MODULE III

COMPLEX VARIABLE - DIFFERENTIATION

Text 2: Relevant portions of sections 13.3, 13.4, 17.1, 17.2, 17.4

Complex function, limit, continuity, derivative, analytic functions, Cauchy-Reimann equations, harmonic functions, finding harmonic conjugate, conformal mappings - mappings $w = z^2$, $w = e^z$ Linear fractional transformation $w = \frac{1}{z}$, fixed points, transformation $w = \sin z$.

Complex Numbers

Complex number z can also be defined as an ordered pair (x,y) of real numbers usually in the form $z = x + iy$. x is called real part of z and y is called imaginary part of z.

ie) $Re(z) = x$ and $Im(z) = y$

* Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal.

ie) If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are equal then $x_1 = x_2$ and $y_1 = y_2$.

* $$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$$
$$= (x_1 + x_2) + i(y_1 + y_2)$$

* $$z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2)$$
$$= (x_1 - x_2) + i(y_1 - y_2)$$

* $$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$$

$$= x_1x_2 + ix_1y_2 + iy_1x_2 + (i)^2y_1y_2$$

$$= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2) \qquad [(i)^2 = -1]$$

* To find $\frac{z_1}{z_2}$, multiply both Numerator and Denominator with the conjugate of z_2.

$$\frac{z_1}{z_2} = \frac{(x_1 + iy_1)}{(x_2 + iy_2)}$$

$$= \frac{(x_1 + iy_1)}{(x_2 + iy_2)} \times \frac{(x_2 - iy_2)}{(x_2 - iy_2)}$$

$$= \frac{x_1x_2 - ix_1y_2 + iy_1x_2 + y_1y_2}{(x_2)^2 - (i)^2y_2^2}$$

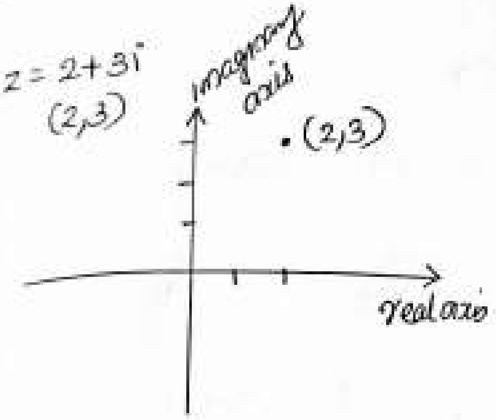
$$= \frac{(x_1x_2 + y_1y_2) + i(y_1x_2 - x_1y_2)}{x_2^2 + y_2^2}$$

* The modulus of a complex number $z = x+iy$ is denoted by $|z|$ and is defined as $|z| = \sqrt{x^2+y^2}$

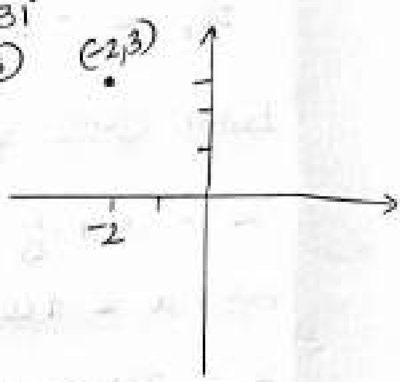
$$|z|^2 = x^2+y^2$$

* Argand plane : A plane in which each point is identified as a complex number is called the complex plane or Argand plane. x-axis is called real axis and y axis is called imaginary axis.

$z = 2+3i$
$(2,3)$



$z = -2+3i$
$(-2,3)$



Q $z_1 = -2+3i \quad z_2 = 5+4i$ Find $z_1+z_2,\ z_1-z_2,\ z_1z_2,\ \frac{z_1}{z_2}$

$\bar{z}_1$

Ans: $z_1+z_2 = (-2+3i)+(5+4i)$

$= 3+7i$

$z_1-z_2 = (-2+3i)-(5+4i)$

$= -7-i$

$z_1z_2 = (-2+3i)(5+4i)$

$= -10-8i+15i-12$

$= -22+7i$

$\frac{z_1}{z_2} = \frac{-2+3i}{5+4i}$

$= \frac{(-2+3i)(5-4i)}{(5+4i)(5-4i)}$

$= \frac{-10+8i+15i+12}{25-16i^2}$

$= \frac{2+23i}{25+16}$

$= \frac{2+23i}{41}$

$\bar{z}_1 = -2-3i$

<u>Polar form of a complex number</u>

$z = x+iy$

put $x = r\cos\theta$, $y = r\sin\theta$

$z = r\cos\theta + ir\sin\theta$

$z = r(\cos\theta + i\sin\theta)$

$z = re^{i\theta}$ is called polar form or modulus amplitude form of a complex number. r is the modulus and θ is the amplitude.

<u>Function of a complex variable</u>

If x and y are real variables, then $z = x+iy$ is called a complex variable. If corresponds to each value of a complex variable, $z(= x+iy)$ in a given region R there corresponds one or more values of another complex variable $w(= u+iv)$, then w is called a function of the complex variable z and is denoted by $w = f(z) = u+iv$

eg: $w = z^2$

$$u+iv = (x+iy)^2$$
$$= x^2 + 2ixy + (iy)^2$$
$$= (x^2-y^2) + i2xy$$

$$u = x^2-y^2 \quad \text{and} \quad v = 2xy$$

Thus u and v are real and imaginary part of w, are functions of real variable x, y

$$\therefore \quad w = f(z) = u(x,y) + iv(x,y)$$

Limit and Continuity

A function $f(z)$ is said to have the limit l as z approaches a point z_0, written $\lim_{z \to z_0} f(z) = l$ or $|f(z) - l| < \epsilon$, ϵ is a positive arbitrary number.

Note

In real variable $x \to x_0$ implies that x approaches x_0 along the number line, either from left or from right.
In complex variable $z \to z_0$ implies that z approaches z_0 along any path straight or curved, since the two points representing z and z_0 in a complex plane can be joined by an infinite number of curves.

A function $f(z)$ is said to be continuous at $z = z_0$ if $f(z_0)$ is defined and $\lim_{z \to z_0} f(z) = f(z_0)$

Q1) Show that $f(z) = \begin{cases} \frac{z}{|z|}, & z \neq 0 \\ 0, & z = 0 \end{cases}$ is discontinuous at $z = 0$

Ans:

$$f(z) = \frac{z}{|z|} = \frac{x + iy}{\sqrt{x^2 + y^2}}$$

$z = x + iy$

$z \to 0 \Rightarrow \begin{matrix} x \to 0 \\ y \to 0 \end{matrix}$

$$\lim_{z \to 0} f(z) = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x + iy}{\sqrt{x^2 + y^2}} \quad \left(\frac{0}{0} \text{ form}\right)$$

$\therefore$ put $y = mx$

$$= \lim_{x \to 0} \frac{x + imx}{\sqrt{x^2 + m^2 x^2}}$$

$$= \lim_{x \to 0} \frac{x(1 + im)}{x\sqrt{1 + m^2}}$$

$$= \lim_{x \to 0} \frac{(1+im)}{\sqrt{1+m^2}}$$

$$= \frac{1+im}{\sqrt{1+m^2}}$$

which depends on the value of m. Therefore the limit does not exist and hence $f(z)$ is not continuous at $z=0$

2) Check whether the function is continuous at $z=0$

$$f(z) = \begin{cases} \dfrac{(\text{Re } z^2)}{|z|} & , z \neq 0 \\ 0 & , z=0 \end{cases}$$

Ans:

$$z = x+iy$$

$$z^2 = (x+iy)^2 = x^2 + i2xy + i^2y^2$$

$$= (x^2-y^2) + i2xy$$

$$\text{Re}(z^2) = x^2-y^2$$

$$|z| = \sqrt{x^2+y^2}$$

$$\therefore f(z) = \frac{(\text{Re } z^2)}{|z|} = \frac{x^2-y^2}{\sqrt{x^2+y^2}}$$

$$\lim_{z \to 0} f(z) = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2-y^2}{\sqrt{x^2+y^2}} \quad \left(\frac{0}{0} \text{ form}\right)$$

put $y = mx$

$$= \lim_{x \to 0} \frac{x^2 - m^2x^2}{\sqrt{x^2+m^2x^2}}$$

$$= \lim_{x \to 0} \frac{x^2(1-m^2)}{x\sqrt{1+m^2}}$$

$$= \lim_{x \to 0} \frac{x(1-m^2)}{\sqrt{1+m^2}}$$

$= 0$, limit exist

$= f(0)$

ie) $\lim_{z \to 0} f(z) = f(0)$ $\therefore$ The given function is continuous at $z=0$

3) check whether the function is continuous or not at $z=0$

$$f(z) = \begin{cases} |z|^2 \operatorname{Im}(1/z) & , z \neq 0 \\ 0 & , z=0 \end{cases}$$

Ans: $z = x+iy$

$|z| = \sqrt{x^2+y^2}$

$|z|^2 = x^2+y^2$

$$\frac{1}{z} = \frac{1}{x+iy}$$

$$= \frac{1}{x+iy}\frac{(x-iy)}{(x-iy)}$$

$$= \frac{(x-iy)}{x^2+y^2}$$

$$\operatorname{Im}\left(\frac{1}{z}\right) = \frac{-y}{x^2+y^2}$$

$$\therefore f(z) = |z|^2 \operatorname{Im}(1/z) = (x^2+y^2)\frac{(-y)}{(x^2+y^2)} = -y$$

$$\underset{z \to 0}{Lt} f(z) = \underset{\substack{x \to 0 \\ y \to 0}}{Lt} -y = 0$$

$= f(0)$

limit exist and is equal to $f(0)$ $\therefore$ The function is continuous.

