

MODULE IV

COMPLEX VARIABLE - INTEGRATION

(Ref 2: Relevant topics from sections 14-1, 14-2, 14-3, 14-4, 15-4)

Complex integration, Line integrals in the complex plane, Basic properties, First evaluation method - indefinite integration and substitution of limit, second evaluation method - use of representation of a path, Contour Integrals, Cauchy integral theorem (without proof) on simply connected domains, Cauchy integral theorem (without proof) on multiply connected domains, Cauchy integral formula (without proof) Cauchy integral formula for derivatives of an analytic function. Taylor's series, and Maclaurin's series.

Line integrals in the complex plane

Complex definite integrals are called line integrals. They are written $\int_C f(z) dz$. If C is a closed curve then this integral is called contour integrals and is denoted by $\oint_C f(z) dz$.

Basic properties

$$\textcircled{1} \quad \int_C [k_1 f_1(z) + k_2 f_2(z)] dz = k_1 \int_C f_1(z) dz + k_2 \int_C f_2(z) dz$$

$$\textcircled{2} \quad \int_a^b f(z) dz = - \int_b^a f(z) dz.$$

\textcircled{3} If C is a point on the arc joining a and b then

$$\int_a^b f(z) dz = \int_a^c f(z) dz + \int_c^b f(z) dz$$

First Evaluation method

Let $\frac{f(z)}{z}$ be analytic in a simply connected domain D . Then there exist an indefinite integral of $f(z)$ in the domain D . That is an analytic function $F(z)$ such that $F'(z) = f(z)$ in D and for all paths in D joining two points z_0 and z_1 in D we have

$$\int_{z_0}^{z_1} \frac{f(z)}{z} dz = [F(z)]_{z_0}^{z_1}$$

$$= F(z_1) - F(z_0)$$

Q Evaluate the following definite integrals.

$$1) \int_0^{1+i} z^2 dz = \left[\frac{z^3}{3} \right]_0^{1+i}$$

$$= \frac{1}{3} [(1+i)^3 - 0^3]$$

$$= \frac{1}{3} [1 + 3i + 3i^2 + i^3]$$

$$= \frac{1}{3} [1 + 3i - 3 - i]$$

$$= \frac{1}{3} [-2 + 2i]$$

$$= -\frac{2}{3} + \frac{2}{3}i$$

$$2) \int_{-\pi i}^{\pi i} \cos z dz = [\sin z]_{-\pi i}^{\pi i}$$

$$= \sin(\pi i) - \sin(-\pi i)$$

$$= \sin(\pi i) + \sin(\pi i)$$

$$= 2 \sin(\pi i)$$

$$\begin{aligned}
 & \text{3) } \int_{z+3\pi i}^{z-3\pi i} e^{\frac{z}{2}} dz \\
 &= [2e^{\frac{z}{2}}]_{z+3\pi i}^{z-3\pi i} \\
 &= 2 \left[e^{\frac{z-3\pi i}{2}} - e^{\frac{z+3\pi i}{2}} \right] \\
 &= 2 \left[e^{4-\frac{3\pi i}{2}} - e^{4+\frac{3\pi i}{2}} \right] \\
 &= 2 \left[e^4 e^{-\frac{3\pi i}{2}} - e^4 e^{\frac{3\pi i}{2}} \right] \\
 &= 2e^4 \left[e^{-\frac{3\pi i}{2}} - e^{\frac{3\pi i}{2}} \right] \\
 &= 2e^4 \left[(\cos(\frac{3\pi}{2}) - i\sin(\frac{3\pi}{2})) - (\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) \right] \\
 &= 2e^4 \left[(0 - i(-1)) - [0 + i] \right] \\
 &= 2e^4 [i - i] \\
 &= \underline{\underline{0}}
 \end{aligned}$$

Second evaluation method : use of representation of a path

Theorem:

Let c be a piecewise smooth path, represented by $z=z(t)$ where $a \leq t \leq b$. Let $f(\gamma)$ be a continuous function on c . Then $\int_c f(\gamma) d\gamma = \int_a^b f[z(t)] \dot{z}(t) dt$ $\left[\dot{z} = \frac{dz}{dt} \right]$

Steps in applying Theorem

1. Represent the path c in the form $z(t)$ ($a \leq t \leq b$)

- Calculate the derivative $\dot{z}(t) = \frac{dz}{dt}$
- Substitute $z(t)$ for every z in $f(z)$ [$x(t)$ for x and $y(t)$ for y]
- Integrate $f[z(t)] \dot{z}(t)$ over t from a to b .

Parametrisation of curves in complex plane.

I) The parametrised representation of a line through (line segment between) two given points z_1 and z_2 is

$$z(t) = z_1 + t(z_2 - z_1) \text{ with } 0 \leq t \leq 1$$

II) The parametrised representation of a full circle or an arc of a circle in the anticlockwise direction is $z(t) = z_0 + Re^{it}$ with $\alpha \leq t \leq \beta$. Here z_0 is the centre of the circle, R is the radius of the circle and α and β are the angle from z_0 to z_1 and z_2 . For full circle $0 \leq t \leq 2\pi$

If t increases or moves in the clockwise direction the representation becomes $z(t) = z_0 + Re^{-it} \quad \alpha \leq t \leq \beta$

III) The parametric representation of curves of the form $y = f(x)$ with $x \in [a, b]$ is $z(t) = (t, f(t)) \quad a \leq t \leq b$

$$(e) z(t) = t + if(t), \quad a \leq t \leq b$$

Q) Evaluate $\int_C \operatorname{Re} z \, dy$, C the shortest path from $1+i^0$ to $5+5i$

$$\operatorname{Re} z = \underline{\underline{f}}(y)$$

Parametric equation of the straight line joining z_1 and z_2 is

$$z(t) = z_1 + t(z_2 - z_1)$$

\therefore Parametric equations of the straight line joining $1+i$ and $5+5i$ is

$$z(t) = (1+i) + t[(5+5i) - (1+i)]$$

$$= (1+i) + t[4+4i]$$

$$z(t) = (1+4t) + i[1+4t]$$

$$\begin{aligned}\dot{z}(t) &= \frac{dz}{dt} = 4+i[0+4] \\ &= 4+4i\end{aligned}$$

$$f(\gamma) = \operatorname{Re} z = (1+4t)$$

$$\int_C f(\gamma) d\gamma = \int_a^b f[z(t)] \dot{z}(t) dt$$

$$\int_C \operatorname{Re} z d\gamma = \int_0^1 (1+4t)(4+4i) dt$$

$$= (4+4i) \int_0^1 (1+4t) dt$$

$$= (4+4i) \left[t + \frac{4t^2}{2} \right]_0^1$$

$$= (4+4i) [t + 2t^2]_0^1$$

$$= (4+4i) [(1+2) - (0+0)]$$

$$= 3(4+4i)$$

$$= 12+12i$$

Q) Integrate $f(\gamma) = \operatorname{Re} z$ from 0 to $1+2i$ along the straight line.

Ans: $f(\gamma) = \operatorname{Re} z$

Equation of the straight line joining 0 and $1+2i$ is

$$z(t) = z_1 + t(z_2 - z_1)$$

$$z(t) = 0 + t[(1+2i) - 0]$$

$$z(t) = t + i2t$$

$$\overset{\circ}{z}(t) = 1 + 2i$$

$$f(z(t)) = \operatorname{Re} z(t) = t$$

$$\int_C f(z) dz = \int_a^b f[z(t)] \dot{z}(t) dt$$

$$= \int_0^1 t(1+2i) dt$$

$$= (1+2i) \int_0^1 t dt$$

$$= (1+2i) \left[\frac{t^2}{2} \right]_0^1$$

$$= (1+2i) \left[\frac{1}{2} \right]$$

$$= \frac{1}{2} + i$$

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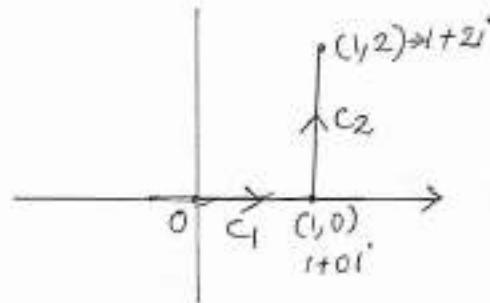
Q3) Integrate $f(z) = \operatorname{Re} z$ from 0 to $1+2i$ along the path C consisting of c_1 and c_2 where $c_1: 0$ to $1+0i$ and $c_2: 1+0i$ to $1+2i$

$$\text{Ans: } \int \operatorname{Re} z dz = \int_{c_1} \operatorname{Re} z dz + \int_{c_2} \operatorname{Re} z dz \quad \text{--- ①}$$

Along c_1

Equation of c_1 : line joining 0 to $1+0i$

$$\begin{aligned} z(t) &= z_1 + t[z_2 - z_1] \\ &= 0 + t[1 - 0] \\ &= t \end{aligned}$$



$$\dot{z}(t) = 1$$

$$f[z(t)] = \operatorname{Re}[z(t)] = t$$

$$\begin{aligned}\int_C f(z) dz &= \int_a^b f[z(t)] \dot{z}(t) dt \\&= \int_0^1 t \cdot 1 dt \\&= \left[\frac{t^2}{2} \right]_0^1 \\&= \frac{1}{2} \quad \text{--- (2)}\end{aligned}$$

Along C_2

Equation of C_2 : line joining $1+0i$ & $1+2i$

$$\begin{aligned}z(t) &= (1+0i) + t[(1+2i) - (1+0i)] \\&= 1 + t2i\end{aligned}$$

$$\dot{z}(t) = 2i$$

$$f[z(t)] = \operatorname{Re} z(t) = 1$$

$$\begin{aligned}\int_C f(z) dz &= \int_a^b f[z(t)] \dot{z}(t) dt \\&= \int_0^1 1 \cdot 2i dt \\&= 2i \int_0^1 dt \\&= 2i [t]_0^1 \\&= 2i(1-0) \\&= 2i \quad \text{--- (3)}\end{aligned}$$

Substituting (2) + (3) in (1)

$$\int_C \operatorname{Re} z dz = \frac{1}{2} + 2i$$

Q4) Evaluate $\int_C \operatorname{Im}(z^2) dz$ where C is the triangle with vertices $0, 1, i$ counter-clockwise.

Sol: Let C be the triangle with vertices $0, 1, i$ in the counter-clockwise direction with sides c_1, c_2, c_3 as shown below.

