1} - \frac{1}{2(z+1)} + \frac{1}{2[2-(z+1)]} \\
&= \frac{1}{(z+1)\left[1-\left(\frac{1}{z+1}\right)\right]} - \frac{1}{2(z+1)} + \frac{1}{4\left[1-\left(\frac{z+1}{2}\right)\right]}
\end{aligned}$$

$$= \frac{1}{(z+1)}\left[1-\left(\frac{1}{z+1}\right)\right]^{-1} - \frac{1}{2(z+1)} + \frac{1}{4}\left[1-\left(\frac{z+1}{2}\right)\right]^{-1}$$
$$= \frac{1}{(z+1)}\left[1+\left(\frac{1}{z+1}\right)+\left(\frac{1}{z+1}\right)^2+\cdots\right] - \frac{1}{2(z+1)}$$
$$+\frac{1}{4}\left[1+\left(\frac{z+1}{2}\right)+\left(\frac{z+1}{2}\right)^2+\cdots\right]$$

Example 6.3.8. *If $f(z) = \frac{1}{z^2}$, find the Taylor series that converges in $|z-i| < R$ and the Laurent's series that converges in $|z-i| > R$.*

$$f(z) = \frac{1}{z^2} = \frac{1}{[(z-i)+i]^2}$$
$$= \frac{1}{i^2\left[1+\left(\frac{z-i}{i}\right)\right]^2}$$
$$= -\left[1+\left(\frac{z-i}{i}\right)\right]^{-2}$$
$$= -\left[1-2\left(\frac{z-i}{i}\right)+3\left(\frac{z-i}{i}\right)^2-\cdots\right]$$
$$\text{provided } \left|\frac{z-i}{i}\right| < 1 \implies |z-i| < |i| = 1$$
$$f(z) = \frac{1}{z^2} - \left[1-2\left(\frac{z-i}{i}\right)+3\left(\frac{z-i}{i}\right)^2-\cdots\right] \text{ for } |z-i|<1$$
$$f(z) = \frac{1}{z^2} = \frac{1}{[(z-i)+i]^2}$$
$$= \frac{1}{(z-i)^2\left[1+\left(\frac{i}{z-i}\right)\right]^2}$$
$$= \frac{1}{(z-i)^2}\left[1+\left(\frac{i}{z-i}\right)\right]^{-2}$$
$$= \frac{1}{(z-i)^2}\left[1-2\left(\frac{i}{z-i}\right)+3\left(\frac{i}{z-i}\right)^2-\cdots\right]$$
$$\text{provided } \left|\frac{i}{z-i}\right| < 1 \implies |i| < |z-i|$$
$$f(z) = \frac{1}{(z-i)^2}\left[1-2\left(\frac{i}{z-i}\right)+3\left(\frac{i}{z-i}\right)^2-\cdots\right] \text{ for } 1<|z-i|$$

Example 6.3.9. *Find Laurent's series expansion for $\frac{z^2-1}{z^2+5z+6}$ in (i) $2<|z|<3$, (ii) $1<|z+1|<2$ and (iii) $|z|>3$*

$$\frac{z^2-1}{z^2+5z+6} = 1 + \frac{(-5z-7)}{z^2+5z+6} \quad \text{(on division)}$$
$$= 1 - \frac{5z+7}{(z+2)(z+3)}$$

Let $$\frac{5z+7}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3}$$
$$= \frac{A(z+3)+B(z+2)}{(z+2)(z+3)}$$
$$\implies 5z+7 = A(z+3)+B(z+2)$$
$$z=-2 \implies A=-3$$
$$z=-3 \implies B=8$$

$$\therefore f(z) = 1 - \left[\frac{(-3)}{z+2} + \frac{8}{z+3}\right]$$
$$= 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

(i) When $2 < |z| < 3$

$$f(z) = 1 + \frac{3}{z\left[1+\left(\frac{2}{z}\right)\right]} - \frac{8}{3\left[1+\left(\frac{z}{3}\right)\right]}$$
$$= 1 + \frac{3}{z}\left[1+\left(\frac{2}{z}\right)\right]^{-1} - \frac{8}{3}\left[1+\left(\frac{z}{3}\right)\right]^{-1}$$
$$= 1 + \frac{3}{z}\left[1-\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2-\cdots\right] - \frac{8}{3}\left[1-\left(\frac{z}{3}\right)+\left(\frac{z}{3}\right)^2-\cdots\right]$$
provided $\left|\frac{2}{z}\right| < 1$ and $\left|\frac{z}{3}\right| < 1 \implies 2 < |z|$ and $|z| < 3 \implies 2 < |z| < 3$

(ii) When $1 < |z+1| < 2$

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$
$$= 1 + \frac{3}{(z+1)+1} - \frac{8}{(z+1)+2}$$
$$= 1 + \frac{3}{(z+1)\left[1+\left(\frac{1}{z+1}\right)\right]} - \frac{8}{2\left[1+\left(\frac{z+1}{2}\right)\right]}$$

$$= 1 + \frac{3}{(z+1)}\left[1 + \left(\frac{1}{z+1}\right)\right]^{-1} - 4\left[1 + \left(\frac{z+1}{2}\right)\right]^{-1}$$

$$= 1 + \frac{3}{(z+1)}\left[1 - \left(\frac{1}{z+1}\right) + \left(\frac{1}{z+1}\right)^2 - \cdots\right]$$

$$- 4\left[1 - \left(\frac{z+1}{2}\right) + \left(\frac{z+1}{2}\right)^2 - \cdots\right]$$

provided $\left|\frac{1}{z+1}\right| < 1$ and $\left|\frac{z+1}{2}\right| < 1 \implies 1 < |z+1|$ and $|z+1| < 2$

$\implies 1 < |z+1| < 2$

(iii) When $|z| > 3$

$$f(z) = 1 + \frac{3}{z\left[1 + \left(\frac{2}{z}\right)\right]} - \frac{8}{z\left[1 + \left(\frac{3}{z}\right)\right]}$$

$$= 1 + \frac{3}{z}\left[1 + \left(\frac{2}{z}\right)\right]^{-1} - \frac{8}{z}\left[1 + \left(\frac{3}{z}\right)\right]^{-1}$$

$$= 1 + \frac{3}{z}\left[1 - \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 - \cdots\right] - \frac{8}{z}\left[1 - \left(\frac{3}{z}\right) + \left(\frac{3}{z}\right)^2 - \cdots\right]$$

provided $\left|\frac{2}{z}\right| < 1$ and $\left|\frac{3}{z}\right| < 1 \implies 2 < |z|$ and $3 < |z| \implies |z| > 3$

Example 6.3.10. *Expand $\frac{e^{2z}}{(z-1)^3}$ about $z = 1$ as a Laurent's series.*

$$\frac{e^{2z}}{(z-1)^3} = \frac{e^{2[(z-1)+1]}}{(z-1)^3}$$

$$= \frac{e^2\, e^{2(z-1)}}{(z-1)^3}$$

$$= \frac{e^2}{(z-1)^3}\left[1 + \frac{2(z-1)}{1!} + \frac{(2(z-1))^2}{2!} + \cdots\right]$$

$$= e^2\left[\frac{1}{(z-1)^3} + \frac{2}{(z-1)^2} + \frac{2^2}{2!(z-1)} + \frac{2^3}{3!} + \frac{2^4(z-1)}{4!} + \cdots\right]$$

Example 6.3.11. *Expand $f(z) = \frac{\sin z}{z - \pi}$ as Laurent's series about $z = \pi$.*

$$
\begin{aligned}
f(z) &= \frac{\sin z}{z-\pi} \\
&= \frac{\sin[(z-\pi)+\pi]}{z-\pi} \\
&= \frac{-\sin(z-\pi)}{z-\pi} \\
&= \frac{-1}{z-\pi}\left[(z-\pi) - \frac{(z-\pi)^3}{3!} + \frac{(z-\pi)^5}{5!} + \cdots\right] \\
&= -1 + \frac{(z-\pi)^2}{3!} - \frac{(z-\pi)^4}{5!} - \cdots
\end{aligned}
$$

SINGULARITIES

A function $f(z)$ is singular or has a singularity at a point $z = z_0$ if $f(z)$ is not analytic at $z = z_0$.

A singularity of a function $f(z)$ is a point at which the function ceases to be analytic.

Isolated and non-isolated singularity

If $z = a$ is a singularity of $f(z)$ and if there is no other singularity with in a small circle surrounding to the point $z = a$, then $z = a$ is said to be an isolated singularity of the function $f(z)$. Otherwise it is called non isolated.

eg: isolated singularity

$$f(z) = \frac{z+1}{z(z-2)}$$

$$z(z-2) = 0$$

$z = 0$, $z = 2$ are the only singularities of the function.