4) check whether the function is continuous or not at $z=0$

$$f(z)=\begin{cases}\dfrac{Re(z)}{1-|z|} & , z\neq 0\\ 0 & , z=0\end{cases}$$

Ans:

$z=x+iy$

$Re(z)=x$

$1-|z| = 1-\sqrt{x^2+y^2}$

$\therefore f(z) = \dfrac{x}{1-\sqrt{x^2+y^2}}$

$$\lim_{z\to 0} f(z) = \underset{\substack{x\to 0\\ y\to 0}}{Lt} \frac{x}{1-\sqrt{\cancel{x^2}+y^2}} = \frac{0}{1} = 0$$

$$= f(0)$$

limit exist and equal to $f(0)$ $\therefore$ The function is continuous.

5) Test the continuity at $z=0$ if $f(z)=\begin{cases}\dfrac{Im\, z}{|z|} & , z\neq 0\\ 0 & , z=0\end{cases}$

Ans: $z=x+iy$

$Im(z) = y \qquad |z|=\sqrt{x^2+y^2}$

$f(z) = \dfrac{Im\, z}{|z|} = \dfrac{y}{\sqrt{x^2+y^2}}$

$$\underset{z\to 0}{Lt} f(z) = \underset{\substack{x\to 0\\ y\to 0}}{Lt} \frac{y}{\sqrt{x^2+y^2}} = \left(\frac{0}{0} \text{ form}\right)$$

put $y=mx$

$$= \lim_{x\to 0} \frac{mx}{\sqrt{x^2+m^2x^2}}$$

$$= \lim_{x\to 0} \frac{mx}{x\sqrt{1+m^2}}$$

$$= \lim_{x\to 0} \frac{m}{\sqrt{1+m^2}}$$ depends on m

$\therefore$ limit doesnot exist $\therefore f(z)$ is not continuous.

Q6) check whether the function $f(z) = \begin{cases} \frac{Re(z^2)}{|z|^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$

is continuous at $z=0$

Ans: $z = x + iy$

$z^2 = (x+iy)^2 = x^2 - y^2 + i2xy$

$Re(z^2) = x^2 - y^2$

$|z|^2 = x^2 + y^2$

$$\therefore f(z) = \frac{Re(z^2)}{|z|^2} = \frac{x^2-y^2}{x^2+y^2}$$

$$\lim_{z\to 0} f(z) = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2-y^2}{x^2+y^2}$$ $\left(\frac{0}{0} \text{ form}\right)$

put $y = mx$

$$= \lim_{x\to 0} \frac{x^2 - m^2x^2}{x^2+m^2x^2}$$

$$= \lim_{x\to 0} \frac{x^2(1-m^2)}{x^2(1+m^2)}$$

$$= \lim_{x\to 0} \frac{1-m^2}{1+m^2}$$ depends on m

Limit does not exist and the function is not continuous.

Derivative

The derivative of a complex function f at a point z is written $f'(z)$ and is defined by

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

Provides, this limit exists and independent of the manner in which $\Delta z \to 0$. Then f is said to be differentiable at z

Differentiate the following

1) $(z-i)/(z+i)$ at i

Ans: $f(z) = \dfrac{z-i}{z+i}$

$$f'(z) = \frac{(z+i)(1) - (z-i)(1)}{(z+i)^2}$$

$$= \frac{(z+i)-(z-i)}{(z+i)^2}$$

$$= \frac{2i}{(z+i)^2}$$

$f'(z)$ at $z=i$ $= \dfrac{2i}{(2i)^2} = \dfrac{1}{2i} = \dfrac{-i}{2}$ $\left[\dfrac{1}{i} = -i\right]$

2) $f(z) = (z-2i)^3$ at $z = 5+2i$

$f'(z) = 3(z-2i)^2(1)$

$f'(5+2i) = 3(5+2i-2i)^2$

$= 75$

3) Prove that $\bar{z} = x - iy$ is not differentiable.

Ans: $f(z) = \bar{z} = x - iy$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(\overline{z+\Delta z}) - (\bar{z})}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(\bar{z} + \overline{\Delta z}) - (\bar{z})}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

if $\Delta y = 0 \Rightarrow \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$

if $\Delta x = 0 \Rightarrow \lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1$

Thus the above equation approches +1 along one path and approches −1 along the second path. Hence by definition the above limit does not exist. as $\Delta z \to 0$

ANALYTIC FUNCTION

If a single valued function $f(z)$ possesses a unique derivative at every point of a neighbourhood of the point z_0 in the region R, then $f(z)$ is analytic at z_0. If $f(z)$ is analytic at all points in a region R, then $f(z)$ is called an analytic function or a regular function or a holomorphic function of z in R

A point where the function ceases to be analytic is called a singular point.

eg: The non negative integer powers $1, z, z^2 \cdots\cdots$ are analytic in the entire complex plane.

polynomial $f(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n$ are also analytic.

The quotient of two polynomials $g(z)$ and $h(z)$

ie) $f(z) = \dfrac{g(z)}{h(z)}$ is called a rational function. This function is analytic except at the points where $h(z)=0$, here we assume that common factors of g and h have been cancelled.

Cauchy – Reimann Equations

The complex function $w = f(z) = u+iv$ is analytic in a domain D if and only if the first partial derivatives of u and v satisfies the two Cauchy-Reimann equations.

$$\boxed{\begin{array}{l} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \quad \text{and} \quad \dfrac{\partial u}{\partial y} = \dfrac{-\partial v}{\partial x} \\ \text{ie) } u_x = v_y \quad \& \quad u_y = -v_x \end{array}}$$

eg 1) $f(z) = z^2$

$= (x^2 - y^2) + i\,2xy$

$u = x^2 - y^2 \qquad v = 2xy$

$u_x = 2x \qquad v_x = 2y$

$u_y = -2y \qquad v_y = 2x$

$u_x = v_y \;\; \& \;\; u_y = -v_x$

$\therefore f(z)$ is analytic.

eg 2) $f(z) = \bar{z}$

$= x - iy$

$u = x \qquad v = -y$

$u_x = 1 \qquad v_x = 0$

$u_y = 0 \qquad v_y = -1$

$u_x \neq v_y$ and $u_y = -v_x$

Cauchy Reimann equation is not satisfied $\therefore f(z) = \bar{z}$ is not analytic.

Theorem

Let $f(z) = u(x,y) + iv(x,y)$ be defined and continuous in some neighbourhood of a point $z = x+iy$ and differentiable at z itself. Then at that point the first order partial derivatives of u and v exist and satisfy the Cauchy Reimann equations.

OR

If $f(z)$ is analytic in a domain D, partial derivatives exist and satisfy C-R equations at all points of D.

Proof

Let $z = x+iy$

$\Delta z = \Delta x + i\Delta y$

$z + \Delta z = (x+\Delta x) + i(y+\Delta y)$

Let $f(z) = u(x,y) + iv(x,y)$

By assumption the derivative $f'(z)$ exist

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{[u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)] - [u(x,y) + iv(x,y)]}{\Delta x + i\Delta y} \quad (1)$$

First we evaluate the derivative along a path parallel to real axis (ie) Along the path $\Delta y = 0$ $\therefore \Delta z = \Delta x$ so that

$$① \Rightarrow f'(z) = \lim_{\Delta x \to 0} \frac{[u(x+\Delta x, y) + iv(x+\Delta x, y)] - [u(x,y) + iv(x,y)]}{\Delta x}$$

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$f'(z) = u_x + iv_x \quad \text{———} ②$$

Next we evaluate the derivative along a path parallel to the imaginary axis (ie) Along the path $\Delta x = 0$ $\therefore \Delta z = i\Delta y$ so that

$$① \Rightarrow f'(z) = \lim_{\Delta y \to 0} \frac{u[(x, y+\Delta y) + iv(x, y+\Delta y)] - [u(x,y) + iv(x,y)]}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{u[(x, y+\Delta y) - u(x,y)]}{i\Delta y} + \frac{i}{i} \lim_{\Delta y \to 0} \frac{[v(x, y+\Delta y) - v(x,y)]}{\Delta y}$$

$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \qquad \left[\frac{1}{i} = -i\right]$$

$$= v_y - iu_y \quad \text{———} ③$$

By the uniqueness of derivative from ② & ③ we get

$u_x = v_y$ and $v_x = -u_y$

(ie) $u_x = v_y$ and $u_y = -v_x$

Q1) Is $f(z) = e^x(\cos y + i\sin y)$ analytic.

Ans: $f(z) = e^x(\cos y + i\sin y)$

$u + iv = e^x\cos y + i e^x \sin y$

$u = e^x \cos y$ $\quad\quad$ $v = e^x \sin y$

$u_x = e^x \cos y$ $\quad\quad$ $v_x = e^x \sin y$

$u_y = -e^x \sin y$ $\quad\quad$ $v_y = e^x \cos y$

$u_x = v_y$ and $u_y = -v_x$

CR equations satisfied $\therefore$ $f(z)$ is analytic for all z.

Q2) Check whether the following functions are analytic or not. Justify your answer (march 17)

(i) $f(z) = z + \bar{z}$ $\quad\quad$ (ii) $f(z) = |z|^2$

Ans: (i) $f(z) = z + \bar{z}$

$= (x+iy) + (x-iy)$

$u + iv = 2x$

$u = 2x$ $\quad\quad$ $v = 0$

$u_x = 2$ $\quad\quad$ $v_x = 0$

$u_y = 0$ $\quad\quad$ $v_y = 0$

$u_x \neq v_y$ $\therefore$ The function is not analytic.