Along c_1

Parametric equation

$$z(t) = 0 + t(1-0)$$

$$z(t) = t$$

$$\dot{z}(t) = 1$$

$$d[z(t)] = \operatorname{Im}[z(t)]^2$$

$$= \operatorname{Im}[t^2]$$

$$= 0$$

$$\therefore \int_{c_1} \operatorname{Im}(z^2) dz = \int_{c_1} d[z(t)] \dot{z}(t) dt$$

$$= \int_C 0 dt$$

$$= \underline{\underline{0}}$$

Along c_2

$$\text{Parametric equation } z(t) = z_1 + t[z_2 - z_1]$$

$$= 1 + t[i-1]$$

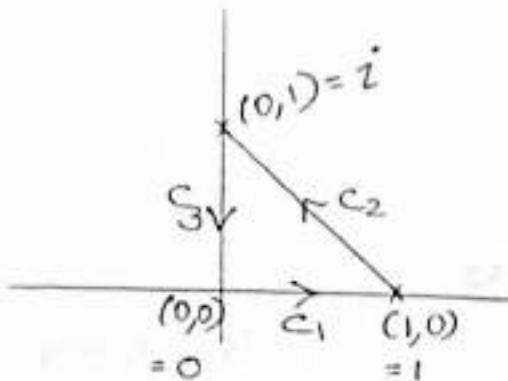
$$=(1-t)+it$$

$$\dot{z}(t) = -1+i$$

$$d[z(t)] = \operatorname{Im}[z(t)]^2$$

$$= \operatorname{Im}[(1-t)+it]^2$$

$$= \operatorname{Im}[-t^2 - t^2 + 2it(1-t)]$$



$$= 2t(1-t)$$

$$\begin{aligned} \int_{C_2} \operatorname{Im}(g^2) dz &= \int_{C_2} f[z(t)] \dot{z}(t) dt \\ &= \int_0^1 2t(1-t)(-1+i) dt \\ &= 2(-1+i) \int_0^1 (t - t^2) dt \\ &= 2(-1+i) \left[\frac{t^2}{2} - \frac{t^3}{3} \right]_0^1 \\ &= 2(-1+i) \left[\left(\frac{1}{2} - \frac{1}{3} \right) - 0 \right] \\ &= 2(-1+i) \frac{1}{6} \\ &= \frac{1}{3}(-1+i) \quad \underline{\underline{=}} \end{aligned}$$

Along C_3

$$\text{parametric equation } z(t) = z_1 + t[z_2 - z_1]$$

$$z(t) = i + t[0-i]$$

$$= i^o - t i^o$$

$$= i^o (1-t)$$

$$\dot{z}(t) = i^o (0-1) = -i^o$$

$$\begin{aligned} f[z(t)] &= \operatorname{Im}[z(t)]^2 \\ &= \operatorname{Im}[i^o(1-t)]^2 \\ &= \operatorname{Im}[-(1-t)^2] \\ &= 0 \end{aligned}$$

$$\begin{aligned} \therefore \int_{C_3} \operatorname{Im}(g^2) dz &= \int_{C_3} f[z(t)] \dot{z}(t) dt \\ &= \int_0^1 i^o(1-t) 0 dt = \underline{\underline{0}} \end{aligned}$$

$$\begin{aligned}
 \int_C \operatorname{Im}(z^2) dz &= \int_{C_1} \operatorname{Im}(z^2) dz + \int_{C_2} \operatorname{Im}(z^2) dz + \int_{C_3} \operatorname{Im}(z^2) dz \\
 &= 0 + \frac{1}{3}(-1+i) + 0 \\
 &= \underline{\underline{\frac{(1-i)}{3}}}
 \end{aligned}$$

H.W Q Evaluate $\oint \operatorname{Re} z^2 dz$ over the boundary C of the square with vertices $0, i, 1+i, 1$ clockwise.

H.W
 Q5) Evaluate $\oint \operatorname{Re} z^2 dz$ over the boundary C of the square with vertices $0, i, 1+i, 1$ clockwise.

Ans: parametric equation of C_1 [0 to i]

$$z(t) = z_1 + t[z_2 - z_1]$$

$$= 0 + t[i - 0]$$

$$= it$$

$$\dot{z}(t) = i$$

$$f(z) = \operatorname{Re} z^2$$

$$f[z(t)] = \operatorname{Re}[z(t)]^2$$

$$= \operatorname{Re}(it)^2$$

$$= \operatorname{Re}(-t^2)$$

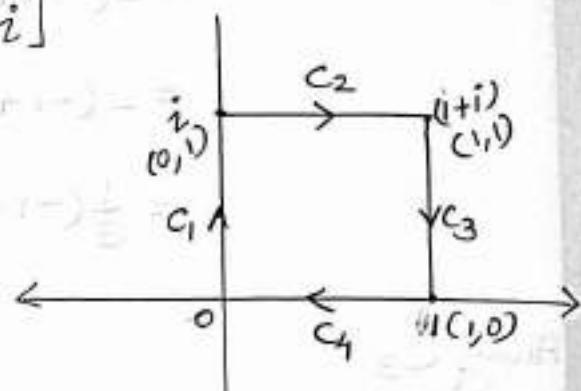
$$= -t^2$$

$$\int_{C_1} \operatorname{Re} z^2 dz = \int_0^1 f[z(t)] \dot{z}(t) dt$$

$$= \int_0^1 -t^2 i dt$$

$$= -i \left[+\frac{t^3}{3} \right]_0^1$$

$$= -i \left[+\frac{1}{3} \right] = -\frac{2}{3}$$



Parametric equation of C_2 [$i \text{ to } 1+i$]

$$z(t) = i + t[(1+i) - i]$$

$$= i + t$$

$$\dot{z}(t) = 1$$

$$\oint_{C_2} \operatorname{Re}(z^2) dz = \operatorname{Re} [z(t)]^2 = \operatorname{Re} (i+t)^2$$
$$= \operatorname{Re} [-1 + t^2 + 2it]$$

$$= -1 + t^2$$

$$\begin{aligned}\oint_{C_2} \operatorname{Re}(z^2) dz &= \int_0^1 (-1 + t^2) \cdot 1 dt \\&= \left[-t + \frac{t^3}{3} \right]_0^1 \\&= \left[-1 + \frac{1}{3} \right] - [0+0] \\&= -\frac{2}{3}\end{aligned}$$

Parametric equation of C_3 [$1+i \text{ to } i$]

$$z(t) = (1+i) + t[1 - (1+i)]$$

$$= (1+i) - it$$

$$= 1 + i(1-t)$$

$$\dot{z}(t) = 0 + i(0-1) = -i$$

$$\begin{aligned}\oint_{C_3} \operatorname{Re}[z(t)]^2 dz &= \operatorname{Re} [z(t)]^2 \\&= \operatorname{Re} [1 + i(1-t)]^2\end{aligned}$$

$$= \operatorname{Re} [1 - (1-t)^2 + 2i(1-t)]$$

$$= 1 - (1-t)^2$$

$$= 1 - [1 + t^2 - 2t]$$

$$= -[t^2 - 2t]$$

$$\begin{aligned}
 \int_{C_3} \operatorname{Re}(z^2) dz &= \int_0^1 +[t^2+2t]i dt \\
 &= +i \int_0^1 (t^2+2t) dt \\
 &= +i \left[\frac{t^3}{3} + t^2 \right]_0^1 \\
 &= +i \left[\left(\frac{1}{3} - 1 \right) - (0 - 0) \right] \\
 &= -\frac{2}{3}i
 \end{aligned}$$

Parametric equations of C_4 :

$$\begin{aligned}
 z(t) &= 1 + t(0 - i) \\
 &= 1 - ti
 \end{aligned}$$

$$\dot{z}(t) = -i$$

$$\begin{aligned}
 f[z(t)] &= \operatorname{Re}[z(t)]^2 \\
 &= \operatorname{Re}[(1-t)^2]
 \end{aligned}$$

$$\begin{aligned}
 \int_{C_4} \operatorname{Re}(z^2) dz &= \int_0^1 f[z(t)] \dot{z}(t) dt \\
 &= \int_0^1 (1-t)^2 - 1 dt \\
 &= -i \int_0^1 (1-t)^2 dt \\
 &= -i \left[\frac{[1-t]^3}{3(-i)} \right]_0^1 \\
 &= \frac{1}{3} \left[[1-t]^3 \right]_0^1 \\
 &= \frac{1}{3}[0 - 1] = -\frac{1}{3}
 \end{aligned}$$

$$\begin{aligned}\therefore \oint_C \operatorname{Re} z^2 dz &= -\frac{1}{3} + -\frac{2}{3} + -\frac{2}{3}^{\circ} - \frac{1}{3} \\ &= -\frac{3}{3} - \frac{3}{3}^{\circ} \\ &= -1 - \underline{\underline{1}}^{\circ}\end{aligned}$$

(Q) Show that $\oint_C \frac{dz}{z} = 2\pi i$ where C is the circle of radius 1 and centre 0 in the counter clockwise direction

Ans: Parametric equation of the circle $z(t) = z_0 + Re^{it}$

$$\text{ie) } z(t) = 0 + e^{it}$$

$$z(t) = e^{it} \quad 0 \leq t \leq 2\pi$$

$$\dot{z}(t) = e^{it} i$$

$$f(z) = \frac{1}{z}$$

$$f[z(t)] = \frac{1}{z(t)} = \frac{1}{e^{it}}$$

$$\begin{aligned}\oint_C \frac{dz}{z} &= \oint_0^{2\pi} f[z(t)] \dot{z}(t) dt \\ &= \int_0^{2\pi} \frac{1}{e^{it}} e^{it} i dt \\ &= i \int_0^{2\pi} dt \\ &= i [t]_0^{2\pi} \\ &= i [2\pi - 0] \\ &= 2\pi i\end{aligned}$$

(Q) Evaluate $\oint_C \left(\frac{5}{(z-2i)} - \frac{6}{(z-2i)^2} \right) dz$ where C is the circle $|z-2i|=4$ clockwise.

Ans: $|z - 2i| = 4$ equation of the circle with centre at $2i$ and radius = 4

parametric equation of the circle is $z(t) = z_0 + Re^{-it}$ [clockwise]

$$(e) \quad z(t) = 2i + 4e^{-it} \quad 0 \leq t \leq 2\pi$$

$$\dot{z}(t) = 4e^{-it}(-i)$$

$$f(z) = \frac{5}{z-2i} - \frac{6}{(z-2i)^2}$$

$$f[z(t)] = \frac{5}{z(t)-2i} - \frac{6}{(z(t)-2i)^2}$$

$$= \frac{5}{4e^{-it}} - \frac{6}{(4e^{-it})^2}$$

$$\int_C f(z) dz = \int_0^{2\pi} f[z(t)] \dot{z}(t) dt$$

$$= \int_0^{2\pi} \left(\frac{5}{4e^{-it}} - \frac{6}{16e^{-2it}} \right) 4e^{-it}(-i) dt$$

$$= \int_0^{2\pi} \left(5 - \frac{6}{4e^{-it}} \right) -i dt$$

$$= -i \int_0^{2\pi} \left(5 - \frac{3}{2} e^{it} \right) dt$$

$$= -i \left[5t - \frac{3}{2} \frac{e^{it}}{i} \right]_0^{2\pi}$$

$$= -i \left\{ \left[10\pi - \frac{3}{2} \frac{e^{i2\pi}}{i} \right] - \left[0 - \frac{3}{2} \frac{1}{i} \right] \right\}$$

$$= -10\pi i^0 + \frac{3}{2} e^{i2\pi} - \frac{3}{2}$$

$$= -10\pi i^0 \left[\frac{3}{2} (\cos 2\pi + i \sin 2\pi) \right] - \frac{3}{2}$$

$$= -10\pi i^0 + \frac{3}{2} [1 + i 0] - \frac{3}{2}$$

$$= -10\pi i^0$$

Q7) Evaluate $\int_C (g + g') dz$, C is the unit circle counter-clockwise.