More over we can draw a small circle around each of these points so that it will contain any other singularities other than itself. Thus $z = 0$ and $z = 2$ are isolated singularities.

eg: non-isolated singularity

Consider the function $f(z) = \dfrac{1}{\tan\left(\frac{\pi}{z}\right)}$

The function is not analytic at the points $\tan\left(\frac{\pi}{z}\right) = 0$

$$\therefore \frac{\pi}{z} = n\pi$$

$$\frac{1}{z} = n$$

$\therefore z = \frac{1}{n} \ (n = 1, 2, 3 \cdots)$

$\therefore z = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\cdots 0$ are the singularities of the function. Here $z=0$ is a non isolated singularity, because in the neighbourhood of $z=0$ there are infinite no of singularities $z = \frac{1}{n}$.

Different types of Isolated Singularities

Let $f(z)$ is analytic with in a domain D except at a point $z=a$, which is an isolated singularity. Draw two concentric circles of centre 'a' both lying in D with $r_1 < r_2$. In the annulus between these two circles we can expand $f(z)$ in a Laurents series in powers of $(z-a)$. Let this expansion be

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

$$= a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2} + \cdots \quad \text{①}$$

The part $\sum_{n=0}^{\infty} a_n (z-a)^n$ is called analytic part while the part $\sum_{n=1}^{\infty} b_n (z-a)^{-n}$ is called the principal part of $f(z)$. In relation to the principal part there are three possibilities.

I Removable Singularity :

All the coefficients b_n in ① be zero, that is there is no principal part in ①. The remaining terms are the Taylor series expansion of $f(z)$ and this function $f(z)$ can be made analytic by suitably defining its value at 'a'. This type of singularity which can be made to disappear by defining the function suitably is called removable singularity.

eg: $f(z) = \dfrac{\sin(z-a)}{(z-a)}$ has a removable singularity at $z=a$

because $\dfrac{\sin(z-a)}{(z-a)} = \dfrac{1}{(z-a)}\left[(z-a) - \dfrac{(z-a)^3}{3!} + \dfrac{(z-a)^5}{5!} - \cdots\right]$

$$= 1 - \frac{(z-a)^2}{3!} + \frac{(z-a)^4}{5!} - \cdots$$

has no term containing negative powers of $(z-a)$. Hence $z=a$ is a removable singularity of $f(z)$.

II <u>Essential Singularity</u>

Let the expansion of ① contains an infinite number of negative powers of $(z-a)$. In this case the point $z=a$ is said to be an essential singularity of $f(z)$.

eg: $f(z) = \sin\left(\dfrac{1}{z-a}\right)$ has essential singularity at $z=a$ because

$$\sin\left(\frac{1}{z-a}\right) = \frac{1}{z-a} - \frac{\frac{1}{(z-a)^3}}{3!} + \frac{\frac{1}{(z-a)^5}}{5!} - \cdots$$

$$= \frac{1}{(z-a)} - \frac{1}{6(z-a)^3} + \frac{1}{120(z-a)^5} - \cdots$$

has infinite number of terms in negative powers of $(z-a)$

III <u>Poles</u>

If the principal part of expansion ① contains only the single term $\dfrac{b_1}{z-a}$, then the singularity at a is known as a simple pole or pole of order 1

If the principal part contains a finite number of terms

as $\frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2} + \cdots + \frac{b_m}{(z-a)^m}$

Then $z=a$ is said to be a pole of order m.

eg: $f(z) = \frac{z}{z-1} \Rightarrow z=1$ is a simple pole

$f(z) = \frac{z}{(z-1)^3} \Rightarrow z=1$ is a pole of order 3.

<u>The zero of an analytic function</u>

A zero of an analytic function $f(z)$ is the value of z such that $f(z)=0$

eg: $f(z) = \frac{(z+1)(z-2)}{(z-3)(z+2)}$

$f(z)=0 \Rightarrow \frac{(z+1)(z-2)}{(z-3)(z+2)} = 0$

$(z+1)(z-2) = 0$

$z = -1, 2$ are the zero's.

Q1) Determine and classify the singular points for the following function.

(i) $f(z) = \frac{\sin z}{(z-\pi)^2}$ (ii) $g(z) = (z+i)^2 e^{\frac{1}{z+i}}$

Ans:

(i) $f(z) = \frac{\sin z}{(z-\pi)^2}$

$$= \frac{\sin[(z-\pi)+\pi]}{(z-\pi)^2}$$

$$= \frac{\sin[\pi + (z-\pi)]}{(z-\pi)^2}$$

$$= \frac{-\sin(z-\pi)}{(z-\pi)^2}$$

$$= \frac{-1}{(z-\pi)^2}\left[(z-\pi) - \frac{(z-\pi)^3}{3!} + \frac{(z-\pi)^5}{5!} - \cdots\right]$$

$$= -\frac{1}{(z-\pi)} + \frac{(z-\pi)}{3!} - \frac{(z-\pi)^3}{5!} + \cdots$$

$z = \pi$ is a simple pole.

(ii) $g(z) = (z+i)^2 \, e^{\left(\frac{1}{z+i}\right)}$

$$= (z+i)^2\left[1 + \frac{\frac{1}{(z+i)}}{1!} + \frac{\frac{1}{(z+i)^2}}{2!} + \frac{\frac{1}{(z+i)^3}}{3!} + \cdots\right]$$

$$= (z+i)^2\left[1 + \frac{1}{(z+i)} + \frac{1}{2(z+i)^2} + \frac{1}{6(z+i)^3} + \cdots\right]$$

$$= (z+i)^2 + (z+i) + \frac{1}{2} + \frac{1}{6(z+i)} + \cdots$$

$z = -i$ is an essential singularity.

Q2) Determine the nature and type of singularities of

(i) $\frac{e^{-z^2}}{z^2}$ (ii) $z \sin\left(\frac{1}{z}\right)$

Ans:

$$f(z) = \frac{e^{-z^2}}{z^2}$$

$$= \frac{1}{z^2}\left[1 + \frac{-z^2}{1!} + \frac{(-z^2)^2}{2!} + \frac{(-z^2)^3}{3!} + \cdots\right]$$

$$= \frac{1}{z^2}\left[1 - \frac{z^2}{1} + \frac{z^4}{2} - \frac{z^6}{6} + \cdots\cdots\right]$$

$$= \frac{1}{z^2} - 1 + \frac{z^2}{2} - \frac{z^4}{6} + \cdots\cdots$$

$z = 0$ is a pole of order 2

(ii) $f(z) = z \sin\left(\frac{1}{z}\right)$

$$= z\left[\frac{1}{z} - \frac{\left(\frac{1}{z}\right)^3}{3!} + \frac{\left(\frac{1}{z}\right)^5}{5!} - \cdots\right]$$

$$= z\left[\frac{1}{z} - \frac{1}{6z^3} + \frac{1}{120z^5} - \cdots\right]$$

$$= 1 - \frac{1}{6z^2} + \frac{1}{120z^4} - \cdots\cdots$$

$z = 0$ is an essential singularity.

RESIDUE INTEGRATION METHOD

Formulas for Residues

A formula for the residue at a simple pole is

$$\boxed{\operatorname*{Res}_{z=z_0} f(z) = b_1 = \lim_{z \to z_0} (z-z_0) f(z)}$$

A second formula for the residue at a simple pole is

$$\boxed{\operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}}$$

here we assume that $f(z) = \dfrac{p(z)}{q(z)}$

The residue of $f(z)$ at an m^{th} order pole at z_0 is

$$\boxed{\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \left\{ \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z)\right] \right\}}$$

In particular for a second order pole ie) $m=2$

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{1!} \lim_{z \to z_0} \frac{d}{dz}\left[(z-z_0)^2 f(z)\right]$$

Q Find all singularities and the corresponding residues

1) $f(z) = \dfrac{9z+i}{z^3+z}$

Ans: poles are obtained by $z^3+z=0$

$$z(z^2+1) = 0$$

$$z=0 \quad \text{or} \quad (z^2+1)=0$$

$z = 0, \quad z^2 = -1$

$z = 0, +i, -i$ are the simple poles.

$$\underset{z=0}{\text{Res}}\, f(z) = \underset{z \to 0}{Lt}\, (z-0)\, f(z)$$

$$= \underset{z \to 0}{Lt}\, (z-0) \frac{9z+i}{z^3+z}$$

$$= \underset{z \to 0}{Lt}\, (z-0) \frac{9z+i}{z(z^2+1)}$$

$$= \underset{z \to 0}{Lt}\, \frac{9z+i}{z^2+1} = \frac{i}{1} = \underline{\underline{i}}$$

<u>OR</u>

$$\underset{z=0}{\text{Res}}\, f(z) = \underset{z=0}{\text{Res}}\, \frac{P(z)}{Q(z)}$$