(ii) $f(z) = |z|^2$

$= \left(\sqrt{x^2+y^2}\right)^2$

$u + iv = x^2 + y^2$

$u = x^2 + y^2$ $\quad\quad$ $v = 0$

$u_x = 2x$ $\quad\quad$ $v_x = 0$

$u_y = 2y$ $\quad\quad$ $v_y = 0$

$u_x \neq v_y$ & $u_y \neq -v_x$

$\therefore$ The function is not analytic.

Q3) Show that $f(z) = \sin z$ is analytic for all z Find $f'(z)$

(March 2017)

Ans: $f(z) = \sin z$

$= \sin(x+iy)$

$= \sin x \cos iy + \cos x \sin iy$

$u+iv = \sin x \cosh y + i \cos x \sinh y$ $\left[\begin{array}{l} \cos iy = \cosh y \\ \sin iy = i \sinh y \end{array}\right]$

$u = \sin x \cosh y$ $v = \cos x \sinh y$

$u_x = \cos x \cosh y$ $v_x = -\sin x \sinh y$

$u_y = \sin x \sinh y$ $v_y = \cos x \cosh y$

$u_x = v_y$ & $u_y = -v_x$

$\therefore$ CR equations satisfies

$\therefore f(z) = \sin z$ is analytic

$f'(z) = \cos z$

Q Show that $f(z) = e^{-x}\cos y - ie^{-x}\sin y$ is differentiable everywhere and find its derivative (Dec 2016)

Ans: $u + iv = e^{-x}\cos y - ie^{-x}\sin y$

$u = e^{-x}\cos y$ $\quad\quad$ $v = -e^{-x}\sin y$

$u_x = -e^{-x}\cos y$ $\quad\quad$ $v_x = e^{-x}\sin y$

$u_y = -e^{-x}\sin y$ $\quad\quad$ $v_y = -e^{-x}\cos y$

$u_x = v_y$ & $u_y = -v_x$

C-R equations satisfies

$\therefore$ $f(z)$ is analytic. every analytic function is differentiable

$\therefore$ $f(z)$ is differentiable.

$$f(z) = u + iv$$

$$f'(z) = u_x + iv_x$$

$$= -e^{-x}\cos y + ie^{-x}\sin y$$

$$= -e^{-x}[\cos y - i\sin y]$$

$$= -e^{-x}e^{-iy}$$

$$= -e^{-(x+iy)}$$

$$= -e^{-z}$$

$$f'(z) = \frac{-1}{e^{z}}$$

Note

To prove that a function is differentiable, it is enough to show that the function is analytic. Since every analytic function is differentiable.

Q Find the points if any, in the complex plane where the function $f(z) = (2x^2+y) + i(y^2-x)$ is (i) differentiable (ii) analytic.

Ans: $u+iv = (2x^2+y) + i(y^2-x)$

$u = 2x^2+y$ $\qquad$ $v = y^2-x$

$u_x = 4x$ $\qquad$ $v_x = -1$

$u_y = 1$ $\qquad$ $v_y = 2y$

f is differentiable only at the points where u & v satisfies CR equations $u_x = v_y$ & $u_y = -v_x$

ie) $4x = 2y$ & $1 = 1$

ie) $2x = y$ only if $x = y = 0$

u & v satisfies CR equations only at $(0,0)$.

$\therefore$ f is differentiable only at $(0,0)$.

$\therefore$ $f(z)$ is not analytic.

Q Find the points where Cauchy-Reimann equations are satisfied for the function $f(z) = xy^2 + ix^2y$. where does $f'(z)$ exist Is the function $f(z)$ analytic at those points.?

Ans: $u+iv = xy^2 + ix^2y$

$u = xy^2$ $\qquad$ $v = x^2y$

$u_x = y^2$ $\qquad$ $v_x = 2xy$

$u_y = 2xy$ $\qquad$ $v_y = x^2$

CR equations $u_x = v_y$ and $u_y = -v_x$

$y^2 = x^2$ and $2xy = -2xy$

C.R equations are true only when $x = y = 0$

$\therefore f'(z)$ exist at $(0,0)$.

$\therefore f(z)$ is not analytic at $(0,0)$.

Q Show that an analytic function of constant Absolute value is constant.

Ans: Let $f(z)$ is an analytic function with $|f(z)| = k$

we have to show that $f(z)$ is a constant.

$$f(z) = u + iv$$

$$|f(z)| = \sqrt{u^2+v^2} = k$$

$$\Rightarrow u^2+v^2 = k^2$$

Differentiating partially w.r.t x and y

$$2uu_x + 2vv_x = 0$$

$$uu_x + vv_x = 0 \quad \text{——} \ (1)$$

$$2uu_y + 2vv_y = 0$$

$$uu_y + vv_y = 0 \quad \text{——} \ (2)$$

By CR equations $u_x = v_y$ and $u_y = -v_x$

$$(1) \Rightarrow uu_x - vu_y = 0 \quad \text{——} \ (3)$$

$$(2) \Rightarrow uu_y + vu_x = 0 \quad \text{——} \ (4)$$

$$(3)\times u + (4)\times v \Rightarrow u^2u_x - uvu_y + uvu_y + v^2u_x = 0$$

$$\text{ie) } (u^2+v^2)u_x = 0$$

$$(3)\times v - (4)\times u \Rightarrow (uvu_x - v^2u_y) - (u^2u_y + uvu_x) = 0$$

$$-v^2u_y - u^2u_y = 0$$

$$u^2u_y + v^2u_y = 0$$

$$(u^2+v^2)u_y = 0$$

If $u^2+v^2=k^2=0$ then $u=v=0$ hence $f(z)=0$

If $u^2+v^2=k^2\neq 0$ then $u_x=u_y=0$

Hence by C-R equations $v_y=v_x=0$

Together this implies $u=$ constant and $v=$ constant

Hence $f(z)$ is a constant.

Q Prove that the function $w=e^z$ is analytic everywhere. Also find its derivative.

Ans: $w=e^z$

$u+iv=e^{x+iy}$

$=e^x e^{iy}$

$=e^x(\cos y+i\sin y)$

$u=e^x\cos y \qquad v=e^x\sin y$

$u_x=e^x\cos y \qquad v_x=e^x\sin y$

$u_y=-e^x\sin y \qquad v_y=e^x\cos y$

$u_x=v_y$ & $u_y=-v_x$

CR equations are satisfied $\therefore f(z)$ is analytic.

$f(z)=u+iv$

$f'(z)=u_x+iv_x$

$=e^x\cos y+ie^x\sin y$

$=e^x(\cos y+i\sin y)$

$=e^x e^{iy}$

$=e^{x+iy}$

$f'(z)=\underline{\underline{e^z}}$

Q Prove that the function $f(z) = \bar{z}$ is not analytic anywhere in the complex plane.

Ans: $f(z) = \bar{z}$

$u + iv = x - iy$

$u = x \qquad v = -y$

$u_x = 1 \qquad v_x = 0$

$u_y = 0 \qquad v_y = -1$

$u_x \neq v_y$

CR equations not satisfied $\therefore \bar{z}$ is not analytic.

Harmonic Functions

A function $u(x,y)$ defined in a domain having continuous first and second order partial derivatives satisfying the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is called a harmonic function.

Theorem

The real and imaginary parts of an analytic function are harmonic functions.

proof

given $f(z) = u + iv$ is analytic

Then by CR equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ — ①

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{— ②}$$

Differentiating ① partially w.r.t x and ② partially w.r.t y

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$$

Adding these $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \left[\because \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}\right]$

Differentiating ① partially w.r.t y and ② partially w.r.t x

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2}$$

Subtracting these $0 = \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2}$

(ie) $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad \left[\because \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}\right]$

Harmonic Conjugate

If $f(z) = u + iv$ is an analytic function then u is called the harmonic conjugate of v and v is called the harmonic conjugate of u.

Problems

1) Show that $v = 3x^2y - y^3$ is harmonic and find the corresponding analytic function $f(z) = u(x,y) + iv(x,y)$.

Ans: $v = 3x^2y - y^3$

$\frac{\partial v}{\partial x} = 6xy \qquad \frac{\partial v}{\partial y} = 3x^2 - 3y^2$

$\frac{\partial^2 v}{\partial x^2} = 6y \qquad \frac{\partial^2 v}{\partial y^2} = -6y$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 6y - 6y = 0$$

$\therefore\ v = 3x^2y - y^3$ is harmonic

$f(z) = u + iv$

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$f'(z) = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$ $\left[\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow \text{CR equation}\right]$

$= (3x^2 - 3y^2) + i\,6xy$

$= 3z^2$ [put $x = z$ & $y = 0$]

integrating $f(z) = \frac{3z^3}{3} + c$

$f(z) = z^3 + (a+ib)$

$= (x+iy)^3 + (a+ib)$

$= x^3 + 3x^2 iy + 3x(iy)^2 + (iy)^3 + (a+ib)$

$= x^3 + i\,3x^2y - 3xy^2 - iy^3 + (a+ib)$

$= (x^3 - 3xy^2 + a) + i(3x^2y - y^3 + b)$

2) Show that $u = y^3 - 3x^2y$ is harmonic. Hence find its harmonic conjugate.