Ans: Parametric equation of unit circle $|z|=1$ in the counter-clockwise direction is $z(t) = 0 + ie^{it}$

$$z(t) = e^{it} \quad 0 \leq t \leq 2\pi$$

$$\dot{z}(t) = ie^{it}$$

$$f(z) = z + \frac{1}{z}$$

$$f[z(t)] = z(t) + \frac{1}{z(t)}$$

$$= \left(e^{it} + \frac{1}{e^{it}} \right)$$

$$\int_C (g + g') dz = \int_0^{2\pi} f[z(t)] \dot{z}(t) dt$$

$$= \int_0^{2\pi} \left(e^{it} + \frac{1}{e^{it}} \right) ie^{it} dt$$

$$= i \int_0^{2\pi} (e^{2it} + 1) dt$$

$$= i \left[\frac{e^{2it}}{2i} + t \right]_0^{2\pi}$$

$$= i \left\{ \left[\frac{1}{2i} e^{i4\pi} + 2\pi \right] - \left[\frac{1}{2i} + 0 \right] \right\}$$

$$= i \left[\frac{1}{2i} (\cos 4\pi + i \sin 4\pi) + 2\pi - \frac{1}{2i} \right]$$

$$= i \left[\frac{1}{2i} [1 + 0] + 2\pi - \frac{1}{2i} \right]$$

$$= \underline{i} \underline{2\pi}$$

Q8) Evaluate $\int_C |z| dz$, where C is the left hand of the unit circle $|z|=1$ from $z=-i$ to $z=i$ ($-\frac{\pi}{2} \leq \arg z \leq \frac{\pi}{2}$) [KJU 2017]

Ans: parametric equations of $|z|=1$

$$z(t) = z_0 + re^{it}$$

$$z(t) = e^{it} \quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$$

$$\dot{z}(t) = ie^{it}$$

$$f(z) = |z|$$

$$f(z(t)) = |z(t)|$$

$$= |e^{it}|$$

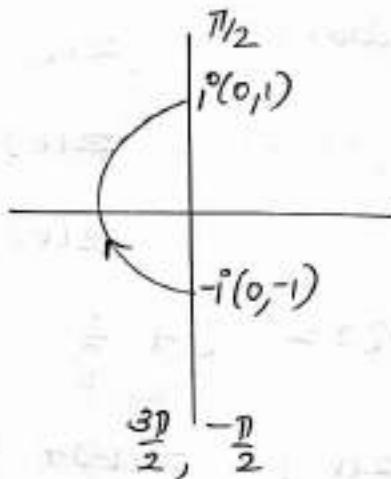
$$= |\cos t + i \sin t|$$

$$= \sqrt{\cos^2 t + \sin^2 t} = \underline{1}$$

$$\int_C |z| dz = \int f[z(t)] \dot{z}(t) dt$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 \cdot ie^{it} dt$$

$$= i \left[\frac{e^{it}}{i} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$



$$\begin{aligned}
 &= [e^{it}]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
 &= e^{i\frac{\pi}{2}} - e^{-i\frac{\pi}{2}} \\
 &= [\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}] - [\cos \frac{\pi}{2} - i \sin \frac{\pi}{2}] \\
 &= (0 + 1 \cdot i) - (0 - 1i) \\
 &= \underline{\underline{2i}}
 \end{aligned}$$

Q9) Evaluate $\int_C \frac{z+2}{z} dz$ where C is the Semicircle $|z|=2$ above the real axis from $(2,0)$ to $(-2,0)$.

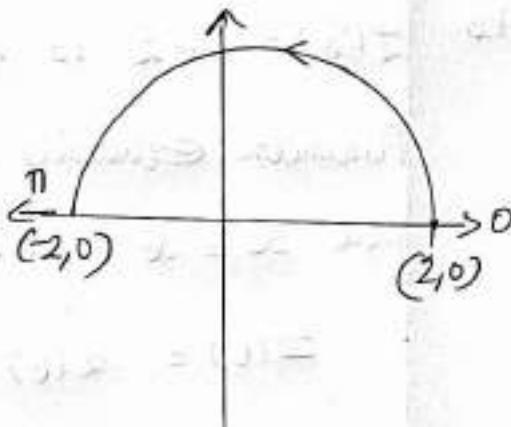
Ans: Parametric equation of the circle $|z|=2$ is

$$z(t) = 2e^{it} \quad 0 \leq t \leq \pi$$

$$\dot{z}(t) = 2ie^{it}$$

$$f(z) = \frac{z+2}{z}$$

$$f[z(t)] = \frac{z(t)+2}{z(t)} = \frac{2e^{it}+2}{2e^{it}}$$



$$\int_C \frac{z+2}{z} dz = \int_0^\pi \left(\frac{2e^{it}+2}{2e^{it}} \right) 2ie^{it} dt$$

$$= i \int_0^\pi (2e^{it} + 2) dt$$

$$= i \left[\frac{2e^{it}}{i} + 2t \right]_0^\pi$$

$$= i^o \left\{ \left[\frac{2e^{i^o \pi}}{i^o} + 2\pi \right] - \left[\frac{2}{i^o} + 0 \right] \right\}$$

$$= i^o \left[\frac{2}{i^o} (\cos \pi + i \sin \pi) + 2\pi - \frac{2}{i^o} \right]$$

$$= i^o \left[\frac{2}{i^o} (-1) + 2\pi - \frac{2}{i^o} \right]$$

$$= i^o \left[-\frac{4}{i^o} + 2\pi \right]$$

$$= -4 + 2\pi i^o$$

Q 10) Evaluate $\int_C \operatorname{Re}(z) dz$, C is the parabola $y = 1 + \frac{1}{2}(x-1)^2$ from $1+i^o$ to $3+3i^o$.

Ans: $f(z) = \operatorname{Re} z$ is not analytic

Polarmetric equation of the parabola is obtained by

$$\text{Put } x = t \quad \therefore y = 1 + \frac{1}{2}(t-1)^2$$

$$\therefore z(t) = x(t) + iy(t)$$

$$z(t) = t + i \left[1 + \frac{1}{2}(t-1)^2 \right]$$

$$\dot{z}(t) = 1 + i \left[0 + \frac{1}{2} \cdot 2(t-1) \right]$$

$$= 1 + i(t-1)$$

$$f(z) = \operatorname{Re} z$$

$$f[z(t)] = \operatorname{Re} z(t)$$

$$= t$$

C is the parabola from $1+i^0$ to $3+3i^0$

i.e) from $(1, 1)$ to $(3, 3)$

$t = \infty \quad \therefore t$ varies from 1 to 3

$$\begin{aligned}\therefore \int_C \operatorname{Re}(z) dz &= \int f[z(t)] z'(t) dt \\&= \int_1^3 t [1 + i(t-1)] dt \\&= \int_1^3 [t + it^2 - i^0 t] dt \\&= \left[\frac{t^2}{2} + i \frac{t^3}{3} - i^0 \frac{t^2}{2} \right]_1^3 \\&= \left[\frac{9}{2} + i^0 9 - i^0 \frac{9}{2} \right] - \left[\frac{1}{2} + i^0 \frac{1}{3} - i^0 \frac{1}{2} \right] \\&= 4 + \frac{14}{3} i^0 \\&= 4 + \frac{14}{3} \end{aligned}$$

Q(i) Evaluate $\int_C |g| dz$ where C is the line segment joining $-i$ and i

Ans: Parametric equation of the line segment is

$$\begin{aligned}z(t) &= z_0 + t(z_1 - z_0) \\&= -i + t(i - -i) \\&= -i + 2it = i(-1 + 2t)\end{aligned}$$

$$z(t) = 2i$$

$$f(z) = |g|$$

$$\begin{aligned}f[z(t)] &= |z(t)| \\&= |i(-1 + 2t)| \\&= \sqrt{(-1 + 2t)^2} = (-1 + 2t)\end{aligned}$$

$$\begin{aligned}
 \int_C |g| dz &= \int_0^1 d[z(t)] \bar{z}(t) dt \\
 &= \int_0^1 (-1+2t) 2^0 dt \\
 &= 2^0 \int_0^1 (-1+2t) dt \\
 &= 2^0 [-t + t^2] \Big|_0^1 \\
 &= 2^0 [(-1+1) - (0+0)] \\
 &= 0
 \end{aligned}$$

Note

If the function $f(z)$ is analytic in a domain, then $\int_C f(z) dz$ is path independent and it depends only on the end points of the curve C .

Q12) Evaluate $\int_C g^2 dz$ where C is given by

- (i) The line $x=2y$ from $(0,0)$ to $(2,1)$
- (ii) The line segment along the real axis from $(0,0)$ to $(2,0)$ and then vertically to $(2,1)$

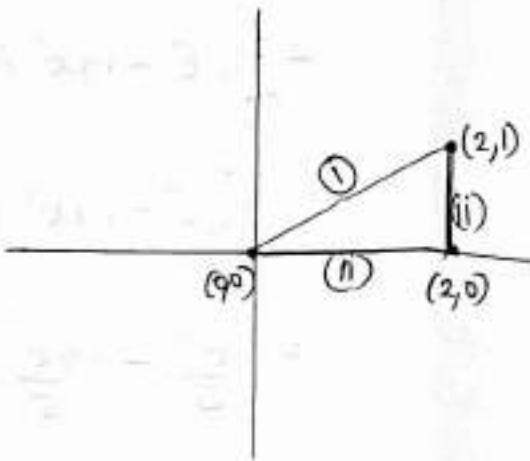
Ans: (i) Along the path $x=2y$ from $(0,0)$ to $(2,1)$

i.e) from $0+0i = 0$ to $2+1i$

$$\begin{aligned}
 \int_0^{2+i} g^2 dz &= \left[\frac{g^3}{3} \right]_0^{2+i} && [g^2 \text{ is analytic}] \\
 &= \frac{1}{3} [(2+i)^3 - 0^3] \\
 &= \frac{1}{3} [8 + 12i^0 + -6 - i^0] = \frac{1}{3} [2 + 11i^0]
 \end{aligned}$$

(ii) from 0 to 2 and from 2 to $2+i^0$

$$\begin{aligned}
 \int_C y^2 dy &= \int_{C_1} y^2 dy + \int_{C_2} y^2 dy \\
 &= \int_0^2 y^2 dy + \int_2^{2+i^0} y^2 dy \\
 &= \left[\frac{y^3}{3} \right]_0^2 + \left[\frac{y^3}{3} \right]_2^{2+i^0} \\
 &= \left[\frac{8}{3} - 0 \right] + \left[\frac{(2+i^0)^3}{3} - \frac{8}{3} \right] \\
 &= \underline{(2+i^0)^3} \\
 &= \underline{\frac{1}{3}[2+11^0]} =
 \end{aligned}$$



Q13) Evaluate $\int_C (x^2 - iy^2) dy$. where C is the parabola $y = 2x^2$ from $(1,2)$ to $(2,8)$

Ans: put $x = t$

$$\therefore y = 2t^2$$

$$\begin{aligned}
 z(t) &= x(t) + iy(t) \\
 &= t + i2t^2
 \end{aligned}$$

$$z'(t) = 1 + 4i^0 t$$

$$f(y) = x^2 - iy^2$$

$$\begin{aligned}
 \int [z(t)] = & [x(t)]^2 - i[y(t)]^2 \\
 & = t^2 - i^0 4t^4
 \end{aligned}$$

$$\int_C (x^2 - iy^2) dy = \int f[z(t)] z'(t) dt$$

$$\begin{aligned}
 &= \int_1^2 (t^2 - 14t^4)(1 + 4t^6) dt \\
 &= \int_1^2 (t^2 - 14t^4 + 4t^{10} + 16t^{12}) dt \\
 &= \left[\frac{t^3}{3} - \frac{14t^5}{5} + t^{11} + \frac{16t^{13}}{13} \right]_1^2 \\
 &= \left[\frac{8}{3} - \frac{128}{5} + 16 + \frac{512}{3} \right] - \left[\frac{1}{3} - \frac{4}{5} + 1 + \frac{8}{3} \right] \\
 &= \frac{7}{3} - \frac{124}{5} + 15 + \frac{504}{3} \\
 &= \frac{511}{3} - \frac{49}{5} \\
 &\quad \underline{\underline{=}}
 \end{aligned}$$

Q8) Evaluate $\int_C |z| dz$, where C is the left hand of the unit circle $|z|=1$ from $z=-i$ to $z=i\left(-\frac{\pi}{2} \leq \arg z \leq \frac{\pi}{2}\right)$ [K70 2017]

Ans: parametric equation of $|z|=1$

$$z(t) = z_0 + re^{it}$$

$$z(t) = e^{it} \quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$$

$$\dot{z}(t) = i e^{it}$$

$$f(z) = |z|$$

$$f(z(t)) = |z(t)|$$

$$= |e^{it}|$$

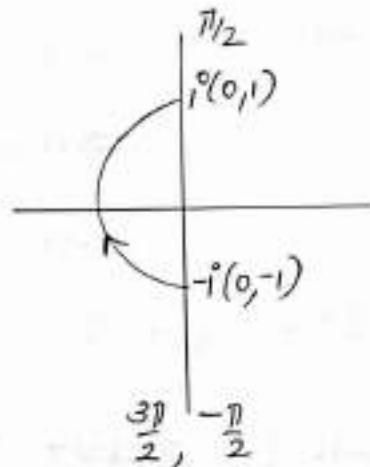
$$= |\cos t + i \sin t|$$

$$= \sqrt{\cos^2 t + \sin^2 t} = 1$$

$$\int_C |z| dz = \int f[z(t)] \dot{z}(t) dt$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 \cdot i e^{it} dt$$

$$= -i \left[\frac{e^{it}}{i} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$



$$\begin{aligned}
 &= \left[e^{-it} \right]_{-\pi/2}^{\pi/2} \\
 &= e^{-i\pi/2} - e^{+i\pi/2} \\
 &= \left[\cos \frac{\pi}{2} - i \sin \frac{\pi}{2} \right] - \left[\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right] \\
 &= (0 - 1i) - (0 + 1i) \\
 &= \underline{\underline{-2i}}
 \end{aligned}$$

Cauchy's Integral Theorem

A simply connected domain D in the complex plane is a domain such that every simple closed path in D encloses only points of D .

e.g. Interior of a circle, ellipse.