$P(z) = 9z + i$

$Q(z) = z^3 + z$

$Q'(z) = 3z^2 + 1$

$$= \frac{P(0)}{Q'(0)}$$

$$= \frac{(9 \times 0 + i)}{(3 \times 0^2 + 1)} = \frac{i}{1} = \underline{\underline{i}}$$

$$\underset{z=i}{\text{Res}}\, f(z) = \underset{z=i}{\text{Res}}\, \frac{P(z)}{Q(z)}$$

$$= \frac{P(i)}{Q'(i)} = \frac{9i+i}{3(i)^2+1}$$

$$= \frac{10i}{-2} = \underline{\underline{-5i}}$$

$$\operatorname{Res}_{z=-i} f(z) = \lim_{z \to -i} \frac{P(z)}{Q(z)}$$

$$= \frac{P(-i)}{Q'(-i)}$$

$$= \frac{9(-i)+i}{3(-i)^2+1}$$

$$= \frac{-8i}{-2}$$

$$= \underline{\underline{4i}}$$

2) $f(z) = \dfrac{50z}{z^3+2z^2-7z+4}$

poles are obtained by $z^3+2z^2-7z+4=0$

$z = -4, 1, 1$

$z=-4$ is a simple pole

$z=1$ is a pole of order 2

$$\operatorname{Res}_{z=-4} f(z) = \lim_{z \to -4} \frac{P(z)}{Q(z)}$$

$P(z) = 50z$

$Q(z) = z^3+2z^2-7z+4$

$Q'(z) = 3z^2+4z-7$

$$= \frac{P(-4)}{Q'(-4)}$$

$$= \frac{50(-4)}{3(-4)^2+4(-4)-7}$$

$$= \frac{-200}{25} = \underline{\underline{-8}}$$

$z=1$ is a pole of order $2(=m)$

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z\to z_0} \left\{ \frac{d^{m-1}}{dz^{m-1}} [(z-z_0)^m f(z)] \right\}$$

$$\operatorname*{Res}_{z=1} f(z) = \frac{1}{1!} \lim_{z\to 1} \frac{d}{dz}\left[(z-1)^2 \frac{50z}{z^3+2z^2-7z+4}\right]$$

$$= \lim_{z\to 1} \frac{d}{dz}\left[\frac{(z-1)^2\, 50z}{(z-1)^2(z+4)}\right]$$

$$= \lim_{z\to 1} \frac{d}{dz}\left[\frac{50z}{z+4}\right]$$

$$= \lim_{z\to 1} \frac{(z+4)50 - 50z(1)}{(z+4)^2}$$

$$= \frac{250-50}{25} = \underline{\underline{8}}$$

3) $f(z) = \dfrac{\sin 2z}{z^6}$

poles are obtained by $z^6=0$

$z=0$ is a pole of order 6

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{5!} \lim_{z\to 0} \frac{d^5}{dz^5}\left[(z-0)^6 \frac{\sin 2z}{z^6}\right]$$

$$= \frac{1}{120} \lim_{z\to 0} \frac{d^5}{dz^5}[\sin 2z]$$

$$= \frac{1}{120} \lim_{z\to 0} \frac{d^4}{dz^4}(2\cos 2z)$$

$$= \frac{1}{120} \lim_{z \to 0} \frac{d^3}{dz^3}(-4 \sin 2z)$$

$$= \frac{1}{120} \lim_{z \to 0} \frac{d^2}{dz^2}(-8 \cos 2z)$$

$$= \frac{1}{120} \lim_{z \to 0} \frac{d}{dz}(16 \sin 2z)$$

$$= \frac{1}{120} \lim_{z \to 0} 32 \cos 2z$$

$$= \frac{1}{120} \times 32 \cos(0)$$

$$= \frac{32}{120} = \frac{4}{15}$$

4) $f(z) = \dfrac{8}{1+z^2}$

poles are obtained by $1+z^2=0$

$z^2 = -1$

$z = +i, -i$ are the simple poles.

$$\operatorname*{Res}_{z=i} f(z) = \lim_{z \to i} \frac{P(z)}{Q(z)}$$

$P(z) = 8$

$Q(z) = 1+z^2$

$Q'(z) = 2z$

$$= \frac{P(i)}{Q'(i)}$$

$$= \frac{8}{2i} = \frac{4}{i}$$

$$= -4i$$

$$\operatorname*{Res}_{z=-i} f(z) = \operatorname*{Res}_{z=-i} \frac{P(z)}{Q(z)}$$

$$= \frac{P(-i)}{Q'(-i)}$$

$$= \frac{8}{2(-i)} = \frac{-4}{i}$$

$$= \underline{\underline{4i}}$$

H.W 5 $f(z) = \dfrac{z+2}{(z+1)^2(z-2)}$

$z=2$ is a simple pole

$z=-1$ is a pole of order 2

$$\operatorname*{Res}_{z=2} f(z) = \frac{4}{9}$$

$$\operatorname*{Res}_{z=-1} f(z) = \frac{-4}{9}$$

H.W 6 $f(z) = \dfrac{e^z}{z^2+\pi^2}$

$z = +\pi i, -\pi i$ are simple poles

$$\operatorname*{Res}_{z=\pi i} f(z) = \frac{i}{2\pi}$$

$$\operatorname*{Res}_{z=-\pi i} f(z) = \underline{\underline{\frac{-i}{2\pi}}}$$

Residue Theorem:

Let $f(z)$ be analytic inside a simple closed path c and on c, except for finitely many singular points $z_1, z_2 \cdots z_k$ inside c. Then the integral of $f(z)$ taken counterclockwise around c equals $2\pi i$ times the sum of residues of $f(z)$ at $z_1, z_2 \cdots z_k$.

ie) $$\oint_c f(z)dz = 2\pi i \sum_{j=1}^{k} \underset{z=z_j}{\text{Res}} f(z)$$

Problems

1 Evaluate $\int_c \frac{z^2 dz}{z-2}$ where c is $|z|=3$

Ans: Singular points are obtained by taking

$z-2=0$

$z=2$

$|2| = 2 < 3$

$\therefore z=2$ lies inside c

$$\underset{z=2}{\text{Res}} f(z) = \frac{P(2)}{q'(2)}$$

$P(z) = z^2$

$q(z) = z-2$

$q'(z) = 1$

$$= \frac{4}{1} = \underline{\underline{4}}$$

By residue Theorem $\int_c \frac{z^2 dz}{(z-2)} = 2\pi i \times$ Sum of Residues

$= 2\pi i \times 4$

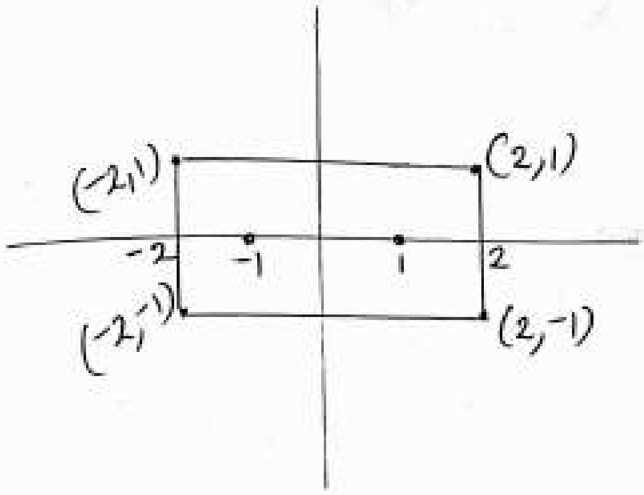
$= \underline{\underline{8\pi i}}$

2) Evaluate $\int_c \frac{\cos \pi z}{z^2-1} dz$ where c is the rectangle with vertices $2 \pm i$, $-2 \pm i$

Ans: poles are obtained by taking $z^2 - 1 = 0$

$z^2 = 1$

$z = +1, -1$ are the simple poles



$z = 1, -1$ both lies inside C

$$\underset{z=1}{\text{Res}}\, f(z) = \frac{P(1)}{Q'(1)}$$

$$\left\{\begin{array}{l} P(z) = \cos \pi z \\ Q(z) = z^2 - 1 \\ Q'(z) = 2z \end{array}\right.$$

$$= \frac{\cos \pi}{2} = \frac{-1}{2}$$

$$\underset{z=-1}{\text{Res}}\, f(z) = \frac{P(-1)}{Q'(-1)}$$

$$= \frac{\cos(-\pi)}{-2}$$

$$= \frac{\cos \pi}{-2} = \frac{-1}{-2} = \frac{1}{2}$$

$\therefore$ By Residue theorem

$$\int_C \frac{\cos \pi z \, dz}{z^2 - 1} = 2\pi i \times \text{Sum of Residues}$$

$$= 2\pi i \times \left[-\frac{1}{2} + \frac{1}{2}\right]$$

$$= 0$$

3) Integrate $f(z) = \frac{\tan z}{z^2-1}$ counterclockwise around the circle $c: |z| = \frac{3}{2}$.