Ans: $u = y^3 - 3x^2y$

$\frac{\partial u}{\partial x} = -6xy$ $\qquad \frac{\partial u}{\partial y} = 3y^2 - 3x^2$

$\frac{\partial^2 u}{\partial x^2} = -6y$ $\qquad \frac{\partial^2 u}{\partial y^2} = 6y$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -6y + 6y = 0$

$\therefore\ u = y^3 - 3x^2y$ is harmonic

$f(z) = u + iv$

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \left[\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\right]$$

$$= -6xy - i(3y^2 - 3x^2)$$

$$f'(z) = -0 - i(0 - 3z^2) \qquad [\text{put } x = z,\ y = 0]$$

$$f'(z) = i3z^2$$

integrating $\quad f(z) = i\dfrac{3z^3}{3} + c$

$$f(z) = iz^3 + (a+ib)$$

$$= i(x+iy)^3 + (a+ib)$$

$$= i\left[x^3 + 3x^2iy + 3xi^2y^2 + (iy)^3\right] + (a+ib)$$

$$= i\left[x^3 + i3x^2y - 3xy^2 - iy^3\right] + (a+ib)$$

$$= ix^3 - 3x^2y - i3xy^2 + y^3 + a + ib$$

$$= (y^3 - 3x^2y + a) + i(x^3 - 3xy^2 + b)$$

harmonic conjugate $\quad v = x^3 - 3xy^2 + b$ //

7) Show that $u = \sin x \cosh y$ is harmonic. Hence find its harmonic conjugate.

Ans: $\quad u = \sin x \cosh y$

$u_x = \cos x \cosh y \qquad\qquad u_y = \sin x \sinh y$

$u_{xx} = -\sin x \cosh y \qquad\qquad u_{yy} = \sin x \cosh y$

$u_{xx} + u_{yy} = -\sin x \cosh y + \sin x \cosh y = 0$

$\therefore u = \sin x \cosh y$ is harmonic

$f(z) = u + iv$

$f'(z) = u_x + iv_x$

$\quad = u_x - iu_y$

$$= \cos x \cosh y - i \sin x \sinh y$$

$\left\{ \begin{array}{l} \cosh 0 = 1 \\ \sinh 0 = 0 \end{array} \right.$

$$f'(z) = \cos z$$

integrabing $f(z) = \sin z + c$

$$= \sin(x+iy) + (a+ib)$$

$$= \sin x \cos iy + \cos x \sin iy + a + ib$$

$$= \sin x \cosh y + i \cos x \sinh y + a + ib$$

$$= (\sin x \cosh y + a) + i (\cos x \sinh y + b)$$

hormonic cogugale $v = \cos x \sinh y \underline{\underline{+ b}}$

4) Prove that the function $u(x,y) = x^3 - 3xy^2 - 5y$ is harmonic everywhere. Also find the harmonic conjugate of u.

Ans: $u = x^3 - 3xy^2 - 5y$

$u_x = 3x^2 - 3y^2$ $\qquad u_y = -6xy - 5$

$u_{xx} = 6x$ $\qquad u_{yy} = -6x$

$u_{xx} + u_{yy} = 6x - 6x = 0$

$\therefore u = x^3 - 3xy^2 - 5y$ is harmonic

$f(z) = u + iv$

$f'(z) = u_x + iv_x$

$= u_x - iu_y \qquad [u_y = -v_x]$

$= (3x^2 - 3y^2) - i(-6xy - 5) \qquad$ (put $x = z$, $y = 0$)

$f'(z) = 3z^2 + 5i$

Integrating $\quad f(z) = \frac{3z^3}{3} + 5iz + c$

$f(z) = z^3 + 5iz + c$

$= (x+iy)^3 + 5i(x+iy) + (a+ib)$

$= x^3 + 3x^2 iy + 3x(iy)^2 + (iy)^3 + 5ix - 5y + (a+ib)$

$= x^3 + i3x^2y - 3xy^2 - iy^3 + i5x - 5y + (a+ib)$

$= (x^3 - 3xy^2 - 5y + a) + i(3x^2y - y^3 + 5x + b)$

harmonic conjugate $v = 3x^2y - y^3 + 5x + \underline{\underline{b}}$

5) If $v = e^x(x\sin y + y\cos y)$ find an analytic function $f(z) = u + iv$

Ans: $v = e^x x\sin y + e^x y\cos y$

$v_x = \sin y[e^x + xe^x] + e^x y\cos y$

$v_y = e^x x\cos y + e^x[-y\sin y + \cos y]$

$f(z) = u + iv$

$f'(z) = u_x + iv_x$

$= v_y + iv_x$

$= e^x x\cos y + e^x[-y\sin y + \cos y] + i[\sin y(e^x + xe^x) + e^x y\cos y]$

put $x = z$ and $y = 0$

$f'(z) = e^z z + e^z[0+1] + i[0+0]$

$f'(z) = e^z z + e^z$

$f'(z) = e^z(z+1)$

integrating

$f(z) = \int (z+1)e^z \, dz$

$= (z+1)e^z - (1)e^z + c$

$= ze^z + e^z - e^z + c$

$= \underline{\underline{ze^z + c}}$

$$f(z) = (x+iy)e^{(x+iy)} + (a+ib)$$

$$= (x+iy)e^{x}e^{iy} + (a+ib)$$

$$= (x+iy)e^{x}(\cos y + i\sin y) + (a+ib)$$

$$= e^{x}[x\cos y + ix\sin y + iy\cos y - y\sin y] + (a+ib)$$

$$= e^{x}[x\cos y - y\sin y] + a + i[e^{x}(x\sin y + y\cos y) + b]$$

6) Show that $u = x^2 - y^2 - y$ is harmonic. Also find the corresponding conjugate harmonic function.

Ans: $u = x^2 - y^2 - y$

$u_x = 2x \qquad u_y = -2y - 1$

$u_{xx} = 2 \qquad u_{yy} = -2$

$u_{xx} + u_{yy} = 2 - 2 = 0$

$\therefore u = x^2 - y^2 - y$ is harmonic

$f(z) = u + iv$

$f'(z) = u_x + iv_x$

$= u_x - iu_y$

$= 2x - i(-2y - 1)$

$f'(z) = 2z + i$

integrating $f(z) = \frac{2z^2}{2} + iz + c$

$f(z) = z^2 + iz + c$

$= (x+iy)^2 + i(x+iy) + (a+ib)$

$= x^2 - y^2 + i2xy + ix - y + a + ib$

$= (x^2 - y^2 - y + a) + i(2xy + x + b)$

harmonic conjugate $v = 2xy + x + b$

7) Show that $u = 2xy + 6x^2y - 2y^3$ can be the real part of an analytic function. Find its harmonic conjugate.

Ans: $u = 2xy + 6x^2y - 2y^3$

$u_x = 2y + 12xy \qquad u_y = 2x + 6x^2 - 6y^2$

$u_{xx} = 12y \qquad u_{yy} = -12y$

$u_{xx} + u_{yy} = 12y - 12y = 0$

$\therefore u = 2xy + 6x^2y - 2y^3$ is harmonic.

$\therefore u$ is the real part of an analytic function.

$f(z) = u + iv$

$f'(z) = u_x + iv_x$

$= u_x - iu_y \qquad [u_y = -v_x]$

$= (2y + 12xy) - i(2x + 6x^2 - 6y^2)$

$f'(z) = (0 + 0) - i(2z + 6z^2 - 0)$

$f'(z) = -i[2z + 6z^2]$

Integrating $f(z) = -i[z^2 + 2z^3] + c$

$= -i[(x+iy)^2 + 2(x+iy)^3] + c$

$= -i[x^2 - y^2 + i2xy + 2(x^3 + i3x^2y - 3xy^2 - iy^3)] + c$

$= -ix^2 + iy^2 + 2xy - i2x^3 + 6x^2y + i6xy^2 - 2y^3 + a + ib$

$= (2xy + 6x^2y - 2y^3 + a) + i(6xy^2 - 2x^3 + y^2 - x^2 + b)$

harmonic conjugate $v = 6xy^2 - 2x^3 + y^2 - x^2 + b$

Conformal mappings or Transformations

Consider the complex valued function $w = f(z) = u+iv$ where $z = x+iy$, $u = u(x,y)$, $v = v(x,y)$ — ① In this case planes known as z - plane and w plane are required to represent each of the complex variable z and w respectively. From equation ① corresponding to each point (x,y) in the z plane, we shall have a point (u,v) in the w plane and these corresponding points are called images of each other. Because of the above correspondance a curve or region of z-plane is said to be transformed into or mapped upon or represented by the corresponding curve or regions in the w- plane.

I **mapping of $w = f(z) = z^2$**

we have $z = x+iy$

$$z^2 = (x+iy)^2$$

$$u+iv = x^2 - y^2 + i2xy$$

$$u = x^2 - y^2 \quad — ① \qquad v = 2xy — ②$$

Q Find the transformation of the vertical line $x = c$ (constant) under the mapping $w = z^2$

Ans: Substituting $x = c$ in ① & ② we get

① ⇒ $u = c^2 - y^2$ ② ⇒ $v = 2cy$

$$y = \frac{v}{2c}$$

$$\therefore \; u = c^2 - \frac{v^2}{4c^2}$$

$$4uc^2 = 4c^4 - v^2$$

$$v^2 = 4c^4 - 4uc^2$$

$$v^2 = 4c^2(c^2 - u)$$

ie) $(v-0)^2 = -4c^2(u - c^2)$ is a parabola whose vertex is at $(c^2, 0)$ and focus at the origin.