A domain that is not simply connected is called multiply connected domain

e.g. Annulus.



Simply connected



Simply connected



not simply connected

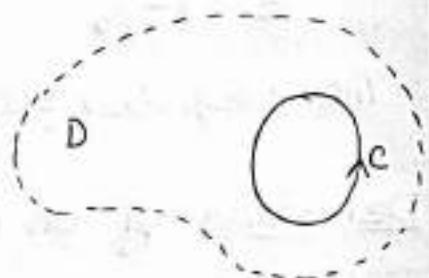


not simply connected.

Cauchy's integral Theorem

If $f(z)$ is analytic in a simply connected domain D , then for every simple closed paths C in D

$$\oint_C f(z) dz = 0$$



Independence of paths

If $f(z)$ is analytic in a simply connected domain D , then the integral of $f(z)$ is independent of path in D

$$(e) \int_{C_1} f(z) dz = \int_{C_2} f(z) dz$$

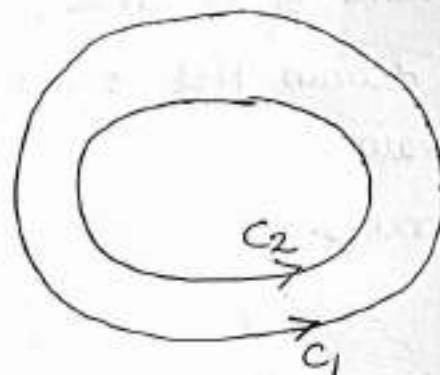


Cauchy's integral theorem for multiply connected Domains

Cauchy's theorem applies to multiply connected domains. Consider a doubly connected domain D with outer boundary C_1 and inner boundary C_2 . If a function $f(z)$ is analytic in domain D^* that contains D and its boundary curves, then

$$\oint_{C_1} f(z) dz = \oint_{C_2} f(z) dz$$

both the integrals being taken counter-clockwise or clockwise and regardless of whether or not the full interior of C_2 belongs to D^* .

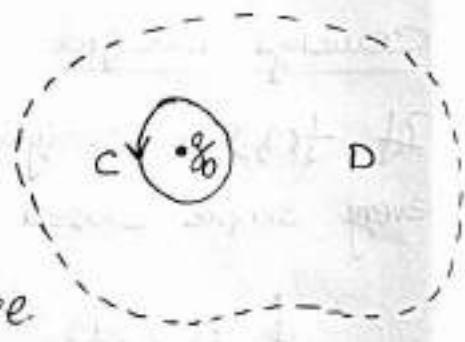


Cauchy's integral formula

Let $f(z)$ be analytic in a simply connected domain D . Then for any point z_0 in D and any simple closed path C in D that encloses z_0 ,

$$\oint_C \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0)$$

The integration being taken counter-clockwise.



Derivatives of an analytic function.

If $f(z)$ is analytic in a domain D , then it has derivatives of all orders in D , which are then also analytic functions in D . The values of these derivatives at a point z_0 in D are given by the formulas,

$$f^{(n)}(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z) dz}{(z-z_0)^{n+1}}$$

[Cauchy's integral formula]

$$f'(z_0) = \frac{1!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^2} dz$$

$$f''(z_0) = \frac{2!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^3} dz$$

and in general

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz$$

here C is any simple closed path in D that encloses z_0 and whose full interior belongs to D , and we integrate counterclockwise around C .

Note:

The point at which $f(z)$ is not analytic is known as singular point.

Problems

- 1) Find $\oint_C e^z dz$, $\oint_C \cos z dz$, $\oint_C z^n dz$ ($n=0, 1, 2, \dots$) where C is any closed path.

Ans: e^z , $\cos z$, z^n are analytic everywhere in the complex plane

\therefore By Cauchy's integral theorem

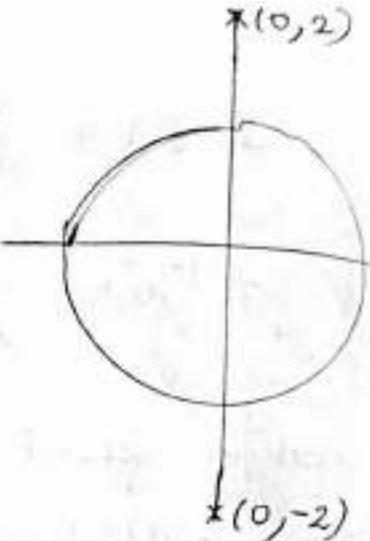
$$\oint_C e^z dz = 0$$

$$\oint_C \cos z dz = 0$$

$$\oint_C z^n dz = 0$$

- 2) Evaluate $\oint_C \frac{dz}{z^2 + 4}$ where C is the unit circle.

Ans: Singular points are obtained by taking denominator = 0



$$z^2 + 4 = 0$$

$$z^2 = -4$$

$$z = \sqrt{-4} = \pm 2i \quad [(0, 2), (0, -2)]$$

both the singular points lies outside C

$\therefore f(z)$ is analytic everywhere in C

\therefore By Cauchy's integral theorem,

$$\oint_C \frac{dz}{z^2+4} = 0$$

3) Evaluate $\int_C \frac{1}{(z^2+4)} dz$ over

$$(a) |z-2| = 2 \quad \text{and} \quad (b) |z-2| = 3$$

Ans (a) Singular points are obtained by taking

$$z^2 + 4 = 0$$

$$z^2 = -4$$

$$z = \pm 2i$$

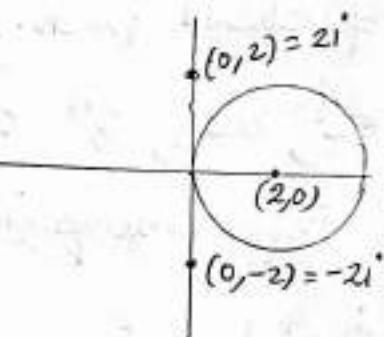
$|z-2|=2$ is a circle with centre at 2

$$\text{Radius} = 2$$

both the singular points lies outside C

\therefore By Cauchy's integral theorem

$$\int_C \frac{1}{(z^2+4)} dz = 0$$

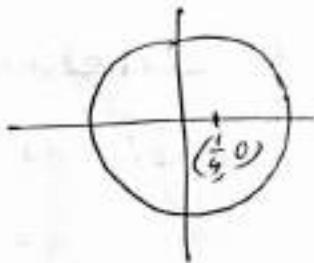


4) Integrate $f(z) = \frac{1}{4z-1}$ counter clockwise around the unit circle

Ans:

$$f(z) = \frac{1}{4z-1}$$

$$4z-1=0 \Rightarrow z=\frac{1}{4} \Rightarrow |z|=\frac{1}{4}$$



$f(z) = \frac{1}{4z-1}$ is not analytic when $z = \frac{1}{4}$, which lies inside $|z|=1$

∴ Cauchy's integral theorem is not applicable. Cauchy's integral formula is used.

$$\oint_C \frac{f(z) dz}{(z-z_0)} = 2\pi i f(z_0)$$

$$\therefore \int_C \frac{1}{4z-1} dz = \int_C \frac{1}{4(z-\frac{1}{4})} dz$$

$$= \frac{1}{4} \times 2\pi i f(\frac{1}{4})$$

$$\begin{aligned} f(z) &= 1 \\ f(\frac{1}{4}) &= 1 \end{aligned}$$

$$= \frac{1}{4} \times 2\pi i \times 1$$

$$= \underline{\underline{\frac{\pi i}{2}}}$$

5) Evaluate $\int_C \frac{1}{z^4-1.2} dz$ where C is the unit circle

counter clockwise

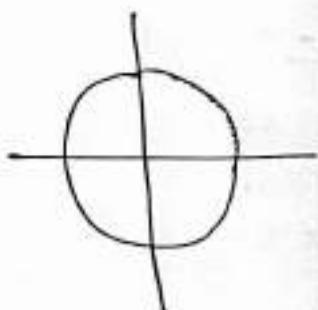
$$f(z) = \frac{1}{z^4-1.2}$$

$$z^4 - 1.2 = 0$$

$$z^4 = 1.2$$

$z = (1.2)^{1/4} > 1$ which lies outside $|z|=1$

∴ $f(z)$ is analytic in $|z|=1$

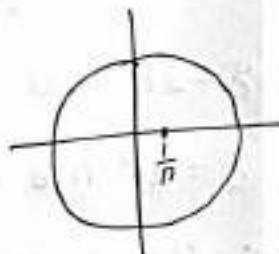


\therefore By Cauchy's integral theorem $\oint_C \frac{dz}{z^4 - 1} = 0$

6) Evaluate $\oint_C \frac{1}{\pi z - 1} dz$ where C is the unit circle $|z|=1$.

Ans: $f(z) = \frac{1}{\pi z - 1}$

$$\pi z - 1 = 0 \Rightarrow \pi z = 1 \Rightarrow z = \frac{1}{\pi}$$



$z = \frac{1}{\pi}$ lies inside $|z|=1$

\therefore By Cauchy's integral theorem formula

$$\oint_C \frac{1}{\pi z - 1} dz = \oint_C \frac{1}{\pi(z - \frac{1}{\pi})} dz$$

$$= \frac{1}{\pi} \cdot \oint_C \frac{1}{(z - \frac{1}{\pi})} dz$$

$$f(z) = 1$$

$$f(\frac{1}{\pi}) = 1$$

$$= \frac{1}{\pi} \times 2\pi i \times f\left(\frac{1}{\pi}\right)$$

$$= 2^{\circ} \times 1$$

$$= \underline{\underline{2^{\circ}}}$$

7) Evaluate $\oint_C \frac{1}{(5z-1)} dz$ where C is the unit circle $|z|=1$

Ans: $f(z) = \frac{1}{5z-1}$

$$5z - 1 = 0 \Rightarrow 5z = 1 \Rightarrow z = \frac{1}{5} \text{ lies inside } |z|=1$$

\therefore By Cauchy's integral formula

$$\oint_C \frac{1}{5z-1} dz = \oint_C \frac{1}{5(z - \frac{1}{5})} dz = \frac{1}{5} \oint_C \frac{1}{(z - \frac{1}{5})} dz$$

$$\begin{cases} f(z) = 1 \\ f\left(\frac{1}{5}\right) = 1 \end{cases}$$

$$= \frac{1}{5} \times 2\pi i f\left(\frac{1}{5}\right)$$

$$= 2\pi i / 5 //$$

8) Evaluate $\oint_C \frac{dz}{z-2i}$ C is the circle $|z|=1$ counterclockwise.