Ans: Singular points are obtained by taking

$z^2 - 1 = 0$

$z^2 = 1$

$z = 1, -1$ are the simple poles

$|1| = 1 < \frac{3}{2}$

$|-1| = 1 < 3/2$

$\therefore z = 1, -1$ both lies inside c

$$\operatorname{Res}_{z=1} f(z) = \frac{P(1)}{Q'(1)} \qquad P(z) = \tan z$$

$$Q(z) = z^2 - 1$$

$$Q'(z) = 2z$$

$$= \frac{\tan 1}{2}$$

$$\operatorname{Res}_{z=-1} f(z) = \frac{P(-1)}{Q'(-1)}$$

$$= \frac{\tan(-1)}{-2}$$

$$= \frac{-\tan(1)}{-2} = \frac{\tan(1)}{2}$$

By Residue theorem,

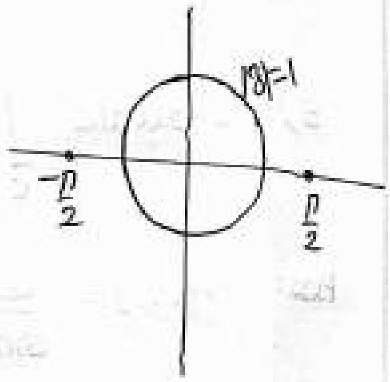
$$\oint_C \frac{\tan z}{z^2-1}\, dz = 2\pi i \left[\Sigma R_i \right]$$

$$= 2\pi i \left[\frac{\tan(1)}{2} + \frac{\tan(1)}{2} \right]$$

$$= 2\pi i \tan(1)$$

4) Evaluate $\int_C \tan z\, dz$ where c is (i) $|z|=1$ and (ii) $|z|=2$

Ans: Let $f(z) = \tan z = \dfrac{\sin z}{\cos z}$



Singular points of $f(z)$ are given by

$$\cos z = 0 \Rightarrow z = (2n+1)\frac{\pi}{2}, \quad n \in Z$$

$$\therefore z = \dots -\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \dots\dots$$

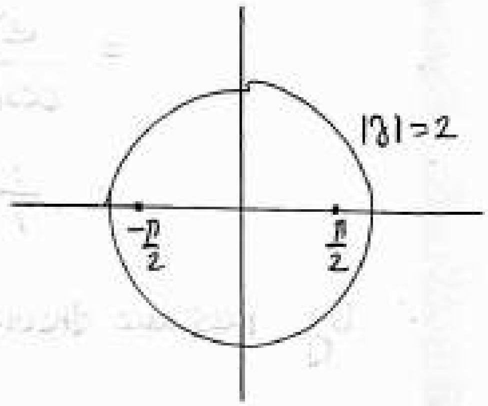
(i) when $C: |z|=1$

All the singular points lies outside c

$\therefore$ By Residue theorem

$$\int_C \tan z\, dz = 2\pi i[\Sigma R_i]$$
$$= 2\pi i \times 0$$
$$= \underline{\underline{0}}$$

(ii) when $C: |z| = 2$



Singular points $z = \frac{\pi}{2}, -\frac{\pi}{2}$ lies inside c

$$\underset{z=\frac{\pi}{2}}{\operatorname{Re}} f(z) = \frac{P(\pi/2)}{Q'(\pi/2)}$$

$$= \frac{\sin(\pi/2)}{-\sin(\pi/2)} = \underline{\underline{-1}}$$

$P(z) = \sin z$
$Q(z) = \cos z$
$Q'(z) = -\sin z$

$$\underset{z=-\frac{\pi}{2}}{\operatorname{Re}} f(z) = \frac{P(-\pi/2)}{Q'(-\pi/2)}$$

$$= \frac{\sin(-\pi/2)}{-\sin(-\pi/2)} = \underline{\underline{-1}}$$

$\therefore$ By Residue Theorem $\int_C \tan z\, dz = 2\pi i[-1+-1]$

$= -4\pi i$

5) Evaluate $\int_C \frac{e^z}{\sin z}\, dz$ where c is $|z|=1$

Ans: $f(z) = \frac{e^z}{\sin z}$

Singular points of $f(z)$ is given by $\sin z = 0$

$z = n\pi, \quad n \in Z$

ie) $z = \ldots\ldots -2\pi, -\pi, 0, \pi, 2\pi \ldots$

$z=0$ lies inside $c : |z|=1$

$\underset{z=0}{\text{Res}}\, f(z) = \frac{P(0)}{Q'(0)}$

$P(z) = e^z$

$Q(z) = \sin z$

$Q'(z) = \cos z$

$= \frac{e^0}{\cos(0)}$

$= \frac{1}{1} = 1$

$\therefore$ By Residue theorem $\int_C \frac{e^z\, dz}{\sin z} = 2\pi i \times 1$

$= 2\pi i$

6) Evaluate $\int_C \frac{e^z}{\cos \pi z}\, dz$ where c is $|z|=1$

Ans: $f(z) = \frac{e^z}{\cos \pi z}$

The singular points is given by

$\cos \pi z = 0 \Rightarrow \pi z = (2n+1)\frac{\pi}{2}, \; n \in \mathbb{Z}$

$\therefore \; z = \left(\frac{2n+1}{2}\right) \; n \in \mathbb{Z}$

$\therefore \; z = \; \ldots\ldots \frac{-3}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2} \ldots$

The singular points $z = \frac{-1}{2}, \frac{1}{2}$ lies inside $|z| = 1$

$\underset{z=\frac{1}{2}}{\text{Res}} \; f(z) = \frac{P(1/2)}{q'(1/2)}$

$P(z) = e^z$
$q(z) = \cos \pi z$
$q'(z) = -\pi \sin \pi z$

$= \frac{e^{1/2}}{-\pi \sin \frac{\pi}{2}}$

$= \frac{e^{1/2}}{-\pi}$

$\underset{z=-1/2}{\text{Res}} \; f(z) = \frac{P(-1/2)}{q'(-1/2)}$

$= \frac{e^{-1/2}}{-\pi \sin(\frac{-\pi}{2})}$

$= \frac{e^{-1/2}}{\pi}$

$\therefore$ By Residue theorem $\int_C \frac{e^z}{\cos \pi z} dz = 2\pi i \left[\frac{e^{1/2}}{-\pi} + \frac{e^{-1/2}}{\pi}\right]$

$= 2i\left[-e^{1/2} + e^{-1/2}\right]$

Q Using Cauchys Residue theorem evaluate

$\int_C \frac{z^2}{(z-1)^2(z+2)} dz$ where C is $|z-2| = 2$

Ans: $f(z) = \dfrac{z^2}{(z-1)^2(z+2)}$

Singular points of $f(z)$ are given by

$(z-1)^2(z+2) = 0$

$z = -2, 1, 1$

$|-2-2| = |-4| = 4 > 2$

$|1-2| = |-1| = 1 < 2$

$\therefore z = -2$ is a simple pole lies outside C

$z = 1$ is a pole of order 2 lies inside C.

$$\cancel{\operatorname*{Res}_{z=2} f(z) = \lim_{z\to 2} (z-2) f(z)}$$

$$\cancel{= \lim_{z\to 2} (z-2) \frac{z^2}{(z-1)^2(z+2)}}$$

$$\operatorname*{Res}_{z=1} f(z) = \frac{1}{1!} \lim_{z\to 1} \frac{d}{dz}\left[(z-1)^2 \frac{z^2}{(z-1)^2(z+2)}\right]$$

$$= \lim_{z\to 1} \frac{d}{dz}\left(\frac{z^2}{z+2}\right)$$

$$= \lim_{z\to 1} \left[\frac{(z+2)(2z)-(z^2)}{(z+2)^2}\right]$$

$$= \lim_{z\to 1} \frac{2z^2+4z-z^2}{(z+2)^2} = \frac{5}{9}$$

$\therefore$ By Residue theorem $\displaystyle\int_C \frac{z^2}{(z-1)^2(z+2)}\,dz = 2\pi i\left[\frac{5}{9}\right] = \frac{10\pi i}{9}$

Example 7.3.9. *Evaluate* $\int_C \frac{dz}{z^2(z-1)}$, *where* C *is* $|z| = 2$.

Let $f(z) = \frac{1}{z^2(z-1)}$.

The singular points of $f(z)$ are given by

$$\begin{aligned} z^2(z-1) &= 0 \\ \implies z &= 0, 1 \end{aligned}$$

The singular points $z = 0, 1$ lies inside the curve $C : |z| = 2$. The singular point $z = 1$ is a simple pole and $z = 0$ is a pole of order 2.

$$\begin{aligned} \operatorname*{Res}_{z=0} f(z) &= \frac{1}{(2-1)!} \lim_{z\to 0} \frac{d}{dz}\left[(z-0)^2 \frac{1}{z^2(z-1)}\right] \\ &= \lim_{z\to 0} \frac{d}{dz}\left[\frac{1}{(z-1)}\right] \\ &= \lim_{z\to 0}\left[-\frac{1}{(z-1)^2}\right] \\ &= -1 \\ \operatorname*{Res}_{z=1} f(z) &= \left.\frac{1}{z^2(1) + (z-1)(2z)}\right]_{z=1} \\ &= 1 \end{aligned}$$

$$\begin{aligned} \therefore \int_C \frac{1}{z^2(z-1)}\, dz &= 2\pi i[-1+1] \\ &= 0 \end{aligned}$$

Example 7.3.10. *Evaluate* $\int_C \frac{e^z}{(z+1)^3}\,dz$*, where* C *is* $|z+1|=2$.