Q Find the transformation of horizontal line $y = k$ (constant) under the mapping $w = z^2$.

Ans: put $y = k$ in ① & ② we get

$u = x^2 - k^2$ $\qquad$ $v = 2xk$

$x = \frac{v}{2k}$

$$\therefore u = \frac{v^2}{4k^2} - k^2$$

$$4k^2u = v^2 - 4k^4$$

$$v^2 = 4k^2u + 4k^4$$

$v^2 = 4k^2(u + k^2)$ is a parabola with vertex at $(-k^2, 0)$ and focus at the origin.

18) Find the image of the line $x=1$, $y=2$ and $x>0$, $y<0$ under the mapping $w=z^2$.

Ans:

$$w = z^2$$

$$u+iv = (x+iy)^2$$

$$= (x^2-y^2) + i2xy$$

$$u = x^2 - y^2 \quad \text{--- (1)} \qquad v = 2xy \quad \text{--- (2)}$$

when $\underline{x=1}$

$$(1) \Rightarrow u = 1-y^2 \quad \& \quad (2) \Rightarrow v = 2y \qquad [y = v/2]$$

$$\therefore u = 1 - \frac{v^2}{4}$$

$$4u = 4 - v^2$$

$$v^2 = -4u + 4$$

$$v^2 = -4(u-1)$$

The line $x=1$ in the z plane corresponds to the parabola $v^2 = -4(u-1)$ whose vertex is at $(1,0)$ and focus at the origin

when $\underline{y=2}$

$$(1) \Rightarrow u = x^2 - 4 \quad \& \quad (2) \Rightarrow v = 4x \qquad [\therefore x = \tfrac{v}{4}]$$

$$\therefore u = \frac{v^2}{16} - 4$$

$$16u = v^2 - 64$$

$$v^2 = 16u + 64$$

$$v^2 = 16(u+4)$$

The line $y=2$ in the z plane corresponds to the parabola $v^2 = 16(u+4)$ whose vertex is at $(-4,0)$ and focus at the origin.

$x>0, y<0$

② $\Rightarrow$ $x>0, y<0$ $\Rightarrow$ $v = 2xy$
$v<0$.

19) Find the image of the semicircle $y=+\sqrt{4-x^2}$ under the transformation $w=z^2$

Ans: $w = z^2$

$u+iv = (x+iy)^2$

$= (x^2-y^2) + i2xy$

$u = x^2-y^2$ — ① $\qquad v = 2xy$ — ②

given $y = +\sqrt{4-x^2}$

squaring $y^2 = 4-x^2$

$x^2+y^2 = 4$

squaring and adding ① & ②

$u^2+v^2 = (x^2-y^2)^2 + (2xy)^2$

$= x^4 + y^4 - 2x^2y^2 + 4x^2y^2$

$= x^4 + y^4 + 2x^2y^2$

$= (x^2+y^2)^2$

$= 4^2$ $\quad$ [Since $x^2+y^2=4$]

$= 16$

The circle $x^2+y^2=4$ in the z plane is transformed on to the circle $u^2+v^2=16$ in the w plane.

Q) Sketch or graph the given region and its image under the given mapping.

(i) $|z| \leq 1/2$, $-\pi/8 < \arg z < \pi/8$, $\omega = z^2$

Ans: $w = z^2$

$Re^{i\phi} = (re^{i\theta})^2$

$Re^{i\phi} = r^2 e^{i2\theta}$

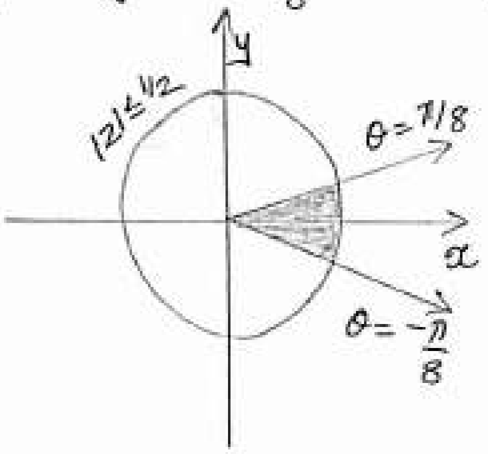
Comparing moduli and arguments we get

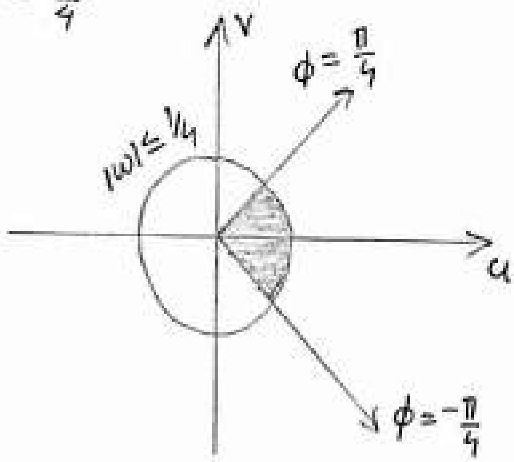
$R = r^2$ and $\phi = 2\theta$

ie) The circles $|z| = r_0$ is mapped onto $|w| = r_0^2$ and rays $\theta = \theta_0$ is mapped to $\phi = 2\theta_0$

$\therefore \ |z| \leq 1/2 \Rightarrow |w| \leq \frac{1}{4}$

$-\frac{\pi}{8} < \arg z < \pi/8 \Rightarrow -\frac{\pi}{4} \leq \arg \omega \leq \frac{\pi}{4}$





(ii) $1 < |z| < 3$ $\quad 0 < \arg z < \frac{\pi}{2}$ $\quad \omega = z^3$

Ans: $\omega = z^3$

$Re^{i\phi} = (re^{i\theta})^3$

$Re^{i\phi} = r^3 e^{i3\theta}$

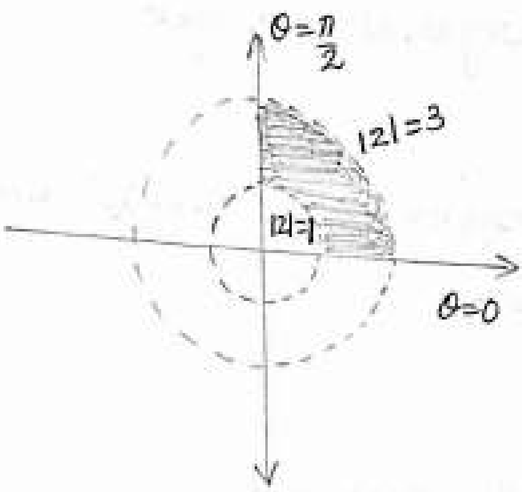
comparing moduli and argements we get

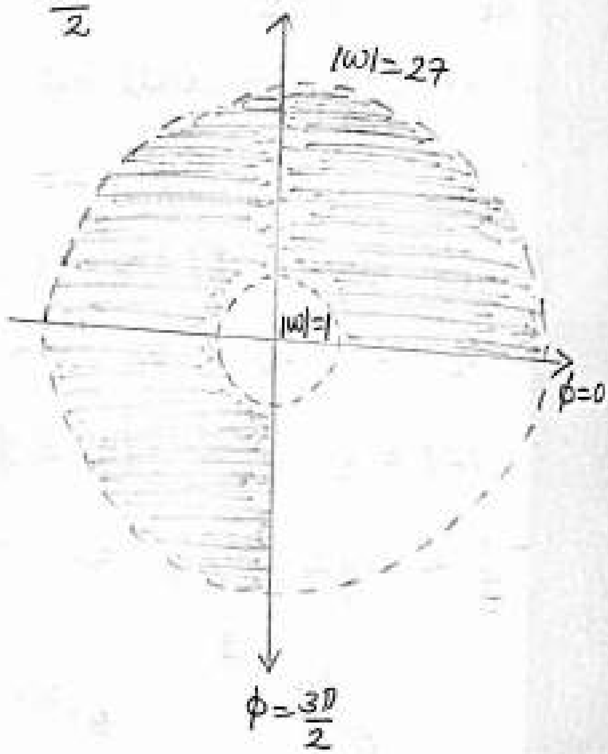
$R = r^3$ & $\phi = 3\theta$

The circle $|z| = r_0$ is mapped on to $|w| = r_0^3$ and the ray $\theta = \theta_0$ is mapped to $\phi = 3\theta_0$