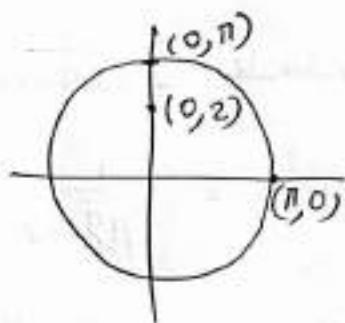
Ans: $f(z) = \frac{1}{z-2i}$

$$z-2i = 0 \Rightarrow z = 2i \quad (0, 2)$$

$$z = 2i \text{ lies inside } |z|=1$$

∴ By Cauchy's integral formula

$$\begin{aligned} \oint_C \frac{dz}{z-2i} &= 2\pi i f(2i) \\ &= 2\pi i \times 1 \\ &= 2\pi i \end{aligned}$$



$$\begin{cases} f(z) = 1 \\ f(2i) = 1 \end{cases}$$

9) Integrate $\frac{z^2}{z^2-1}$ by Cauchy's formula counter clockwise around the circle. $|z+1| = 3/2$.

Ans: $f(z) = \frac{z^2}{z^2-1}$

$$z^2-1 = 0 \Rightarrow (z-1)(z+1) = 0$$

$$(z-1) = 0 \text{ or } (z+1) = 0$$

$$z = 1, -1$$

$$z = 1 \Rightarrow |1+1| = 2 > 3/2 \quad \therefore z = 1 \text{ lies outside } C$$

$$z = -1 \Rightarrow |-1+1| = 0 < 3/2 \quad \therefore z = -1 \text{ lies inside } C$$

$$\oint_C \frac{z^2}{z^2-1} dz = \oint_C \frac{z^2}{(z-1)(z+1)} dz$$

$$= \int_{\gamma} \frac{z^2/(z-1)}{(z-(-1))} dz$$

$$= \int_{\gamma} \frac{g(z) dz}{(z-(-1))} \quad \left[g(z) = \frac{z^2}{z-1} \text{ is analytic in } \gamma \right]$$

By Cauchy's integral formula

$$g(-1) = \frac{(-1)^2}{-1-1} = -\frac{1}{2}$$

$$= 2\pi i^0 g(-1)$$

$$= 2\pi i^0 \times \frac{1}{-2}$$

$$= -\underline{\underline{\pi i^0}}$$

10) Evaluate $\oint_C \frac{\sin \theta}{\theta + 4i^0} d\theta$ where $C: |\theta - 4-2i^0| = 6.5$

Ans: $\theta + 4^0 = 0$

$\theta = -4^0$ is the singular point.

$C: |\theta - (4+2i^0)| = 6.5$ is a circle with centre at $4+2i^0$ and radius 6.5

$$\begin{aligned}\theta = -4^0 &\Rightarrow |\theta - (4+2i^0)| \\&= |-4^0 - 4 - 2i^0| \\&= |-4 - 6i^0| \\&= \sqrt{16+36} \\&= \sqrt{52} > 6.5\end{aligned}$$

$\therefore \theta = -4^0$ lies outside the circle $C: |\theta - 4-2i^0| = 6.5$

\therefore By Cauchy's integral theorem

$$\oint_C \frac{\sin \theta}{\theta + 4i^0} d\theta = 0$$

11) Evaluate $\oint_C \frac{z}{z^2+5z+4} dz$ where C is the circle with centre and radius 2.

$$\text{Ans. } z^2 + 5z + 4 = 0$$

$$(z+4)(z+1) = 0$$

$$z = -4, -1$$

$$C: |z - -1| = 2$$

$$\text{ie) } C: |z+1| = 2$$

$$z = -4 \Rightarrow |-4+1| = 3 > 2$$

$\therefore z = -4$ lies outside C

$$z = -1 \Rightarrow |-1+1| = 0 < 2$$

$\therefore z = -1$ lies inside C

$$\therefore \oint_C \frac{z dz}{z^2+5z+4} = \oint_C \frac{z dz}{(z+4)(z+1)}$$

$$= \oint_C \frac{z/(z+4)}{(z+1)} dz$$

$$= 2\pi i^0 \times f(-1)$$

$$= 2\pi i^0 \times \frac{(-1)}{(-1+4)}$$

$$= 2\pi i^0 \times \frac{-1}{3}$$

$$= -\frac{2\pi i^0}{3}$$

12) Evaluate $\int_C \frac{4-3z}{z(z-1)} dz$ over the circle $|z| = 3/2$

Ans: $z(z-1) = 0$

$z = 0$ & $z-1 = 0$

$z = 0, 1$

both $z=0, 1$ lies inside C

\therefore By partial fraction

$$\frac{4-3z}{z(z-1)} = \frac{A}{z} + \frac{B}{(z-1)}$$

\times by $z(z-1)$

$$4-3z = A(z-1) + Bz$$

$$z=1 \Rightarrow 1 = A(0) + B \quad \therefore B=1$$

$$z=0 \Rightarrow 4 = A(-1) \quad \therefore A=-4$$

$$\therefore \frac{4-3z}{z(z-1)} = \frac{-4}{z} + \frac{1}{(z-1)}$$

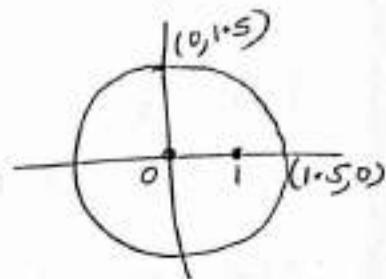
$$\begin{aligned} \therefore \int_C \frac{4-3z}{z(z-1)} dz &= \int_C -\frac{4}{z} dz + \int_C \frac{1}{(z-1)} dz \\ &= \int_C \frac{-4}{z-0} dz + \int_C \frac{1}{(z-1)} dz \end{aligned}$$

$$= 2\pi i f(0) + 2\pi i f(1)$$

$$= 2\pi i (-4) + 2\pi i (1)$$

$$= -8\pi i + 2\pi i$$

$$= -6\pi i$$



13) Evaluate $\int \frac{3z^2 + 7z + 1}{z+1} dz$ over the circle $|z+i|=1$

Ans.

$$z+1=0$$

$z = -1$ is the singular point

$$C: |z+i|=1$$

$$z = -1 \Rightarrow |-1+i|$$

$$= \sqrt{1+1} = \sqrt{2} > 1$$

$\therefore z = -1$ lies outside $C \therefore f(z)$ is analytic inside C

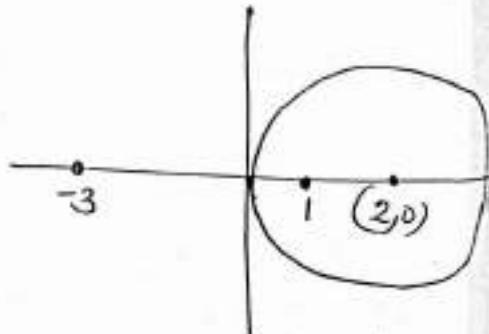
$$\therefore \text{By Cauchy's integral theorem } \int \frac{3z^2 + 7z + 1}{z+1} dz = 0$$

14) Evaluate $\int_C \frac{5z+7}{z^2+2z-3} dz$ where C is the circle $|z-2|=2$

Ans. $z^2+2z-3=0$

$$(z+3)(z-1)=0$$

$z = -3, 1$ are the singular points.



$z=1$ lies inside C & $z=-3$ lies outside C

$$\therefore \int_C \frac{5z+7}{z^2+2z-3} dz = \int_C \frac{5z+7}{(z+3)(z-1)} dz$$

$$= \int_C \frac{\frac{5z+7}{(z+3)}}{(z-1)} dz$$

$$= 2\pi i \times f(1)$$

$$= \underline{\underline{6\pi i}}$$

$$\begin{cases} f(z) = \frac{5z+7}{z+3} \\ f(1) = \frac{12}{4} = 3 \end{cases}$$

15 Integrate $\frac{z^2}{z^2-1}$ counterclockwise around the circle $|z-1-i|^0 = \frac{\pi}{2}$
by Cauchy's integral formula.

$$\text{Ans: } z^2 - 1 = 0$$

$z^2 = 1 \Rightarrow z = +1, -1$ are the singular points.

$$z = 1 \Rightarrow |1-1-i|^0 = |-i|^0 = \sqrt{(-1)^2} = 1 < \frac{\pi}{2}$$

$\therefore z = 1$ lies inside C.

$$z = -1 \Rightarrow |-1-1-i|^0$$

$$= |-2-i|^0 = \sqrt{4+1} = \sqrt{5} > \frac{\pi}{2}$$

$\therefore z = -1$ lies outside C

$$\begin{aligned}\therefore \int_C \frac{z^2}{z^2-1} dz &= \int \frac{z^2}{(z-1)(z+1)} dz \\ &= \int \frac{z^2/(z+1)}{(z-1)} dz \\ &= 2\pi i^0 f(1) \\ &= 2\pi i^0 \times \frac{1}{2} \\ &= \underline{\underline{\pi i^0}}\end{aligned}$$

$$f(z) = \frac{z^2}{(z-1)}$$

$$f(1) = \frac{1}{2}$$

15 Integrate $\frac{z^2}{z^2-1}$ counterclockwise around the circle $|z-1-i|^0 = \frac{\pi}{2}$
by Cauchy's integral formula.

$$\text{Ans: } z^2 - 1 = 0$$

$z^2 = 1 \Rightarrow z = +1, -1$ are the singular points.

$$z=1 \Rightarrow |1-1-i|^0 = |-i|^0 = \sqrt{(-1)^2} = 1 < \frac{\pi}{2}$$

$\therefore z=1$ lies inside C .

$$z=-1 \Rightarrow |-1-1-i|^0$$

$$= |-2-i|^0 = \sqrt{4+1} = \sqrt{5} > \frac{\pi}{2}$$

$\therefore z=-1$ lies outside C

$$\begin{aligned}\oint_C \frac{z^2}{z^2-1} dz &= \int \frac{z^2}{(z-1)(z+1)} dz \\ &= \int \frac{z^2}{(z-1)} dz \\ &= 2\pi i^0 f(1) \\ &= 2\pi i^0 \times \frac{1}{2} \\ &= \underline{\underline{\pi i^0}}\end{aligned}$$

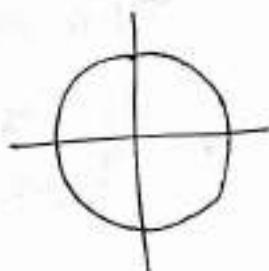
$$f(z) = \frac{z^2}{(z-1)}$$

$$f(1) = \frac{1}{2}$$

16) Evaluate $\oint_C \frac{\sin z}{z^4} dz$ $C: |z|=1$

$$\text{Ans: } z^4 = 0$$

$z = 0, 0, 0, 0$ lies inside C



$$\oint_C \frac{\sin z}{z^4} dz = \oint_C \frac{\sin z}{(z-0)^4} dz$$

By Cauchy's integral formula for derivatives

$$f^n(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(\gamma) d\gamma}{(\gamma - z_0)^{n+1}} \quad \therefore \oint_C \frac{f(\gamma) d\gamma}{(\gamma - z_0)^{n+1}} = \frac{2\pi i}{n!} f^n(z_0)$$

$$\begin{aligned} \therefore \oint_C \frac{\sin 2\gamma}{\gamma^4} d\gamma &= \oint_C \frac{8 \sin 2\gamma}{(\gamma - 0)^4} d\gamma & f(\gamma) &= 8 \sin 2\gamma \\ &= \frac{2\pi i}{3!} f'''(0) & f'(0) &= 2 \cos 2\gamma \\ &= \frac{2\pi i}{6} \times -8 & f''(0) &= -4 \sin 2\gamma \\ &= -\frac{8\pi i}{3} & f'''(0) &= -8 \cos 2\gamma \\ &&&= -8 \end{aligned}$$

17) Evaluate $\oint_C \frac{z^6}{(2z-1)^6} dz$ $c: |z|=1$

Ans: $(2z-1)^6 = 0$

$$2z - 1 = 0$$

$$2z = 1$$

$$z = \frac{1}{2} \text{ lies inside } |z|=1$$

$$\oint_C \frac{f(\gamma) d\gamma}{(\gamma - z_0)^{n+1}} = \frac{2\pi i}{n!} f^n(z_0)$$

$$\therefore \oint_C \frac{z^6}{(2z-1)^6} dz = \oint_C \frac{z^6}{[2(z-\frac{1}{2})]^6} dz$$

$$= \frac{1}{64} \oint_C \frac{z^6}{(z-\frac{1}{2})^6} dz$$

$$\begin{aligned}
 &= \frac{1}{64} \times \frac{2\pi^10}{5!} f'(1/2) \\
 &= \frac{1}{64} \times \frac{2\pi^10}{120} \times 360 \\
 &= \underline{\underline{\frac{3\pi^10}{32}}}
 \end{aligned}$$

$$\begin{aligned}
 f(z) &= z^6 \\
 f'(z) &= 6z^5 \\
 f''(z) &= 30z^4 \\
 f'''(z) &= 120z^3 \\
 f^{IV}(z) &= 360z^2 \\
 f^V(z) &= 720z \\
 f'(1/2) &= 720 \times \frac{1}{2} \\
 &= 360
 \end{aligned}$$

18) Evaluate $\oint_C \frac{dz}{(z-21^0)(z-\frac{1}{2})^2}$. $c \text{ is } 181 = 1$

Ans: $(z-21^0)^2(z-\frac{1}{2})^2 = 0$

$(z-21^0) = 0$ or $(z-\frac{1}{2}) = 0$

$z = 21^0$, $\frac{1}{2}$ are the singular points.