Let $f(z) = \frac{e^z}{(z+1)^3}$.
Singular points of $f(z)$ are given by $(z+1)^3 = 0 \implies z = -1, -1, -1$. The point $z = -1$ lies inside the curve $C : |z+1| = 2$ and it is a pole of order 3.

$$\begin{aligned}
\operatorname*{Res}_{z=-1} f(z) &= \frac{1}{(3-1)!} \lim_{z\to -1} \frac{d^2}{dz^2}\left[[z-(-1)]^2 \, \frac{e^z}{(z+1)^3}\right] \\
&= \frac{1}{2} \lim_{z\to -1} \frac{d^2}{dz^2}\left[e^z\right] \\
&= \frac{e^{-1}}{2} \\
\therefore \int_C \frac{e^z}{(z+1)^3}\,dz &= 2\pi i\left[\frac{e^{-1}}{2}\right] \\
&= \pi i\, e^{-1}
\end{aligned}$$

Example 7.3.11. *Evaluate* $\int_C \frac{\sin^2 z}{(z-\pi/6)^3}\,dz$*, where* C *is* $|z|=1$.

Let $f(z) = \frac{\sin^2 z}{(z-\pi/6)^3}$.
Singular points of $f(z)$ are given by $(z-\pi/6)^3 = 0 \implies z = \pi/6, \pi/6, \pi/6$. The point $z = \pi/6$ lies inside the curve $C : |z| = 1$ and it is a pole of order 3.

$$\begin{aligned}
\operatorname*{Res}_{z=\pi/6} f(z) &= \frac{1}{(3-1)!} \lim_{z\to \pi/6} \frac{d^2}{dz^2}\left[(z-\pi/6)^3 \, \frac{\sin^2 z}{(z-\pi/6)^3}\right] \\
&= \frac{1}{2} \lim_{z\to \pi/6} \frac{d^2}{dz^2}\left[\sin^2 z\right] \\
&= \frac{1}{2} \lim_{z\to \pi/6} \frac{d}{dz}\left[2\sin z\cos z\right] \\
&= \frac{1}{2} \lim_{z\to \pi/6} \left[2\cos 2z\right] \\
&= \frac{1}{2} \\
\therefore \int_C \frac{\sin^2 z}{(z-\pi/6)^3}\,dz &= 2\pi i\left[\frac{1}{2}\right] \\
&= \pi i
\end{aligned}$$

Example 7.3.12. *Evaluate* $\int_C \frac{e^{-z}}{z^3}\,dz$*, where* C *is* $|z|=1$.

Let $f(z) = \frac{e^{-z}}{z^3}$. We can see that $z = 0$ is a singular point of $f(z)$ which lies inside the curve $C : |z| = 1$.

$$f(z) = \frac{e^{-z}}{z^3} = \frac{1 - \frac{z}{1!} + \frac{z^2}{2!} - \frac{z^3}{3!} \cdots}{z^3}$$
$$= \frac{1}{z^3} - \frac{1}{1!z^2} + \frac{1}{2!z} - \cdots$$
$$\operatorname*{Res}_{z=0} f(z) = \text{coefficient of } \frac{1}{z-0} = \frac{1}{2}$$
$$\therefore \int_C \frac{e^{-z}}{z^3}\, dz = 2\pi i \left[\frac{1}{2}\right] = \pi i$$

Example 7.3.13. *Evaluate* $\int_C z^4 e^{1/z^2}\, dz$*, where* C *is* $|z| = 1$.

Let
$$f(z) = z^4 e^{1/z^2}$$
$z = 0$ is a singular point of $f(z)$ which lies inside the curve $C : |z| = 1$.

$$f(z) = z^4 e^{\frac{1}{z^2}}$$
$$= z^4 \left[1 + \frac{1}{1!z^2} + \frac{1}{2!z^4} + \frac{1}{3!z^6} \cdots\right]$$
$$= z^4 + \frac{z^2}{1!} + \frac{1}{2!} + \frac{1}{3!z^2} \cdots$$
$$\operatorname*{Res}_{z=0} f(z) = \text{coefficient of } \frac{1}{z-0}$$
$$= 0$$
$$\therefore \int_C z^4 e^{1/z^2}\, dz = 2\pi i[0]$$
$$= 0$$

Example 7.3.14. *Evaluate* $\int_C \frac{\sin z}{z^4}\, dz$*, where* C *is* $|z| = 1$.

Let
$$f(z) = \frac{\sin z}{z^4}$$
$z = 0$ is a singular point of $f(z)$ which lies inside the curve $C : |z| = 1$.

$$f(z) = \frac{\sin z}{z^4}$$
$$= \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots}{z^4}$$
$$= \frac{1}{z^3} - \frac{1}{3!z} + \frac{z}{5!} - \cdots$$

$z = 0$ is a pole order 3.

$$
\begin{aligned}
\operatorname*{Res}_{z=0} f(z) &= \text{coefficient of } \frac{1}{z-0} \\
&= -\frac{1}{3!} \\
&= -\frac{1}{6} \\
\therefore \int_C \frac{\sin z}{z^4}\, dz &= 2\pi i\left[-\frac{1}{6}\right] \\
&= -\frac{\pi i}{3}
\end{aligned}
$$

RESIDUE INTEGRATION OF REAL INTEGRALS

I Integrals of Rational Functions of $\cos\theta$ and $\sin\theta$

The integral is of the form $\int_0^{2\pi} f(\sin\theta, \cos\theta)\,d\theta$ where $f(\sin\theta, \cos\theta)$ is rational function of $\sin\theta$ and $\cos\theta$ can be evaluated by taking the transformation $z = e^{i\theta}$ $\therefore |z| = 1$

$z = e^{i\theta} \Rightarrow z = \cos\theta + i\sin\theta$ —— ①

$\frac{1}{z} = e^{-i\theta} \Rightarrow \frac{1}{z} = \cos\theta - i\sin\theta$ —— ②

①+② $\Rightarrow$ $z + \frac{1}{z} = 2\cos\theta$

$$\therefore \cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$\cos\theta = \frac{z^2+1}{2z}$$

①−② $\Rightarrow$ $z - \frac{1}{z} = 2i\sin\theta$

$$\therefore \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$= \frac{z^2-1}{2iz}$$

$z = e^{i\theta} \Rightarrow \frac{dz}{d\theta} = e^{i\theta} \times i$

$$\text{ie)}\ \frac{dz}{d\theta} = iz$$

$$\therefore d\theta = \frac{dz}{iz}$$

Substituting these values in $\int_0^{2\pi} f(\sin\theta, \cos\theta)\,d\theta$ we get,

$$\int_0^{2\pi} f(\sin\theta, \cos\theta)\, d\theta = \int_{|z|=1} f\left(\frac{z^2-1}{2iz}, \frac{z^2+1}{2z}\right)\frac{dz}{iz}$$

$$= \oint_{|z|=1} f(z)\, dz$$ which can be evaluated

by cauchys Residue Theorem.

Note

$$\cos 2\theta = \frac{z^4+1}{2z^2} \qquad \cos 3\theta = \frac{z^6+1}{2z^3}$$

$$\sin 2\theta = \frac{z^4-1}{2iz^2} \qquad \sin 3\theta = \frac{z^6-1}{2iz^3}$$ and so on...

Q1) using contour integration evaluate $\int_0^{2\pi} \frac{1}{5+4\cos\theta}\, d\theta$

Ans Consider the unit circle $|z|=1$, $z = e^{i\theta}$, $0 \le \theta \le 2\pi$

$$\cos\theta = \frac{z^2+1}{2z}, \quad \sin\theta = \frac{z^2-1}{2iz}, \quad d\theta = \frac{dz}{iz}$$

$$\int_0^{2\pi} \frac{1}{5+4\cos\theta}\, d\theta = \int_{|z|=1} \frac{1}{5+4\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz}$$

$$= \int \frac{1}{\frac{10z+4z^2+4}{2z}} \frac{dz}{iz}$$

$$= \frac{1}{i}\int \frac{dz}{2z^2+5z+2}$$

$$= \frac{1}{i}\int_C f(z)\, dz \quad \text{——— (1)}$$

$$f(z) = \frac{1}{2z^2+5z+2}$$

poles are obtained by taking $2z^2+5z+2=0$

$$z = \frac{-5 \pm \sqrt{25-16}}{4}$$

$$= \frac{-5 \pm 3}{4}$$

$$= -2, -\frac{1}{2} \text{ are simple poles}$$

$z=-2$ lies outside $|z|=1$

$z=-\frac{1}{2}$ lies inside $|z|=1$

$$\therefore \operatorname*{Res}_{z=-\frac{1}{2}} f(z) = \frac{P(-1/2)}{Q'(-1/2)}$$

$P(z)=1$

$Q(z)=2z^2+5z+2$

$Q'(z)=4z+5$

$$= \frac{1}{-2+5}$$

$$= \frac{1}{3}$$

By Residue Theorem

$$\therefore \int_C f(z)\,dz = 2\pi i \times \text{sum of Residues}$$

$$= 2\pi i \times \frac{1}{3}$$

$$= \frac{2\pi i}{3}$$

Substituting in ①

$$\int_0^{2\pi} \frac{1}{5+4\cos\theta}\,d\theta = \frac{1}{i} \times \frac{2\pi i}{3}$$

$$= \underline{\underline{\frac{2\pi}{3}}}$$

Q2) using contour integration evaluate $\int_0^{2\pi} \frac{1}{2+\cos\theta}\,d\theta$.