$\therefore 1 < |z| < 3 \Rightarrow 1 < |w| < 27$

$0 < \arg z < \pi/2 \Rightarrow 0 < \arg w < \frac{3\pi}{2}$





II mapping of $w = f(z) = e^z$

$w = e^z$

$Re^{i\phi} = e^{x+iy}$

$= e^x e^{iy}$

$R = e^x$ —— ① and $\phi = y$ —— ②

(i) when $x = 0$

① ⇒ $R = e^0$

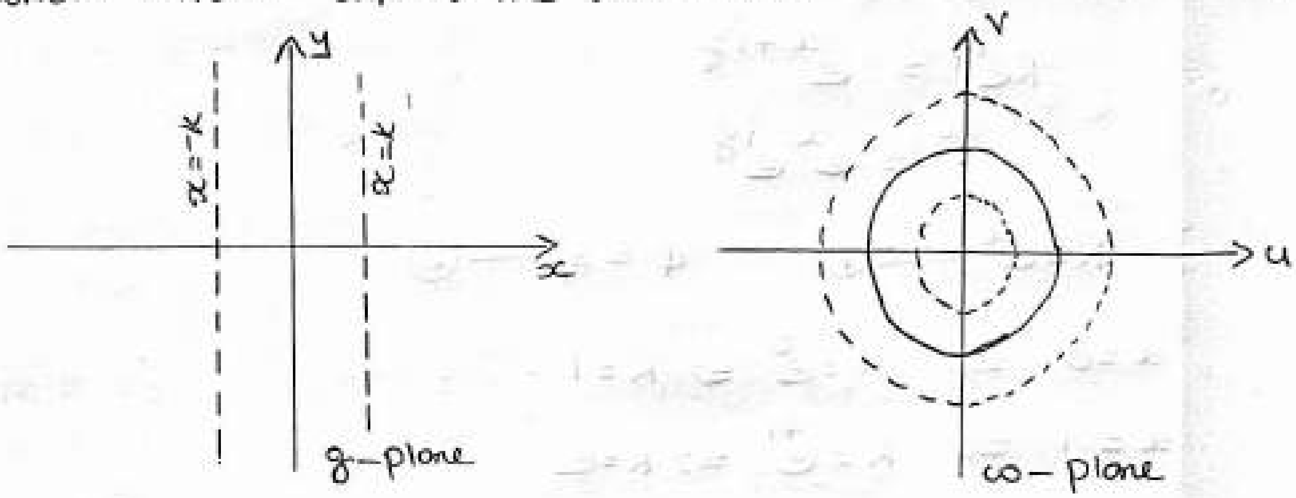
$R = 1$

ie) $x = 0$ is transformed in to a unit circle in the w plane.

(ii) when $x = k$ (k = constant)

① ⇒ $R = e^k$

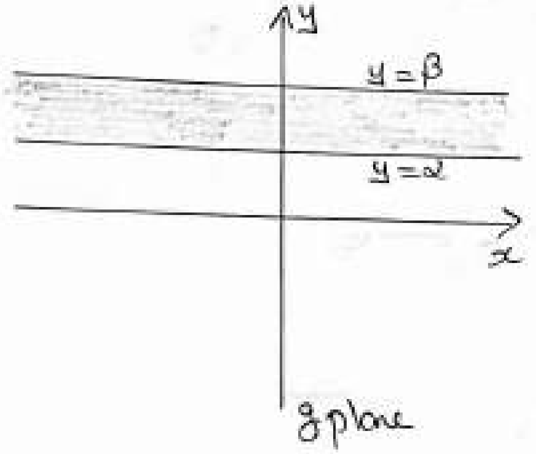
when $k > 0$ transforms in to a concentric circles surrounding the unit circle. where as $x = -k$ where $k > 0$ transforms into concentric circles with in the unit circle.

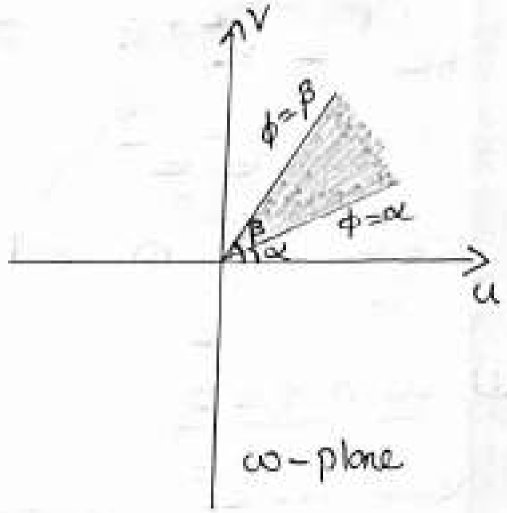


(iii) when $y = \alpha$ (constant)

② ⇒ $\phi = \alpha$ which is a line through the origin in the w plane. Hence the lines parallel to x axis in the z plane corresponds to radial lines in the w-plane.

iv) The area in the z-plane between the lines $y=\alpha$ and $y=\beta$ Corresponds to the area in the ω plane betwen the lines $\phi=\alpha$ and $\phi=\beta$





v) In particular if $\alpha=0$, $\beta=\pi$ the region in the z-plane between the lines $y=0$, $y=\pi$ corresponds to the half the ω-plane above real axis.

Q) Find the image of $0<x<1$, $\frac{1}{2}<y<1$ under the mapping $\omega=e^{z}$

Ans: Let $\omega=e^{z}$

$$Re^{i\phi} = e^{x+iy}$$

$$= e^{x}e^{iy}$$

$R=e^{x}$ —— ① $\phi=y$ —— ②

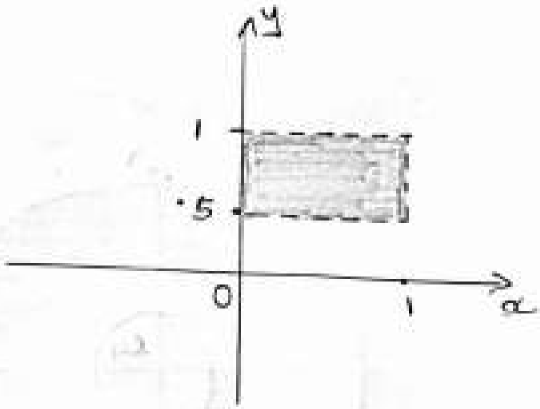
$x=0 \Rightarrow R=e^{0} \Rightarrow R=1$

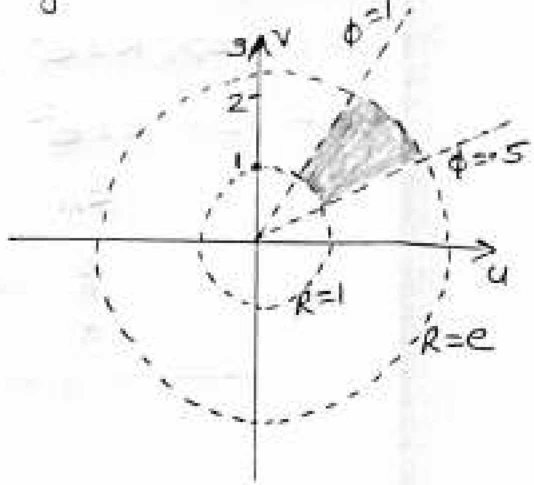
$x=1 \Rightarrow R=e^{+1} \Rightarrow R=e$

$y=\frac{1}{2} \Rightarrow \phi=\frac{1}{2}$

$y=1 \Rightarrow \phi=1$

The image of the given rectangle is the region included between the two circles and the two rays.





Q) Find the image of the infinite strip $0 \leq y \leq 2\pi$ under the transformation $w = e^z$. Also find the image of $0 \leq x \leq 2$ under $w = e^z$.

Ans: Let $w = e^z$

$$Re^{i\phi} = e^{x+iy}$$

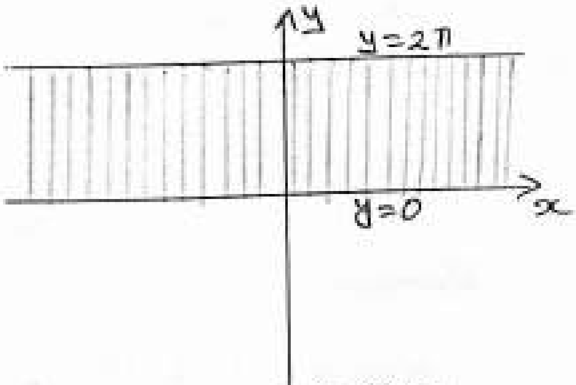
$$= e^x e^{iy}$$

$R = e^x$ —— ① $\phi = y$ —— ②

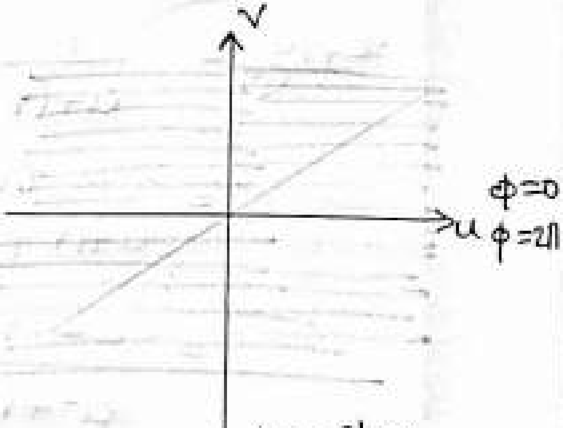
<u>$0 \leq y \leq 2\pi$</u>

$y = 0 \Rightarrow \phi = 0$

$y = 2\pi \Rightarrow \phi = 2\pi$



z plane



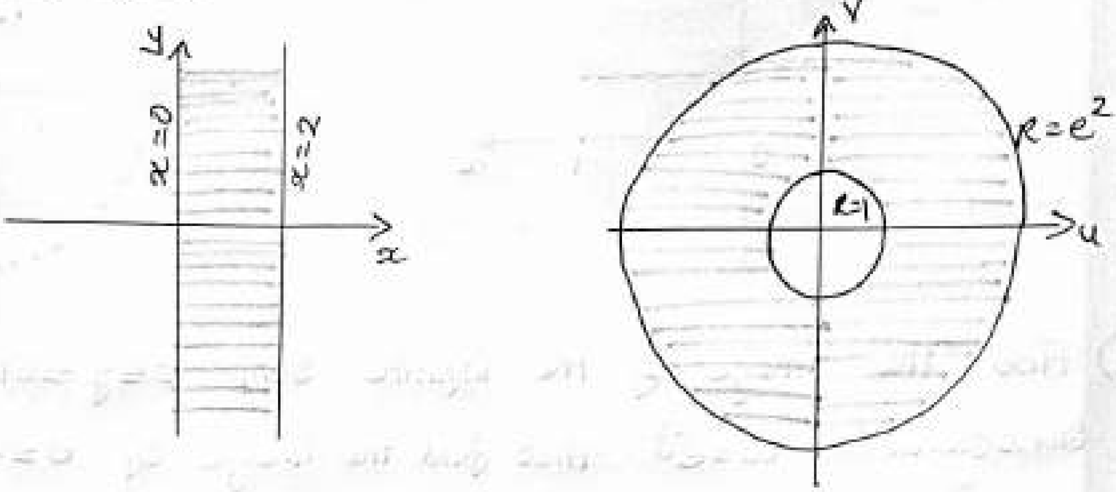
w - plane

Infinite strip $0 \leq y \leq 2\pi$ is transformed in to the entire w-plane.