$z = 21^0$ lies outside C

$z = \frac{1}{2}$ lies inside C

$$\begin{aligned}
 \oint_C \frac{dz}{(z-21^0)(z-\frac{1}{2})^2} &= \oint_C \frac{\frac{dz}{(z-21^0)^2}}{(z-\frac{1}{2})^2} \\
 &= \frac{2\pi^10}{11} f'(\frac{1}{2}) \quad \text{--- (1)}
 \end{aligned}$$

$$f(z) = \frac{1}{(z-21^0)^2}$$

$$f'(z) = \frac{(z-21^0)^2 \times 0 - 1 \times 2(z-21^0)}{(z-21^0)^4} = \frac{-2}{(z-21^0)^3}$$

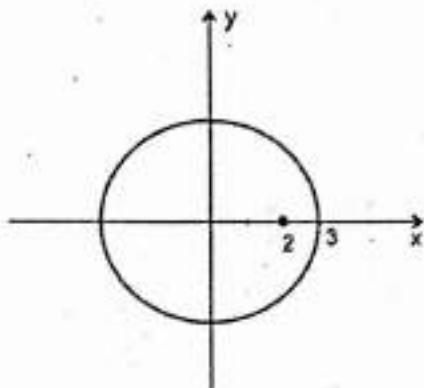
$$\begin{aligned}
 f'\left(\frac{1}{2}\right) &= \frac{-2}{\left(\frac{1}{2} - 2i\right)^3} \\
 &= \frac{-2}{\left(-\frac{3}{2}i\right)^3} \\
 &= \frac{-2}{-27(-1^0)} \\
 &\quad \underline{\underline{8}} \\
 &= \frac{16}{27(-1^0)} \\
 &= -\frac{16}{27i^0} \\
 &\underline{\underline{=}}
 \end{aligned}$$

Satzableitung ist ①

$$\begin{aligned}
 \oint_C \frac{dz}{(z-2i)^2(z-\frac{1}{2})^2} &= \frac{2\pi i^0}{1!} \times -\frac{16}{27i^0} \\
 &= -\frac{32\pi}{27} \\
 &\underline{\underline{=}}
 \end{aligned}$$

functions.

Example 6.2.17. Using Cauchy's integral formula, evaluate $\int_C \frac{\cos \pi z}{z-2} dz$, where C is $|z| = 3$.

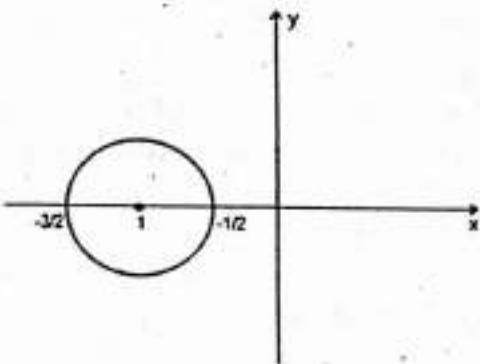


Let $F(z) = \frac{\cos \pi z}{z-2}$.

Singular points of $F(z)$ are given by $z-2=0 \implies z=2$. The point $z=2$ lies inside the curve $C : |z|=3$.

$$\begin{aligned}\therefore \int_C \frac{\cos \pi z}{z-2} dz &= \int_C \frac{f(z)}{z-2} dz \\ &\quad \text{where } f(z) = \cos \pi z \text{ is analytic inside } C \\ &= 2\pi i f(2), \text{ by Cauchy's integral formula} \\ &= 2\pi i \cos 2\pi \\ &= 2\pi i\end{aligned}$$

Example 6.2.18. Evaluate $\int_C \frac{e^z}{z+1} dz$, where C is $|z+1| = \frac{1}{2}$.



Let $F(z) = \frac{e^z}{z+1}$.

Singular points of $F(z)$ are given by $z+1=0 \implies z=-1$. The point $z=-1$ lies inside the curve $C : |z+1| = \frac{1}{2}$.

$$\therefore \int_C \frac{e^z}{z+1} dz = \int_C \frac{f(z)}{z+1} dz$$

$$\begin{aligned} &\text{where } f(z) = e^z \text{ is analytic inside } C \\ &= 2\pi i f(-1) \text{ , by Cauchy's integral formula} \\ &= 2\pi i e^{-1} \end{aligned}$$

Example 6.2.19. Evaluate $\int_C \frac{e^z}{z} dz$, where C is $|z|=1$.

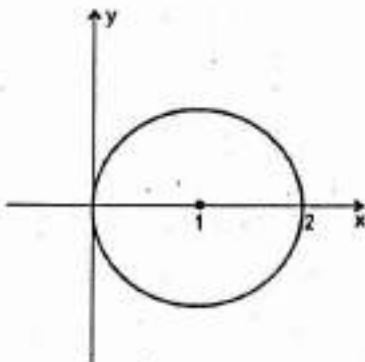
Here the given function is $F(z) = \frac{e^z}{z}$.

Singular points of $F(z)$ are given by $z=0$. The point $z=0$ lies inside the curve $C : |z|=1$.

$$\therefore \int_C \frac{e^z}{z} dz = \int_C \frac{f(z)}{z-0} dz$$

$$\begin{aligned} &\text{where } f(z) = e^z \text{ is analytic inside } C \\ &= 2\pi i f(0) \text{ , by Cauchy's integral formula} \\ &= 2\pi i e^0 \\ &= 2\pi i \end{aligned}$$

Example 6.2.20. Evaluate $\int_C \frac{z^2 + 2z + 3}{z^2 - 1} dz$, where C is $|z-1|=1$.



Let $F(z) = \frac{z^2 + 2z + 3}{z^2 - 1}$.

Singular points of $F(z)$ are given by $z^2 - 1 = 0 \implies z = \pm 1$. The point $z=1$ lies inside

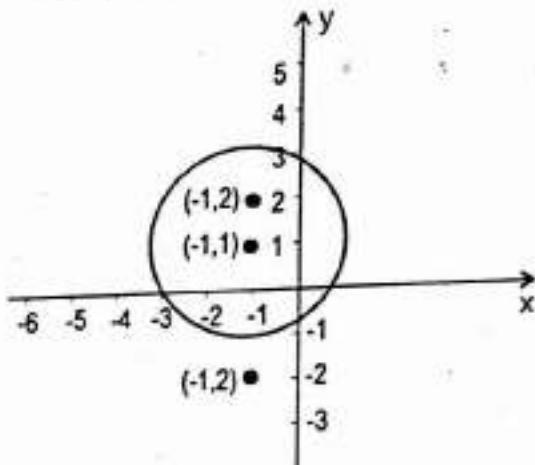
the curve $C : |z - 1| = 1$ and $z = -1$ lies outside C .

$$\begin{aligned} \therefore \int_C \frac{z^2 + 2z + 3}{z^2 - 1} dz &= \int_C \frac{z^2 + 2z + 3}{(z+1)(z-1)} dz \\ &= \int_C \frac{f(z)}{z-1} dz \\ &\quad \text{where } f(z) = \frac{z^2 + 2z + 3}{z+1} \text{ is analytic inside } C \\ &= 2\pi i f(1) \quad , \text{ by Cauchy's integral formula} \\ &= 2\pi i \times 3 \\ &= 6\pi i \end{aligned}$$

$$= 2\pi i \wedge \omega$$

$$= 6\pi i$$

Example 6.2.21. Evaluate $\int_C \frac{z+4}{z^2+2z+5} dz$, where C is $|z+1-i|=2$.



$$\text{Let } F(z) = \frac{z+4}{z^2+2z+5}.$$

Singular points of $F(z)$ are given by $z^2 + 2z + 5 = 0 \implies z = -1 \pm 2i$. The point $z = -1 + 2i$ lies inside the curve $C : |z - (-1+i)| = 2$ and $z = -1 - 2i$ lies outside C .

$$\therefore \int_C \frac{z+4}{z^2+2z+5} dz = \int_C \frac{z+4}{[z - (-1+2i)][z - (-1-2i)]} dz$$

$$= \int_C \frac{f(z)}{z - (-1+2i)} dz$$

where $f(z) = \frac{z+4}{z - (-1-2i)}$ is analytic inside C

$= 2\pi i f(-1+2i)$, by Cauchy's integral formula

$$= 2\pi i \times \frac{3+2i}{4i}$$

$$= \frac{(3+2i)\pi}{2}$$

Example 6.2.22. If $f(a) = \int_C \frac{3z^2+7z+1}{z-a} dz$, where C is the circle $x^2+y^2=4$, find $f(3), f(1), f'(1-i)$ and $f''(1+i)$

$$\begin{aligned}
 f(a) &= \int_C \frac{3z^2 + 7z + 1}{z - a} dz \\
 &= \int_C \frac{g(z)}{z - a} dz \quad \text{where } g(z) = 3z^2 + 7z + 1 \\
 &= \begin{cases} 2\pi i g((a)) & \text{if } a \text{ lies inside } C \\ 0 & \text{if } a \text{ lies outside } C \end{cases} \quad , \text{ by Cauchy's integral formula}
 \end{aligned}$$

$a = 3$ lies outside the curve $C : x^2 + y^2 = 4$

$$\therefore f(3) = 0$$

$a = 1$ lies inside the curve $C : x^2 + y^2 = 4$

$$\begin{aligned}
 \therefore f(1) &= 2\pi i g(1) \\
 &= 2\pi i \times 11 \\
 &= 22\pi i
 \end{aligned}$$

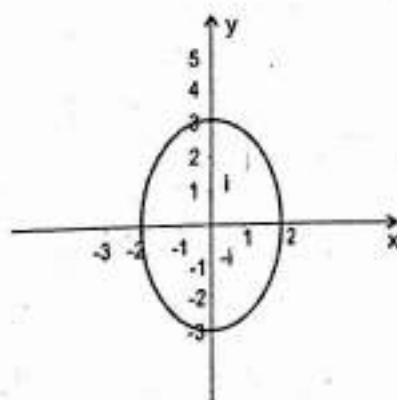
$a = 1 - i = (1, -1)$ lies inside the curve $C : x^2 + y^2 = 4$

$$\begin{aligned}
 \therefore f'(1 - i) &= 2\pi i g'(1 - i) \\
 &= 2\pi i(13 - 6i) = 2\pi i(6 + 13i)
 \end{aligned}$$