Ans: Consider the unit circle $|z|=1$, $z=e^{i\theta}$

$$\cos\theta = \frac{z^2+1}{2z}, \quad \sin\theta = \frac{z^2-1}{2iz} \quad \text{and} \quad d\theta = \frac{dz}{iz}$$

$$\therefore \int_0^{2\pi} \frac{1}{2+\cos\theta}\, d\theta = \int_{|z|=1} \frac{1}{2+\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz}$$

$$= \int \frac{1\times 2z}{4z+z^2+1} \times \frac{dz}{iz}$$

$$= \frac{2}{i}\int \frac{dz}{z^2+4z+1}$$

$$= \frac{2}{i}\int_C f(z)\,dz \quad \text{——— (1)}$$

$$f(z) = \frac{1}{z^2+4z+1}$$

poles are obtained by $z^2+4z+1 = 0$

$$z = \frac{-4 \pm \sqrt{16-4}}{2}$$

$$= \frac{-4 \pm \sqrt{12}}{2}$$

$$= \frac{-4 \pm 2\sqrt{3}}{2}$$

$= -2+\sqrt{3}$, $-2-\sqrt{3}$ are simple poles

$z = -2+\sqrt{3}$ lies inside $|z|=1$

$z = -2-\sqrt{3}$ lies outside $|z|=1$

$$\underset{z=-2+\sqrt{3}}{\text{Res}}\ f(z) = \frac{P(-2+\sqrt{3})}{Q'(-2+\sqrt{3})}$$

$P(z) = 1$

$Q(z) = z^2+4z+1$

$Q'(z) = 2z+4$

$$= \frac{1}{2(-2+\sqrt{3})+4} = \frac{1}{2\sqrt{3}}$$

By Residue Theorem $\int_C f(z)\,dz = 2\pi i \times$ sum of Residues

$$= 2\pi i \times \frac{1}{2\sqrt{3}}$$

$$= \frac{\pi i}{\sqrt{3}}$$

Substituting in (1)

$$\int_0^{2\pi} \frac{1}{2+\cos\theta}\,d\theta = \frac{2}{i} \times \frac{\pi i}{\sqrt{3}}$$

$$= \frac{2\pi}{\sqrt{3}}$$

Q 3) using contour integration, evaluate $\int_0^{2\pi} \frac{1}{5-3\sin\theta} d\theta$

Ans: Consider the unit circle $|z|=1$ then $z = e^{i\theta}$

$$\sin\theta = \frac{z^2-1}{2iz}, \quad \cos\theta = \frac{z^2+1}{2z}, \quad d\theta = \frac{dz}{iz}$$

$$\int_0^{2\pi} \frac{1}{5-3\sin\theta} d\theta = \int_{|z|=1} \frac{1}{5-3\left[\frac{z^2-1}{2iz}\right]} \frac{dz}{iz}$$

$$= \int_{|z|=1} \frac{1 \times 2iz}{10iz-3z^2+3} \frac{dz}{iz}$$

$$= \int_{|z|=1} \frac{2\,dz}{-[3z^2-10iz-3]}$$

$$= -2 \int_{|z|=1} \frac{dz}{3z^2-10iz-3}$$

$$= -2 \int_C f(z)\,dz \quad \text{—— ①}$$

$$f(z) = \frac{1}{3z^2 - 10iz - 3}$$

poles are obtained by taking $3z^2 - 10iz - 3 = 0$

$$z = \frac{10i \pm \sqrt{-100+36}}{6}$$

$$z = \frac{10i \pm 8i}{6}$$

$$z = 3i, \frac{i}{3}$$ are the simple poles

The point $z = 3i$ $(0, 3)$ lies outside $|z| = 1$

$z = \frac{i}{3}$ $(0, \frac{1}{3})$ lies inside $|z| = 1$

$$\operatorname*{Res}_{z=\frac{i}{3}} f(z) = \frac{P(i/3)}{Q'(i/3)}$$

$P(z) = 1$

$Q(z) = 3z^2 - 10iz - 3$

$Q'(z) = 6z - 10i$

$$= \frac{1}{6\left(\frac{i}{3}\right) - 10i} = \frac{1}{-8i}$$

By Residue theorem

$$\int_c f(z)\,dz = 2\pi i \times \text{sum of residues}$$

$$= 2\pi i \times \frac{1}{-8i}$$

$$= -\frac{\pi}{4}$$

$$① \Rightarrow \int_0^{2\pi} \frac{1}{5 - 3\sin\theta}\,d\theta = -2 \times -\frac{\pi}{4}$$

$$= \underline{\underline{\pi/2}}$$

4) using contour integrals evaluate $\displaystyle\int_0^{2\pi} \frac{1}{(5-3\cos\theta)^2}\,d\theta$

Ans: Consider the unit circle $|z| = 1$ then $z = e^{i\theta}$

$\sin\theta = \frac{z^2-1}{2iz}$, $\cos\theta = \frac{z^2+1}{2z}$, $d\theta = \frac{dz}{iz}$

$$\int_0^{2\pi} \frac{1}{(5-3\cos\theta)^2}\, d\theta = \int_{|z|=1} \frac{1}{\left[5 - 3\left(\frac{z^2+1}{2z}\right)\right]^2} \frac{dz}{iz}$$

$$= \int_{|z|=1} \frac{1}{\left(\frac{10z - 3z^2 - 3}{2z}\right)^2} \frac{dz}{iz}$$

$$= \int_{|z|=1} \frac{4z^2}{\left(-[3z^2 - 10z + 3]\right)^2} \frac{dz}{iz}$$

$$= \int_{|z|=1} \frac{4z}{i(3z^2 - 10z + 3)^2}\, dz$$

$$= \frac{4}{i} \int \frac{z\, dz}{(3z^2 - 10z + 3)^2}$$

$$= \frac{4}{i} \int_C f(z)\, dz \quad \text{—— (1)}$$

poles are obtained by taking $(3z^2 - 10z + 3)^2 = 0$

ie) $3z^2 - 10z + 3 = 0$

$$z = \frac{10 \pm \sqrt{100 - 36}}{6}$$

$$= \frac{10 \pm 8}{6}$$

$$= 3, \tfrac{1}{3}, 3, \tfrac{1}{3}$$

$z = 3$ is a pole of order 2 lies outside $|z| = 1$

$z = 1/3$ is a pole of order 2 lies inside $|z| = 1$

$$\underset{z = 1/3}{\text{Res}}\, f(z) = \frac{1}{1!} \lim_{z \to \frac{1}{3}} \frac{d}{dz}\left[\left(z - \tfrac{1}{3}\right)^2 f(z)\right]$$

$$= \lim_{z \to \frac{1}{3}} \frac{d}{dz} \left(\frac{3z-1}{3}\right)^2 \frac{z}{[(z-3)(3z-1)]^2}$$

$$= \lim_{z \to \frac{1}{3}} \frac{d}{dz} \frac{z}{9(z-3)^2}$$

$$= \frac{1}{9} \lim_{z \to \frac{1}{3}} \frac{(z-3)^2 - z\,2(z-3)}{(z-3)^4}$$

$$= \frac{1}{9} \lim_{z \to \frac{1}{3}} \frac{(z-3) - 2z}{(z-3)^3}$$

$$= \frac{1}{9} \frac{\left(\frac{1}{3}-3\right) - 2\left(\frac{1}{3}\right)}{\left(\frac{1}{3}-3\right)^3}$$

$$= \frac{1}{9} \times \frac{-10}{3} \times \frac{-27}{512} = \frac{5}{256}$$

By Residue theorem

$$\int_C f(z)\,dz = 2\pi i \times \frac{5}{256} = \frac{10\pi i}{256}$$

Substituting in (1)

$$\int_0^{2\pi} \frac{1}{(5-3\cos\theta)^2}\,d\theta = \frac{4}{i} \times \frac{10\pi i}{256} = \frac{5\pi}{32}$$

Q5) using contour integration evaluate $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\,d\theta$