$0 \le x \le 2$

$x = 0 \Rightarrow R = e^0 \Rightarrow R = 1$

$x = 2 \Rightarrow R = e^2$



$0 \le x \le 2$ is mapped onto the region between two concentric circles.

III) Transformation $\omega = f(z) = \frac{1}{z}$

Q) Find the transformation $x \ge 1$ under the mapping $\omega = 1/z$.

Ans: $\omega = \frac{1}{z}$

$z = \frac{1}{\omega}$

$x + iy = \frac{1}{u + iv}$

$= \frac{1}{u + iv} \frac{(u - iv)}{(u - iv)}$

$= \frac{u - iv}{u^2 + v^2}$

$x = \frac{u}{u^2 + v^2} \quad y = \frac{-v}{u^2 + v^2}$

$$x \geq 1 \Rightarrow \frac{u}{u^2+v^2} \geq 1$$

$$u \geq u^2+v^2$$

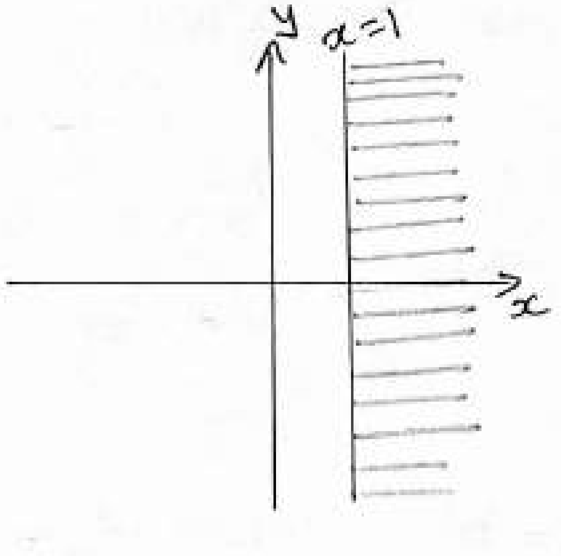
$$u^2+v^2 \leq u$$

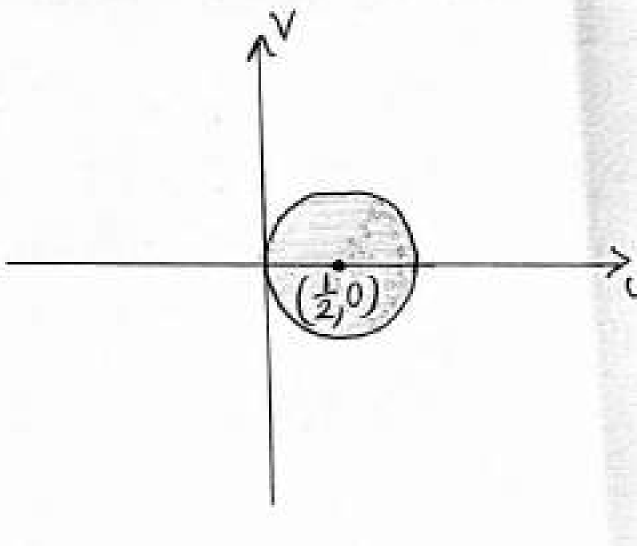
$$u^2+v^2-u \leq 0$$

$$u^2-u+\frac{1}{4}+v^2 \leq \frac{1}{4}$$

$$\left(u-\frac{1}{2}\right)^2+v^2 \leq \left(\frac{1}{2}\right)^2$$

which is a circular disk with centre $\left(\frac{1}{2}, 0\right)$ and radius $\frac{1}{2}$





21 Find the image of $|z-\frac{1}{2}| \le \frac{1}{2}$ under the transformation $w=\frac{1}{z}$. Also find the fixed points of the transformation $w = 1/z$.

Ans: $w = \frac{1}{z}$

$\therefore z = \frac{1}{w}$

$x+iy = \frac{1}{u+iv}$

$= \frac{(u-iv)}{(u+iv)(u-iv)}$

$= \frac{u-iv}{u^2+v^2}$

$x = \frac{u}{u^2+v^2} \qquad y = \frac{-v}{u^2+v^2}$

$|z-\frac{1}{2}| \le \frac{1}{2}$

$|x+iy-\frac{1}{2}| \le \frac{1}{2}$

$|(x-\frac{1}{2})+iy| \le 1/2$

$\sqrt{(x-\frac{1}{2})^2+y^2} \le \frac{1}{2}$

$(x-\frac{1}{2})^2+y^2 \le \frac{1}{4}$

$x^2-x+\frac{1}{4}+y^2 \le \frac{1}{4}$

$x^2-x+y^2 \le 0$

$\frac{u^2}{(u^2+v^2)^2} - \frac{u}{(u^2+v^2)} + \frac{v^2}{(u^2+v^2)^2} \le 0$

$\frac{u^2-u(u^2+v^2)+v^2}{(u^2+v^2)^2} \le 0$

$(u^2+v^2) - u(u^2+v^2) \leq 0$

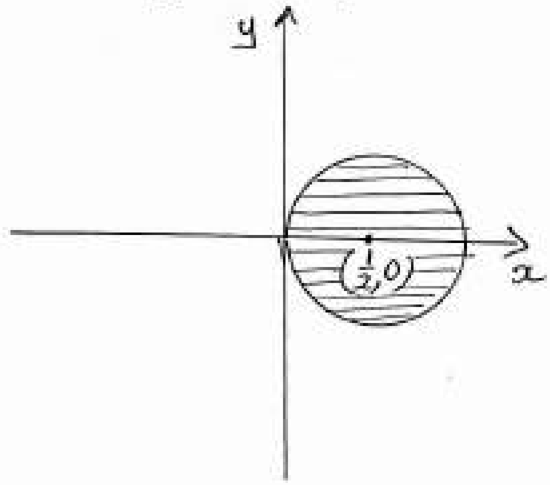
$(1-u)(u^2+v^2) \leq 0$

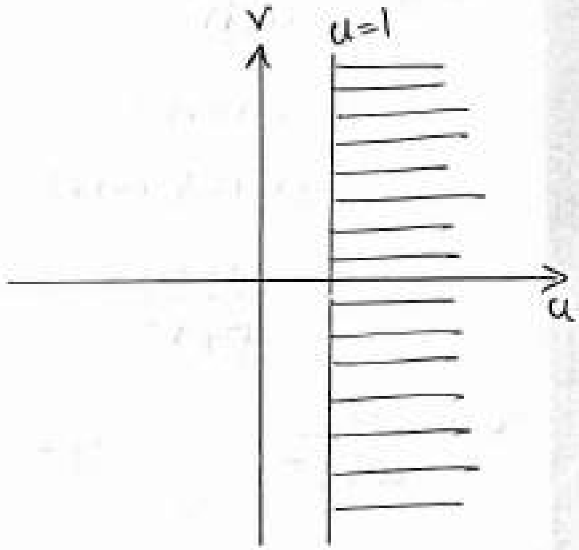
$1-u \leq 0$

$1 \leq u$

$u \geq 1$

$|z - \frac{1}{2}| \leq \frac{1}{2}$ is mapped to $u \geq 1$





22) Find the image of the region $|z - \frac{1}{3}| \leq \frac{1}{3}$ under the transformation $w = \frac{1}{z}$.