$a = 1 + i = (1, 1)$ lies inside the curve $C : x^2 + y^2 = 4$

$$\begin{aligned}
 \therefore f''(1 + i) &= 2\pi i g''(1 + i) \\
 &= 2\pi i \times 6 \\
 &= 12\pi i
 \end{aligned}$$

Example 6.2.23. If $f(a) = \int_C \frac{4z^2 + z + 5}{z - a} dz$, where C is the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$, find $f(i), f'(-1)$ and $f''(-1)$



$$\begin{aligned}
 f(a) &= \int_C \frac{4z^2 + z + 1}{z - a} dz \\
 &= \int_C \frac{g(z)}{z - a} dz \quad \text{where } g(z) = 4z^2 + z + 1 \\
 &= \begin{cases} 2\pi i g((a)) & \text{if } a \text{ lies inside } C \\ 0 & \text{if } a \text{ lies outside } C \end{cases} \quad , \text{ by Cauchy's integral formula}
 \end{aligned}$$

$a = i = (0, 1)$ lies inside the curve $C : \frac{x^2}{4} + \frac{y^2}{9} = 1$

$$\begin{aligned}\therefore f(i) &= 2\pi i g(i) \\ &= 2\pi i(i - 3) = -2\pi(1 + 3i)\end{aligned}$$

$a = -1$ lies inside the curve C .

$$\begin{aligned}\therefore f'(-1) &= 2\pi i g'(-1) \\ &= 2\pi i \times -7 \\ &= -14\pi i\end{aligned}$$

$a = -i = (0, -1)$ lies inside the curve C .

$$\begin{aligned}\therefore f''(-i) &= 2\pi i g''(-i) \\ &= 2\pi i \times 8 \\ &= 16\pi i\end{aligned}$$

$a = 1 + i = (1, 1)$ lies inside the curve $C : x^2 + y^2 = 4$

$$\begin{aligned}\therefore f''(1+i) &= 2\pi i g''(1+i) \\ &= 2\pi i \times 6 \\ &= 12\pi i\end{aligned}$$

Example 6.2.24. Using Cauchy's integral formula evaluate

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$$

where C is $|z| = 3$.

Let $F(z) = \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}$.

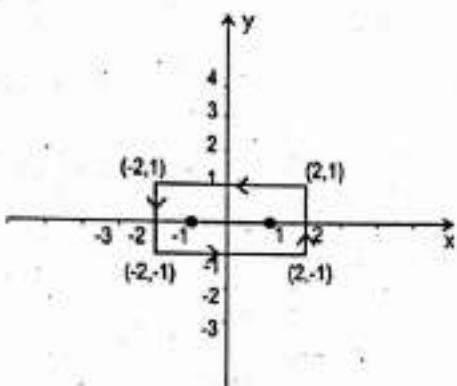
Singular points of $F(z)$ are given by $(z-1)(z-2) = 0 \implies z = 1, 2$. The points $z = 1, 2$ lie inside the curve $C : |z| = 3$.

$$\begin{aligned}\text{Let } \frac{1}{(z-1)(z-2)} &= \frac{A}{z-1} + \frac{B}{z-2} \\ &\quad \frac{A(z-2) + B(z-1)}{(z-1)(z-2)} \\ &\implies 1 = A(z-2) + B(z-1) \\ &\implies A = -1 \text{ and } B = 1 \\ \implies \frac{1}{(z-1)(z-2)} &= \frac{-1}{z-1} + \frac{1}{z-2}\end{aligned}$$

$$\begin{aligned} \therefore \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz &= \int_C \frac{-[\sin \pi z^2 + \cos \pi z^2]}{z-1} dz + \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2} dz \\ &= - \int_C \frac{f(z)}{z-1} dz + \int_C \frac{f(z)}{z-2} dz \end{aligned}$$

where $f(z) = \sin \pi z^2 + \cos \pi z^2$ is analytic inside C
 $= -2\pi i f(1) + 2\pi i f(2)$, by Cauchy's integral formula
 $= 2\pi i(-1) + 2\pi i(1)$
 $= 4\pi i$

Example 6.2.25. Evaluate $\int_C \frac{\cos \pi z}{z^2 - 1}$, where C is the rectangle with vertices $2 \pm i$, $-2 \pm i$



Let $F(z) = \frac{\cos \pi z}{z^2 - 1}$. Singular points of $F(z)$ are given by $z^2 - 1 = 0 \implies z = \pm 1$. The points $z = 1, -1$ lie inside the curve C : rectangle with vertices $2 \pm i$, $-2 \pm i$.

$$\begin{aligned} \text{Let } \frac{1}{z^2 - 1} &= \frac{1}{(z-1)(z+1)} = \frac{A}{z-1} + \frac{B}{z+1} \\ &= \frac{A(z+1) + B(z-1)}{(z-1)(z+1)} \\ \implies 1 &= A(z+1) + B(z-1) \implies A = \frac{1}{2} \text{ and } B = -\frac{1}{2} \end{aligned}$$

$$\begin{aligned} \implies \frac{1}{z^2 - 1} &= \frac{\left(\frac{1}{2}\right)}{z-1} + \frac{\left(-\frac{1}{2}\right)}{z+1} \\ \therefore \int_C \frac{\cos \pi z}{z^2 - 1} dz &= \int_C \frac{\left(\frac{1}{2} \cos \pi z\right)}{z-1} dz + \int_C \frac{\left(-\frac{1}{2} \cos \pi z\right)}{z+1} dz \end{aligned}$$

$$\begin{aligned} &= \int_C \frac{f(z)}{z-1} dz - \int_C \frac{f(z)}{z-(-1)} dz \\ \text{where } f(z) &= \frac{1}{2} \cos \pi z \text{ is analytic inside } C \\ &= 2\pi i f(1) - 2\pi i f(-1) \text{, by Cauchy's generalized integral formula} \\ &= 2\pi i \left(-\frac{1}{2}\right) - 2\pi i \left(-\frac{1}{2}\right) = 0 \end{aligned}$$

Example 6.2.26. Evaluate $\int_C \frac{z^2 + 5z + 3}{(z - 2)^2} dz$, where C is $|z| = 3$.

Let $F(z) = \frac{z^2 + 5z + 3}{(z - 2)^2}$. Singular points of $F(z)$ are given by $(z - 2)^2 = 0 \Rightarrow z = 2$. The point $z = 2$ lies inside the curve $C : |z| = 3$.

$$\begin{aligned}\therefore \int_C \frac{z^2 + 5z + 3}{(z - 2)^2} dz &= \int_C \frac{f(z)}{(z - 2)^2} dz \\ &\quad \text{where } f(z) = z^2 + 5z + 3 \text{ is analytic inside } C \\ &= \frac{2\pi i}{1!} f'(2), \quad \text{by Cauchy's generalized integral formula} \\ &= 2\pi i \times 9 \\ &= 18\pi i\end{aligned}$$

Example 6.2.27. Using Cauchy's integral formula, evaluate $\int_C \frac{z^2}{(z - 1)^2(z + 2)} dz$, where C is $|z - 2| = 2$.

Let $F(z) = \frac{z^2}{(z - 1)^2(z + 2)}$. Singular points of $F(z)$ are given by $(z - 1)^2(z + 2) = 0 \Rightarrow z = 1, 1, -2$. The point $z = 1$ lies inside the curve $C : |z - 2| = 2$ and $z = -2$ lies outside C .

$$\begin{aligned}\therefore \int_C \frac{z^2}{(z - 1)^2(z + 2)} dz &= \int_C \frac{\left(\frac{z^2}{z + 2}\right)}{(z - 1)^2} dz \\ &= \int_C \frac{f(z)}{z - 1} dz \\ &\quad \text{where } f(z) = \frac{z^2}{z + 2} \text{ is analytic inside } C \\ &= \frac{2\pi i}{1!} f'(1) \quad f'(z) = \frac{z^2 + 4z}{(z + 2)^2} \\ &\quad , \text{ by Cauchy's generalized integral formula} \\ &= 2\pi i \times \frac{5}{9} \\ &= \frac{10\pi i}{9}\end{aligned}$$

Example 6.2.28. Evaluate $\int_C \frac{e^z}{(z + 1)^3} dz$, where C is $|z + 1| = 1$.

Let $F(z) = \frac{e^z}{(z + 1)^3}$.

Singular points of $F(z)$ are given by $(z + 1)^3 = 0 \Rightarrow z = -1, -1, -1$. The point $z = -1$ lies inside the curve $C : |z + 1| = 1$.

es inside the curve $C : |z + 1| = 1$.

$$\begin{aligned} \therefore \int_C \frac{e^z}{(z+1)^3} dz &= \int_C \frac{f(z)}{(z+1)^3} dz \\ &\quad \text{where } f(z) = e^z \text{ is analytic inside } C \\ &= \frac{2\pi i}{2!} f''(-1), \quad \text{by Cauchy's generalized integral formula} \\ &= \pi i e^{-1} \end{aligned}$$

Taylor and MacLaurin Series

The Taylor series of a function $f(x)$ is

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \text{ where } a_n = \frac{1}{n!} f^{(n)}(x_0)$$

Taylor series can be written as

$$\begin{aligned} f(x) &= f(x_0) + \frac{(x-x_0)}{1!} f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \dots \\ &\quad + \frac{(x-x_0)^n}{n!} f^n(x_0) + \dots \end{aligned}$$

MacLaurin Series

MacLaurin series is a Taylor's series about $x_0 = 0$ is given by

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^n}{n!} f^n(0) + \dots$$

MacLaurin series of some elementary functions are given below:

$$1. e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$2. \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$3. \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$4. \tan x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$5. \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

$$6. (1-x)^{-1} = 1 + x + x^2 + x^3 + \dots \text{ where } |x| < 1$$

$$7. (1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$$

$$8. (1-g)^{-2} = 1 + 2g + 3g^2 + 4g^3 + \dots \quad \text{where } |g| < 1$$

$$9. (1+g)^{-2} = 1 - 2g + 3g^2 - 4g^3 + \dots$$

$$10. \log(1+g) = g - \frac{g^2}{2} + \frac{g^3}{3} - \dots$$

$$11. -\log(1-g) = g + \frac{g^2}{2} + \frac{g^3}{3} + \dots$$

$$12. \log\left[\frac{1+g}{1-g}\right] = 2\left[g + \frac{g^3}{3} + \frac{g^5}{5} + \dots\right]$$