Ans: Consider the unit circle $|z|=1$. Then $z = e^{i\theta}$

$$\cos\theta = \frac{z^2+1}{2z} \qquad \cos 2\theta = \frac{z^4+1}{2z^2} \qquad d\theta = \frac{dz}{iz}$$

$$\therefore \int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\,d\theta = \int_{|z|=1} \frac{\left[\frac{z^4+1}{2z^2}\right]}{5+4\left[\frac{z^2+1}{2z}\right]} \frac{dz}{iz}$$

$$= \int_{|z|=1} \frac{[z^4+1]\, 2z}{2z^2[10z+4z^2+4]} \frac{dz}{iz}$$

$$= \frac{1}{i} \int_{|z|=1} \frac{(z^4+1)\, dz}{z^2[4z^2+10z+4]}$$

$$= \frac{1}{i} \int_C f(z)\, dz. \quad \text{—— } 1$$

$$f(z) = \frac{(z^4+1)}{z^2(4z^2+10z+4)}$$

poles are obtained by $z^2(4z^2+10z+4)=0$

$z^2=0$, $\quad 4z^2+10z+4=0$

$z=0,0 \qquad z = \dfrac{-10 \pm \sqrt{100-64}}{8}$

$$= \frac{-10 \pm 6}{8}$$

$$= -2, -\tfrac{1}{2}$$

$z = 0, 0, -\tfrac{1}{2}, -2$

$z=0$ is a pole of order 2 lies inside $|z|=1$

$z=-\tfrac{1}{2}$ is a pole of order 1 lies inside $|z|=1$

$z=-2$ is a pole of order 1 lies outside $|z|=1$

$$\underset{z=0}{\text{Res}}\, f(z) = \frac{1}{1!} \lim_{z \to 0} \frac{d}{dz}\left[(z-0)^2 \frac{(z^4+1)}{z^2(4z^2+10z+4)}\right]$$

$$= \lim_{z \to 0} \frac{d}{dz}\left[\frac{z^4+1}{4z^2+10z+4}\right]$$

$$\lim_{z\to 0} \frac{(4z^2+10z+4)(4z^3)-(z^4+1)(8z+10)}{(4z^2+10z+4)^2}$$

$$= \frac{0-10}{16} = -\frac{5}{8}$$

$\operatorname{Res}_{z=-1/2} f(z) = \frac{P(-1/2)}{Q'(-1/2)}$

$P(z) = z^4+1$

$Q(z) = 4z^4+10z^3+4z^2$

$Q'(z) = 16z^3+30z^2+8z$

$$= \frac{\left(-\frac{1}{2}\right)^4+1}{16\left(-\frac{1}{2}\right)^3+30\left(-\frac{1}{2}\right)^2+8\left(-\frac{1}{2}\right)}$$

$$= \frac{+\frac{1}{16}+1}{-2+7\cdot5-4} = \frac{\frac{17}{16}}{1\cdot5}$$

$$= \frac{17}{16} \times \frac{+1}{1\cdot5}$$

$$= \frac{17}{24}$$

By Residue theorem,

$$\int_C f(z)\,dz = 2\pi i \times \text{Sum of Residues}$$

$$= 2\pi i\left[-\frac{5}{8}+\frac{17}{24}\right]$$

$$= 2\pi i \times \frac{1}{12} = \frac{\pi i}{6}$$

Substituting in ①

$$\int_0^{2\pi} \frac{\cos 2\theta\, d\theta}{5+4\cos\theta} = \frac{1}{i} \times \frac{\pi i}{6} = \underline{\underline{\frac{\pi}{6}}}$$

Q6) using contour integration evaluate $\int_0^{2\pi} \frac{\cos 3\theta}{5+4\cos\theta} d\theta$

Ans: Consider the unit circle $|z|=1$, Then $z = e^{i\theta}$

$$\cos\theta = \frac{z^2+1}{2z} \quad \cos 3\theta = \frac{z^6+1}{2z^3} \quad \text{and} \quad d\theta = \frac{dz}{iz}$$

$$\therefore \int_0^{2\pi} \frac{\cos 3\theta}{5+4\cos\theta} d\theta = \int_{|z|=1} \frac{\left[\frac{z^6+1}{2z^3}\right]}{5+4\left[\frac{z^2+1}{2z}\right]} \frac{dz}{iz}$$

$$= \int_{|z|=1} \frac{\frac{(z^6+1)}{2z^3}}{\frac{10z+4z^2+4}{2z}} \frac{dz}{iz}$$

$$= \int_{|z|=1} \left(\frac{z^6+1}{2z^3}\right) \times \frac{2z}{(4z^2+10z+4)} \cdot \frac{dz}{iz}$$

$$= \frac{1}{i}\int_C \frac{(z^6+1)\,dz}{z^3(4z^2+10z+4)}$$

$$= \frac{1}{i}\int_C f(z)\,dz$$

$$f(z) = \frac{z^6+1}{z^3(4z^2+10z+4)}$$

poles are obtained by taking $z^3(4z^2+10z+4)=0$

$z^3 = 0$ or $4z^2+10z+4=0$

$z = 0,0,0 \qquad z = \frac{-10 \pm \sqrt{100-64}}{8}$

$$z = \frac{-10 \pm 6}{8} = -2, \frac{-1}{2}$$

$z = 0,0,0, -\frac{1}{2}, -2$

$z=-2$ is a pole of order 1 lies outside C

$z=0$ is a pole of order 3 lies inside C

$z=-\frac{1}{2}$ is a pole of order 1 lies inside C

$$\underset{z=0}{\text{Res}}\, f(z) = \frac{1}{2!} \lim_{z\to 0} \frac{d^2}{dz^2}\left[\frac{(z-0)^3 (z^6+1)}{z^3(4z^2+10z+4)}\right]$$

$$= \frac{1}{2} \lim_{z\to 0} \frac{d^2}{dz^2}\left[\frac{(z^6+1)}{4z^2+10z+4}\right]$$

$$= \frac{1}{2} \lim_{z\to 0} \frac{d}{dz}\left[\frac{(4z^2+10z+4)6z^5-(z^6+1)(8z+10)}{(4z^2+10z+4)^2}\right]$$

$$= \frac{1}{2} \lim_{z\to 0} \frac{d}{dz}\left[\frac{(24z^7+60z^6+24z^5)-(8z^7+8z+10z^6+10)}{(4z^2+10z+4)^2}\right]$$

$$= \frac{1}{2} \lim_{z\to 0} \frac{d}{dz}\left[\frac{16z^7+50z^6+24z^5-8z-10}{(4z^2+10z+4)^2}\right]$$

$$= \frac{1}{2} \lim_{z\to 0}\left[\frac{(4z^2+10z+4)^2(112z^6+300z^5+120z^4-8) - (16z^7+50z^6+24z^5-8z-10)\,2(4z^2+10z+4)(8z+10)}{(4z^2+10z+4)^4}\right]$$

$$= \frac{1}{2}\left[\frac{(16)(-8)-(-10)(2)(4)(10)}{256}\right]$$

$$= \frac{1}{2}\left[\frac{-128+800}{256}\right] = \underline{\underline{\frac{21}{16}}}$$

$$\underset{z=-1/2}{\text{Re}}\, f(z) = \frac{P(-1/2)}{Q'(-1/2)}$$

$Q(z) = 4z^5+10z^4+4z^3$

$Q'(z) = 20z^4+40z^3+12z^2$

$$= \frac{(-\frac{1}{2})^6+1}{\frac{5}{4}-5+3}$$

$$= \frac{\frac{1}{64}+1}{\frac{-3}{4}}$$

$$= \frac{65}{64} \times \frac{-4}{3}$$

$$= -\frac{65}{48}.$$

By Residue Theorem

$$\int_C f(z)\,dz = 2\pi i \times \left[\frac{21}{16} - \frac{65}{48}\right]$$

$$= 2\pi i \left[\frac{63-65}{48}\right]$$

$$= 2\pi i \left[\frac{-2}{48}\right]$$

$$= -\frac{\pi i}{12}$$

$$① \Rightarrow \int_0^{2\pi} \frac{\cos 3\theta \, d\theta}{5+4\cos\theta} = \frac{1}{i} \times \frac{-\pi i}{12}$$

$$= \underline{\underline{\frac{-\pi}{12}}}$$

II Integrals of the type $\int_{-\infty}^{\infty} \frac{f(x)}{g(x)} dx$

$$\int_{-\infty}^{\infty} \frac{f(x)}{g(x)} dx = \int_C \frac{f(z)dz}{g(z)} \quad \text{or} \quad \int_C f(z)dz$$

This can be evaluated with the help of cauchys residue theorem, considering all poles of $f(z)$ lying above real axis.