Ans: $w = \frac{1}{z}$

$z = \frac{1}{w}$

$x+iy = \frac{1}{u+iv}$

$x+iy = \frac{(u-iv)}{(u+iv)(u-iv)}$

$= \frac{u-iv}{u^2+v^2}$

$x = \frac{u}{u^2+v^2}$, $y = \frac{-v}{u^2+v^2}$

$|z-\frac{1}{3}| \le \frac{1}{3}$

$|x+iy-\frac{1}{3}| \le \frac{1}{3}$

$|(x-\frac{1}{3})+iy| \le \frac{1}{3}$

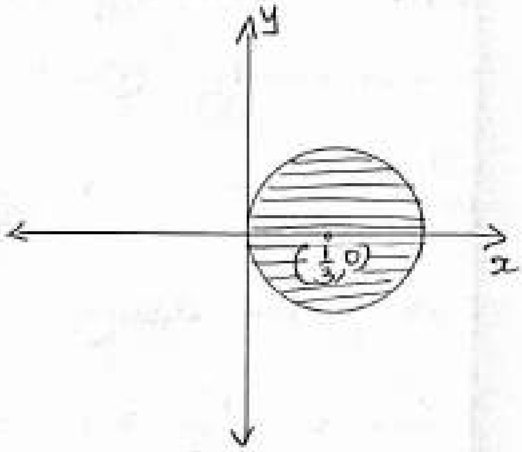
$\sqrt{(x-\frac{1}{3})^2+y^2} \le \frac{1}{3}$

$(x-\frac{1}{3})^2+y^2 \le \frac{1}{9}$

$x^2-\frac{2}{3}x+\frac{1}{9}+y^2 \le \frac{1}{9}$

$x^2-\frac{2x}{3}+y^2 \le 0$

$\frac{u^2}{(u^2+v^2)^2} - \frac{2}{3}\frac{u}{(u^2+v^2)} + \frac{v^2}{(u^2+v^2)^2} \le 0$

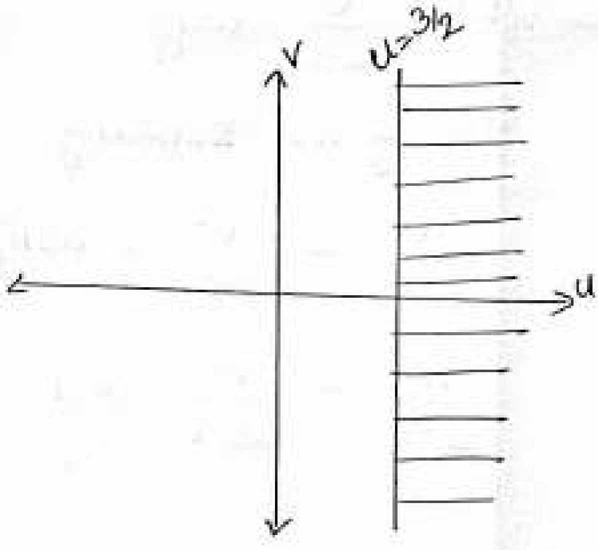


$\frac{3u^2-2u(u^2+v^2)+3v^2}{3(u^2+v^2)^2} \le 0$

$3u^2+3v^2-2u(u^2+v^2) \le 0$

$3(u^2+v^2)-2u(u^2+v^2) \le 0$

$(3-2u)(u^2+v^2) \le 0$



$3-2u \le 0$

$3 \le 2u$

$\frac{3}{2} \le u$

$\therefore u \ge \frac{3}{2}$

$|z-\frac{1}{3}| \le \frac{1}{3}$ is mapped on to $u \ge \frac{3}{2}$

IV MAPPING OF $w = \sin z$

$w = \sin z$

$u + iv = \sin(x+iy)$

$= \sin x \cos iy + \cos x \sin iy$

$= \sin x \cosh y + i \cos x \sinh y$

$u = \sin x \cosh y$ —— ① $v = \cos x \sinh y$ —— ②

① $\Rightarrow \sin x = \dfrac{u}{\cosh y}$ ② $\Rightarrow \cos x = \dfrac{v}{\sinh y}$

squaring and adding

$$\sin^2 x + \cos^2 x = \frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y}$$

ie) $$\frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y} = 1 \text{ —— ③}$$

① $\Rightarrow \cosh y = \dfrac{u}{\sin x}$ ② $\Rightarrow \sinh y = \dfrac{v}{\cos x}$

squaring and subtraction

$$\cosh^2 y - \sinh^2 y = \frac{u^2}{\sin^2 x} - \frac{v^2}{\cos^2 x}$$

ie) $$\frac{u^2}{\sin^2 x} - \frac{v^2}{\cos^2 x} = 1 \text{ —— ④}$$

Q Find the transformation of real axis $y=0$ in the z plane.

Ans: $y=0 \Rightarrow$ ① $\Rightarrow u = \sin x \cosh(0) = \sin x$ $[\cosh 0 = 1]$

② $\Rightarrow v = \cos x \sinh(0) = 0$ $[\sinh 0 = 0]$

$\sin x$ varies from -1 to 1

$\therefore$ u varies from -1 to 1

$\therefore$ real axis in the z plane $(y=0)$ is mapped on to the section between -1 and $+1$ of the real axis in the w plane.

Q: Find the transformation of imaginary axis $x=0$ in the z plane.

Ans: $x=0 \Rightarrow$ ① $\Rightarrow$ $u = \sin 0 \cosh y = 0$

$$v = \cos 0 \sinh y = \sinh y \qquad \left[\sinh y = \frac{e^{y}-e^{-y}}{2}\right.$$

$\sinh y$ varies from $-\infty$ to ∞

$u=0$ is the v axis

Hence the y axis in the z plane transformed in to the entire v axis in the w plane.

Q. Find the transformation of $y=k$ (lines parallel to x axis)

Ans: when $y=k$

$$③ \Rightarrow \frac{u^2}{\cosh^2 k} + \frac{v^2}{\sinh^2 k} = 1$$

which represents an ellipse. Hence lines parallel to x axis transform in to confocal ellipses.

Q. Find the transformation of $x=k$ (lines parallel to y axis)

Ans: when $x=k$

$$④ \Rightarrow \frac{u^2}{\sin^2 k} - \frac{v^2}{\cos^2 k} = 1$$

represents a hyperbola.

Hence lines parallel to y axis transforms in to a system of confocal hyperbolas.

u.Q Find the image of the line $x=c$ and $y=k$ where c and k are constants under the mapping $w=\sin z$.

Ans: $w = \sin z$

$$u+iv = \sin(x+iy)$$

$$= \sin x \cos iy + \cos x \sin iy$$

$$= \sin x \cosh y + i \cos x \sinh y$$

$u = \sin x \cosh y$ —— ① $v = \cos x \sinh y$ —— ②

① $\Rightarrow \sin x = \dfrac{u}{\cosh y}$ ② $\Rightarrow \cos x = \dfrac{v}{\sinh y}$

Squaring and adding

$$\sin^2 x + \cos^2 x = \frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y}$$

ie) $\dfrac{u^2}{\cosh^2 y} + \dfrac{v^2}{\sinh^2 y} = 1$ —— ③

① $\Rightarrow \cosh y = \dfrac{u}{\sin x}$ ② $\Rightarrow \sinh y = \dfrac{v}{\cos x}$

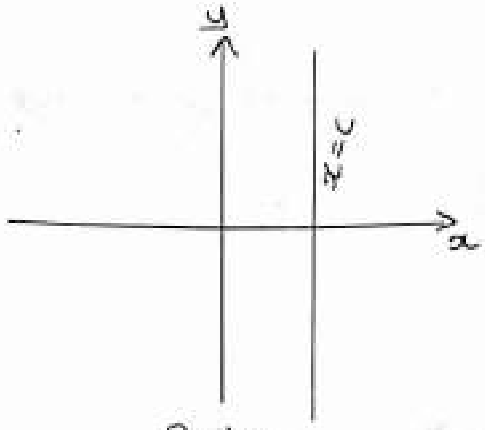
Squaring and subtracting

$$\cosh^2 y - \sinh^2 y = \frac{u^2}{\sin^2 x} - \frac{v^2}{\cos^2 x}$$

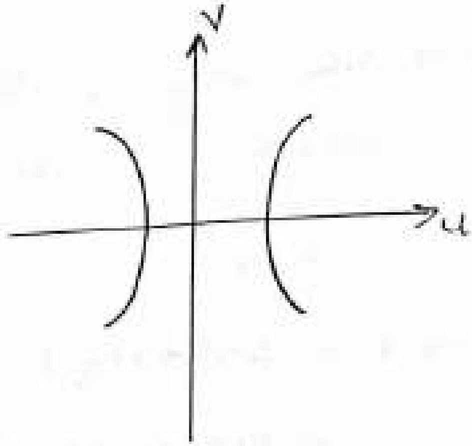
ie) $\dfrac{u^2}{\sin^2 x} - \dfrac{v^2}{\cos^2 x} = 1$ —— ④

$\underline{x=c}$

④ $\Rightarrow \dfrac{u^2}{\sin^2 c} - \dfrac{v^2}{\cos^2 c} = 1$ hyperbola in the w plane.



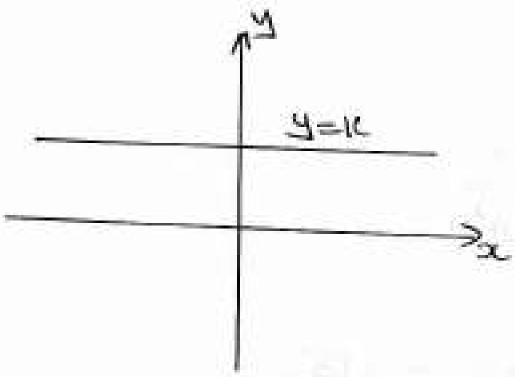
z plane



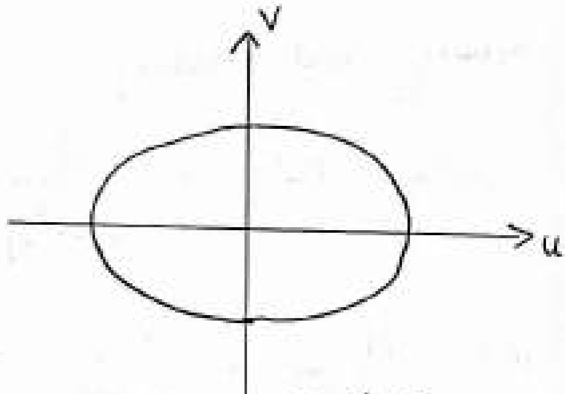
ω plane

$\underline{y = k}$

③ $\Rightarrow \quad \dfrac{u^2}{\cosh^2 k} + \dfrac{v^2}{\sinh^2 k} = 1$ is an ellipse.



z plane



ω - plane