Find the macclaurin series of the following

$$1. \frac{1}{(1+g^2)}$$

$$= \frac{1}{(1-g^2)}$$

$$= (1-g^2)^{-1}$$

$$= 1 + (-g^2) + (-g^2)^2 + (-g^2)^3 + \dots$$

$$= 1 - g^2 + g^4 - g^6 + \dots$$

$$2. \sin\left(\frac{g^2}{2}\right)$$

$$\text{we know } \sin g = g - \frac{g^3}{3!} + \frac{g^5}{5!} - \frac{g^7}{7!} + \dots$$

$$\sin\left(\frac{g^2}{2}\right) = \left(\frac{g^2}{2}\right) - \frac{\left(\frac{g^2}{2}\right)^3}{3!} + \frac{\left(\frac{g^2}{2}\right)^5}{5!} - \dots$$

$$= \frac{g^2}{2} - \frac{g^8}{8 \cdot 3!} + \frac{g^{10}}{32 \cdot 5!} - \dots$$

$$③ \frac{z+2}{1-z^2} = \frac{z+2}{(1-z)(1+z)} = \frac{A}{1+z} + \frac{B}{1-z}$$

$$z+2 = A(1-z) + B(1+z)$$

$$z=1 \Rightarrow 3 = 2B \Rightarrow B = \frac{3}{2}$$

$$z=-1 \Rightarrow 1 = 2A \Rightarrow A = \frac{1}{2}$$

$$\frac{z+2}{(1-z)(1+z)} = \frac{\frac{1}{2}}{1+z} + \frac{\frac{3}{2}}{1-z}$$

$$= \frac{1}{2}(1+z)^{-1} + \frac{3}{2}(1-z)^{-1}$$

$$= \frac{1}{2} \left[1 - z + z^2 - z^3 + \dots \right] + \frac{3}{2} \left[1 + z + z^2 + z^3 + \dots \right]$$

$$= \left[\frac{1}{2} - \frac{3}{2} + \frac{z^2}{2} - \frac{z^3}{2} + \dots \right] + \left[\frac{3}{2} + \frac{3z}{2} + \frac{3z^2}{2} + \frac{3z^3}{2} + \dots \right]$$

$$= \underline{\underline{z + z + 2z^2 + z^3 + \dots}}$$

$$④ \frac{1}{8+z^4} = \frac{1}{8(1+\frac{z^4}{8})}$$

$$= \frac{1}{8} (1+\frac{z^4}{8})^{-1}$$

$$= \frac{1}{8} \left[1 - \frac{z^4}{8} + \left(\frac{z^4}{8}\right)^2 - \left(\frac{z^4}{8}\right)^3 + \dots \right]$$

$$= \frac{1}{8} - \frac{z^4}{8^2} + \frac{z^8}{8^3} - \frac{z^{12}}{8^4} + \dots$$

$$⑤ \frac{1}{1+2^0z} = (1+2^0z)^{-1}$$

$$= 1 - 2^0z + (2^0z)^2 - (2^0z)^3 + \dots$$

$$= 1 - 2^0z - 4z^2 + 8^0z^3 + \dots$$

$$6) f(\theta) = 2 \sin^2 \frac{\theta}{2} \quad 2 \sin^2 \theta = 1 - \cos 2\theta$$

$$\begin{aligned} 2 \sin^2 \frac{\theta}{2} &= 1 - \cos \theta \\ &= 1 - \left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots \right] \\ &= \frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \dots \\ &\quad \underline{\underline{\quad}} \end{aligned}$$

$$\begin{aligned} 7) f(\theta) &= \sin^2 \theta \\ &= \frac{1 - \cos 2\theta}{2} \\ &= \frac{1}{2} \left[1 - \left\{ 1 - \frac{(2\theta)^2}{2!} + \frac{(2\theta)^4}{4!} - \frac{(2\theta)^6}{6!} + \dots \right\} \right] \\ &= \frac{1}{2} \left[\frac{4\theta^2}{2!} - \frac{16\theta^4}{4!} + \frac{64\theta^6}{6!} - \dots \right] \\ &= \frac{2\theta^2}{2!} - \frac{8\theta^4}{4!} + \frac{32\theta^6}{6!} - \dots \\ &\quad \underline{\underline{\quad}} \end{aligned}$$

Q) Find the Taylor series for the following functions with centre at θ_0 .

$$8) f(\theta) = \frac{1}{\theta}, \quad \theta_0 = 1^\circ$$

$$\begin{aligned} f(\theta) &= \frac{1}{\theta - 1^\circ + 1^\circ} \\ &= \frac{1}{1^\circ \left[1 + \left(\frac{\theta - 1^\circ}{1^\circ} \right) \right]} \\ &= \frac{1}{1^\circ [1 - 1^\circ(\theta - 1^\circ)]} \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{1-p} \left[1 - p(g-p) \right]^{-1} \\
 &= \frac{1}{1-p} \left[1 + p(g-p) + p^2(g-p)^2 + p^3(g-p)^3 + \dots \right] \\
 &= -p \left[1 + p(g-p) - (g-p)^2 - p(g-p)^3 + \dots \right] \\
 &\quad + -p + (g-p) + p(g-p)^2 - (g-p)^3 + \dots
 \end{aligned}$$

$$2) d(g) = \frac{1}{1-g} \quad g_0 = -p$$

$$\begin{aligned}
 d(g) &= \frac{1}{1-g} = \frac{1}{1-g-p+p} \\
 &= \frac{1}{(1-p)(g+\bar{p})} \\
 &= \frac{1}{(1-p) \left[1 + \frac{g+\bar{p}}{1-p} \right]} \\
 &= \frac{1}{(1-p)} \left[1 + \frac{(g+\bar{p})}{(1-p)} \right]^{-1} \\
 &= \frac{1}{(1-p)} \left[1 - \left(\frac{g+\bar{p}}{1-p} \right) + \left(\frac{g+\bar{p}}{1-p} \right)^2 - \left(\frac{g+\bar{p}}{1-p} \right)^3 + \dots \right] \\
 &= \frac{1}{(1-p)} - \frac{(g+\bar{p})}{(1-p)^2} + \frac{(g+\bar{p})^2}{(1-p)^3} - \frac{(g+\bar{p})^3}{(1-p)^4} + \dots
 \end{aligned}$$

$$3) d(g) = \cos^2 \theta \quad , \quad \theta = \frac{\pi}{2}$$

$$\begin{aligned}
 d(g) &= \cos^2 \theta = \frac{1 + \cos 2\theta}{2} \\
 &= \frac{1}{2} \left[1 + \cos 2\theta - \frac{\pi}{2} + \frac{\pi}{2} \right]
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2} \left[1 + \cos(2\beta - \pi + \pi) \right] \\
 &= \frac{1}{2} \left[1 + -\cos(2\beta - \pi) \right] \\
 &= \frac{1}{2} \left[1 - \cos 2(\beta - \frac{\pi}{2}) \right] \\
 &= \frac{1}{2} \left[1 - \left\{ 1 - \frac{(2(\beta - \frac{\pi}{2}))^2}{2!} + \frac{(2(\beta - \frac{\pi}{2}))^4}{4!} - \dots \right\} \right] \\
 &= \frac{1}{2} \left[\frac{2^2(\beta - \frac{\pi}{2})^2}{2!} - \frac{2^4(\beta - \frac{\pi}{2})^4}{4!} + \dots \right] \\
 &= (\beta - \frac{\pi}{2})^2 - \frac{2^4(\beta - \frac{\pi}{2})^4}{4!} + \dots
 \end{aligned}$$

4) $f(\beta) = \cos \beta \quad \text{at } \beta = \pi$

$$\begin{aligned}
 f(\beta) &= \cos(\beta - \pi + \pi) \\
 &= -\cos(\beta - \pi) \\
 &= - \left[1 - \frac{(\beta - \pi)^2}{2!} + \frac{(\beta - \pi)^4}{4!} - \dots \right] \\
 &= -1 + \frac{(\beta - \pi)^2}{2!} - \frac{(\beta - \pi)^4}{4!} + \dots
 \end{aligned}$$

5) $f(\beta) = \cosh(\beta - \pi/2) \quad \text{at } \beta = \pi/2$

$$= 1 + \frac{(\beta - \pi/2)^2}{2!} + \frac{(\beta - \pi/2)^4}{4!} + \dots$$

6) $f(\beta) = \frac{1}{(\beta - \rho)^2} \quad \beta = -\rho$

$$f(\beta) = \frac{1}{(\beta + i^0 + -i^0 - i^0)^2}$$

$$\begin{aligned}
 &= \frac{1}{[(8+i) - 2i]^2} \\
 &= \frac{1}{(-2i)^2 \left[1 - \frac{(8+i)}{2i} \right]^2} \\
 &= -\frac{1}{4} \left[1 - \frac{(8+i)}{2i} \right]^{-2} \\
 &= -\frac{1}{4} \left[1 + \frac{2(8+i)}{2i} + \frac{3(8+i)^2}{(2i)^2} + \frac{4(8+i)^3}{(2i)^3} + \dots \right] \\
 &= -\frac{1}{4} - \frac{1}{2} \frac{(8+i)}{2i} + \frac{3}{16} (8+i)^2 - \underline{\frac{1}{8} (8+i)^3} + \dots
 \end{aligned}$$

7) $f(z) = e^{z(z-2)}$ $z_0 = 1$

$$\begin{aligned}
 f(z) &= e^{z^2 - 2z + 1 - 1} \\
 &= e^{(z-1)^2 - 1} \\
 &= e^1 e^{(z-1)^2} \\
 &= e^1 \left[1 + \frac{(z-1)^2}{1!} + \frac{(z-1)^4}{2!} + \frac{(z-1)^6}{3!} + \dots \right]
 \end{aligned}$$

8) $f(z) = 8 \sinh(2z - i)$ $z_0 = \frac{i}{2}$

$$\begin{aligned}
 f(z) &= 8 \sinh[2(z - \frac{i}{2})] \\
 &= 2(z - \frac{i}{2}) + \frac{[2(z - \frac{i}{2})]^3}{3!} + \frac{[2(z - \frac{i}{2})]^5}{5!} + \dots \\
 &= 2(z - \frac{i}{2}) + \frac{8}{3!} (z - \frac{i}{2})^3 + \frac{32}{5!} (z - \frac{i}{2})^5 + \dots
 \end{aligned}$$

9) Find Taylor series for the function with centre $\gamma_0 = 1$

$$f(\gamma) = \frac{2\gamma^2 + 9\gamma + 5}{\gamma^3 + \gamma^2 - 8\gamma - 12}$$

Ans: $f(\gamma) = \frac{2\gamma^2 + 9\gamma + 5}{\gamma^3 + \gamma^2 - 8\gamma - 12} = \frac{2\gamma^2 + 9\gamma + 5}{(\gamma+2)^2(\gamma-3)}$

$$= \frac{A}{(\gamma+2)} + \frac{B}{(\gamma+2)^2} + \frac{C}{(\gamma-3)}$$

\times by $\frac{2\gamma^2 + 9\gamma + 5}{(\gamma+2)^2(\gamma-3)}$

$$2\gamma^2 + 9\gamma + 5 = A(\gamma+2)(\gamma-3) + B(\gamma-3) + C(\gamma+2)^2$$

$$\gamma = -2 \Rightarrow 2(4) - 18 + 5 = -5B$$

$$-5 = -5B \Rightarrow B = \underline{\underline{1}}$$

$$\gamma = 3 \Rightarrow 18 + 27 + 5 = 25C \Rightarrow C = \underline{\underline{2}}$$

$$\gamma = 0 \Rightarrow 5 = -6A - 3 + 8 \Rightarrow A = 0$$

$$f(\gamma) = \frac{1}{(\gamma+2)^2} + \frac{2}{(\gamma-3)}$$

$$= \frac{1}{(\gamma-1+3)^2} + \frac{2}{(\gamma-1-2)}$$

$$= \frac{1}{3\left[1 + \frac{\gamma-1}{3}\right]^2} + \frac{2}{-2\left[1 - \frac{\gamma-1}{2}\right]}$$

$$= \frac{1}{9} \left[1 + \left(\frac{\gamma-1}{3} \right) \right]^2 - \left[1 - \left(\frac{\gamma-1}{2} \right) \right]^{-1}$$

$$= \frac{1}{9} \left[1 - 2\left(\frac{\gamma-1}{3}\right) + 3\left(\frac{\gamma-1}{3}\right)^2 - 4\left(\frac{\gamma-1}{3}\right)^3 + \dots \right]$$

$$\begin{aligned}
 & - \left[1 + \left(\frac{g-1}{2}\right) + \left(\frac{g-1}{2}\right)^2 + \left(\frac{g-1}{2}\right)^3 + \dots \right] \\
 &= \left(\frac{1}{9}-1\right) + \left(\frac{-2}{27}-\frac{1}{2}\right)(g-1) + \left(\frac{1}{27}-\frac{1}{4}\right)(g-1)^2 \\
 &\quad + \left(\frac{-4}{9 \times 27} - \frac{1}{8}\right)(g-1)^3 + \dots \\
 &= -\frac{8}{9} - \frac{31}{54}(g-1) + \frac{-23}{108}(g-1)^2 - \frac{275}{1944}(g-1)^3 + \dots
 \end{aligned}$$

Laurent Series