Q: using contour integration evaluate $\int_0^{\infty} \frac{1}{(x^2+a^2)^2} dx$

Ans: Let $f(z) = \frac{1}{(z^2+a^2)^2}$

singular points are obtained by taking

$$(z^2+a^2)^2 = 0$$

$$z^2+a^2 = 0$$

$$z^2 = -a^2$$

$$z = \pm ai$$

$$z = +ai, +ai, -ai, -ai$$

$z = +ai$ is a pole of order 2 lies above the real axis

$z = -ai$ is a pole of order 2 lies below the real axis

$$\operatorname{Res}_{z=ai} f(z) = \frac{1}{1!} \lim_{z \to ai} \frac{d}{dz}\left[(z-ai)^2 \frac{1}{(z^2+a^2)^2}\right]$$

$$= \lim_{z \to ai} \frac{d}{dz}\left[(z-ai)^2 \frac{1}{\left[(z-ai)(z+ai)\right]^2}\right]$$

$$= \lim_{z \to ai} \frac{d}{dz}\left[\frac{1}{(z+ai)^2}\right]$$

$$= \lim_{z \to ai} \frac{d}{dz}\left[(z+ai)^{-2}\right]$$

$$= \lim_{z \to ai} (-2)(z+ai)^{-3}(1)$$

$$= \lim_{z \to ai} \frac{-2}{(z+ai)^{+3}}$$

$$= \frac{-2}{(2ai)^3} = \frac{-2}{-8a^3 i} = \frac{1}{4a^3 i}$$

By Residue theorem

$$\int_C \frac{1}{(z^2+a^2)^2} dz = 2\pi i \times \frac{1}{4a^3 i}$$

$$= \frac{\pi}{2a^3}$$

$$\therefore \int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2} dx = \frac{\pi}{2a^3}$$

$$\therefore \int_{0}^{\infty} \frac{1}{(x^2+a^2)^2} dx = \frac{1}{2}\left[\frac{\pi}{2a^3}\right]$$

$$= \underline{\underline{\frac{\pi}{4a^3}}}$$

Q2) Using contour integration evaluate $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2} dx$

Ans: Let $f(z) = \frac{z^2}{(z^2+1)^2}$

poles are obtained by taking $(z^2+1)^2 = 0$

$$z^2+1 = 0$$

$$z^2 = -1$$

$$z = \pm i$$

$z = +i, +i, -i, -i$

$z = +i$ is a pole of order 2 lies above real axis

$z = -i$ is a pole of order 2 lies below real axis

$$\operatorname*{Res}_{z=i} f(z) = \frac{1}{1!} \lim_{z \to i} \frac{d}{dz}\left[(z-i)^2 f(z)\right]$$

$$= \lim_{z\to i} \frac{d}{dz}\left[(z-i)^2 \frac{z^2}{[(z-i)(z+i)]^2}\right]$$

$$= \lim_{z\to i} \frac{d}{dz}\left[\frac{z^2}{(z+i)^2}\right]$$

$$= \lim_{z\to i} \frac{(z+i)^2 2z - z^2 2(z+i)}{(z+i)^4}$$

$$= \lim_{z\to i} \frac{(z+i)\left[2z(z+i) - 2z^2\right]}{(z+i)^4}$$

$$= \lim_{z\to i} \frac{2zi}{(z+i)^3}$$

$$= \frac{2(i)^2}{(2i)^3} = \frac{-2}{-8i} = \frac{1}{4i}$$

By Residue Theorem,

$$\int_C \frac{z^2}{(z^2+1)^2} dz = 2\pi i \times \frac{1}{4i} = \frac{\pi}{2}$$

$$\therefore \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2} dx = \underline{\underline{\frac{\pi}{2}}}$$

Q3) Using contour integration evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx$

Let $f(z) = \dfrac{z^2 - z + 2}{z^4 + 10z^2 + 9}$

Singular points are obtained by taking

$$z^4+10z^2+9=0$$

$$(z^2+1)(z^2+9)=0$$

$$z^2+1=0 \quad \text{or} \quad z^2+9=0$$

$$z^2=-1 \quad \text{or} \quad z^2=-9$$

$$z=\pm i \qquad z=\pm 3i$$

$$z=i, -i, 3i, -3i$$

$z=i, 3i$ are simple poles lies above real axis

$z=-i, -3i$ are simple poles lies below real axis

$$\operatorname*{Res}_{z=i} f(z) = \frac{P(i)}{Q'(i)}$$

$$\left[\begin{array}{l} P(z)= z^2-z+2 \\ Q(z)= z^4+10z^2+9 \\ Q'(z)=4z^3+20z \end{array}\right.$$

$$= \frac{(i)^2-(i)+2}{4(i)^3+20(i)}$$

$$= \frac{-1-i+2}{-4i+20i}$$

$$= \frac{1-i}{16i}$$

$$\operatorname*{Res}_{z=3i} f(z) = \frac{P(3i)}{Q'(3i)}$$

$$= \frac{(3i)^2-(3i)+2}{4(3i)^3+20(3i)}$$

$$= \frac{-9-3i+2}{-108i+60i}$$

$$= \frac{-7-3i}{-48i} = \frac{-(7+3i)}{-48i} = \frac{7+3i}{48i}$$

By Residue theorem

$$\int_C \frac{z^2-z+2}{z^4+10z^2+9}\,dz = 2\pi i\left[\frac{1-i}{16i} + \frac{7+3i}{48i}\right]$$

$$= 2\pi i\left[\frac{3-3i+7+3i}{48i}\right]$$

$$= 2\pi i\left[\frac{10}{48i}\right]$$

$$= \frac{5\pi}{12}$$

$$\therefore \int_{-\infty}^{\infty} \frac{(x^2-x+2)}{(x^4+10x^2+9)}\,dx = \underline{\underline{\frac{5\pi}{12}}}$$

Q4) using contour integration evaluate $\int_0^\infty \frac{1}{1+x^4}\,dx$

Ans: $f(z) = \frac{1}{1+z^4}$

singular points are obtained by $1+z^4=0$

$$z^4 = -1$$

$$z^4 = (e^{(2n+1)\pi i})$$

$$z = (e^{(2n+1)\frac{\pi}{4}i}) \quad n=0,1,2,3$$

$$z = e^{\frac{\pi}{4}i}, e^{\frac{3\pi}{4}i}, e^{\frac{5\pi}{4}i}, e^{\frac{7\pi}{4}i}$$

are simple poles

$z = e^{\frac{\pi}{4}i}, e^{\frac{3\pi}{4}i}$ lies above the real axis

$$\operatorname*{Res}_{z=e^{\frac{\pi}{4}i}} f(z) = \frac{P(e^{\frac{\pi}{4}i})}{Q'(e^{\frac{\pi}{4}i})}$$

$P(z) = 1$

$Q(z) = 1+z^4$

$Q'(z) = 4z^3$

$$= \frac{1}{4(e^{\frac{\pi}{4}i})^3}$$

$$= \frac{1}{4\, e^{\frac{3\pi}{4} i}}$$

$$= \frac{1}{4} e^{-\frac{3\pi}{4} i}$$

$$= \frac{1}{4}\left[\cos\frac{3\pi}{4} - i \sin\frac{3\pi}{4}\right]$$

$$= \frac{1}{4}\left[-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right]$$

$$\underset{z = e^{\frac{3\pi}{4} i}}{\text{Res}} f(z) = \frac{P(e^{\frac{3\pi}{4} i})}{Q'(e^{\frac{3\pi}{4} i})}$$

$$= \frac{1}{4\left(e^{\frac{3\pi}{4} i}\right)^3}$$

$$= \frac{1}{4 e^{\frac{9\pi}{4} i}}$$

$$= \frac{1}{4}\left[e^{-\frac{9\pi}{4} i}\right]$$

$$= \frac{1}{4}\left[\cos\frac{9\pi}{4} - i \sin\frac{9\pi}{4}\right]$$

$$= \frac{1}{4}\left[\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right]$$

By Residue theorem

$$\int_C \frac{1}{1+z^4}\, dz = 2\pi i\left[\frac{1}{4}\left(\frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) + \frac{1}{4}\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\right]$$

$$= \frac{2\pi i}{4}\left[-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right]$$

$$= \frac{\pi i}{2}\left[-\frac{2i}{\sqrt{2}}\right] = \frac{\pi}{\sqrt{2}}$$

$$\therefore \int_{-\infty}^{\infty} \frac{1}{1+x^4}\,dx = \frac{\pi}{\sqrt{2}}$$

$$\therefore \int_{0}^{\infty} \frac{1}{1+x^4}\,dx = \frac{\pi}{2\sqrt{2}}$$

Q5) Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^3}$

Ans: Let $f(z) = \dfrac{1}{(1+z^2)^3}$

poles are obtained by $(1+z^2)^3 = 0$

$$1+z^2 = 0$$
$$z^2 = -1$$
$$z = \pm i$$

$z = +i$ is a pole of order 3 lies above real axis

$z = -i$ is a pole of order 3 lies below real axis

$$\underset{z=i}{\text{Res}}\, f(z) = \frac{1}{2!} \lim_{z\to i} \frac{d^2}{dz^2}\left[(z-i)^3 \frac{1}{(1+z^2)^3}\right]$$

$$= \frac{1}{2} \lim_{z\to i} \frac{d^2}{dz^2}\left[(z-i)^3 \frac{1}{[(z-i)(z+i)]^3}\right]$$

$$= \frac{1}{2} \lim_{z\to i} \frac{d^2}{dz^2}\left[\frac{1}{(z+i)^3}\right]$$

$$= \frac{1}{2} \lim_{z\to i} \frac{d^2}{dz^2}\left[(z+i)^{-3}\right]$$

$$= \frac{1}{2} \lim_{z\to i} \frac{d}{dz}\left[-3(z+i)^{-4}\right]$$

$$= \frac{1}{2} \lim_{z\to i} 12(z+i)^{-5}$$

$$= \frac{1}{2} \lim_{z \to i} \left[\frac{12}{(z+i)^5} \right]$$

$$= \frac{1}{2} \left[\frac{12}{(2i)^5} \right]$$

$$= \frac{6}{32i} = \frac{3}{16i}$$

By Residue theorem $\int_C \frac{dz}{(1+z^2)^3} = 2\pi i \times \frac{3}{16i} = \frac{3\pi}{8}$

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^3} = \frac{3\pi}{8}$$