MODULE 5

Matrix representation of graphs- Adjacency matrix, Incidence Matrix, Circuit matrix, Fundamental Circuit matrix and Rank, Cut set matrix, Path matrix

Introduction

- Graph is a set of edges and vertices.
- Graph can be represented in the form of matrix.
- Different matrix that can be formed are:

1. Incidence Matrix
2. Adjacency Matrix
3. Cut-Set Matrix
4. Circuit Matrix
5. Path Matrix

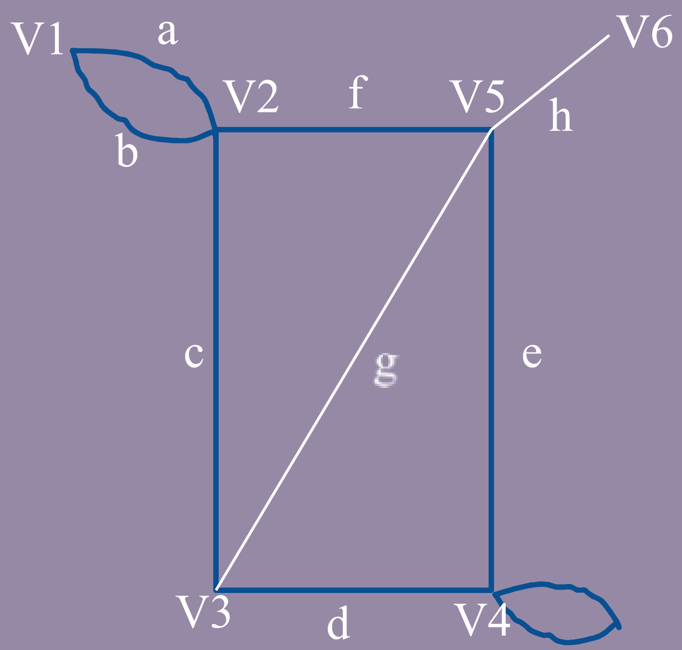
Incidence Matrix

- Edge connected to the vertex is known as incidence edge to that vertex.

- If vertex is incident on vertex then put 1 else 0.

$$a_{ij} = 1, \text{ if edge } e_j \text{ is incident on vertex } v_i$$
$$= 0, \text{ otherwise}$$

Vertex	**Edges**
1	a, b
2	a, b, c, f
3	c, d, g
4	d, e
5	d, e, f, g, h
6	h

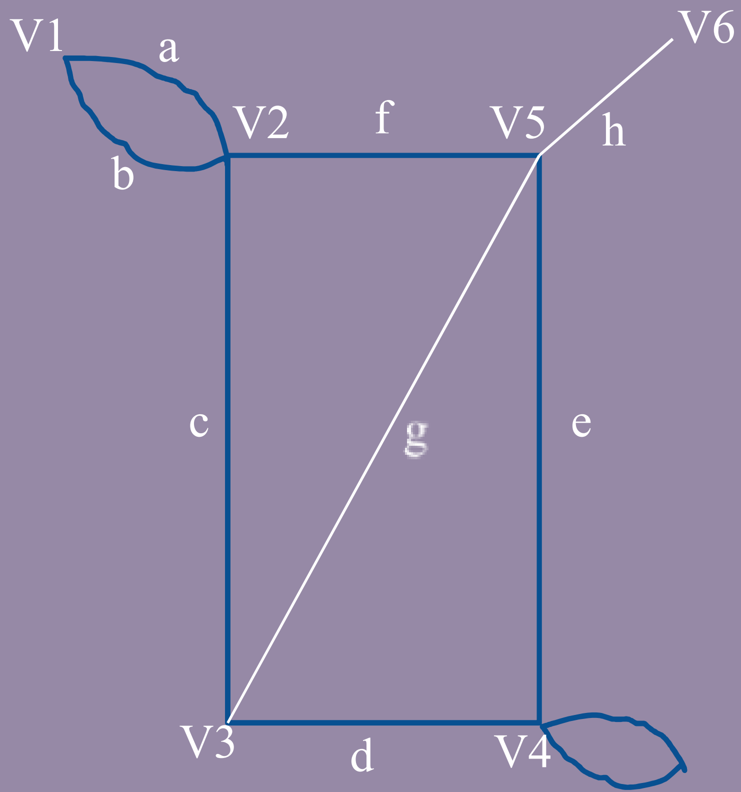


Edges

$$\text{Vertex}\quad
\begin{array}{c}
 \\ V1 \\ V2 \\ V3 \\ V4 \\ V5 \\ V6
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{array}$$

Adjacency Matrix

- If two vertices are connected by single path than they are known as adjacent vertices.
- If vertex is connected to itself then vertex is said to be adjacent to itself.
- If vertex is adjacent then put 1 else 0.



Vertices

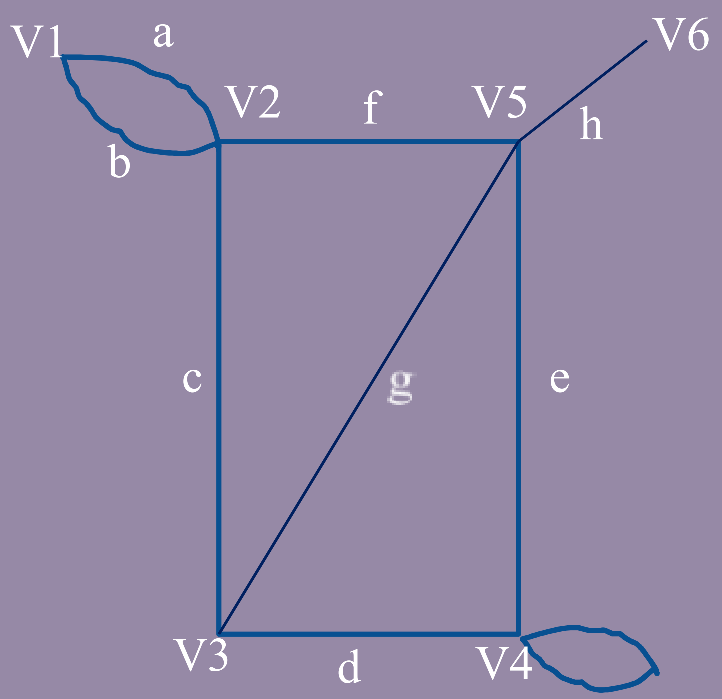
$$
\text{Vertices}\quad
\begin{array}{c}
 \\ V1 \\ V2 \\ V3 \\ V4 \\ V5 \\ V6
\end{array}
\begin{array}{c}
\begin{array}{cccccc} V1 & V2 & V3 & V4 & V5 & V6 \end{array} \\
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{array}
$$

Cut-Set Matrix

- Cut set is a "Set of edges in a graph whose removal leaves the graph disconnected".
- If edge of graph is a part of given cut set then put 1 else 0.

$$C_{ij} = \begin{cases} 1, & \text{if } j^{th} \text{ cutset contains edge} \\ 0, & \text{otherwise} \end{cases}$$

Cut Set	Edges
1	f, g, d
2	c, g, e
3	h
4	a, b



Edges

$$\text{Cut Sets}\quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

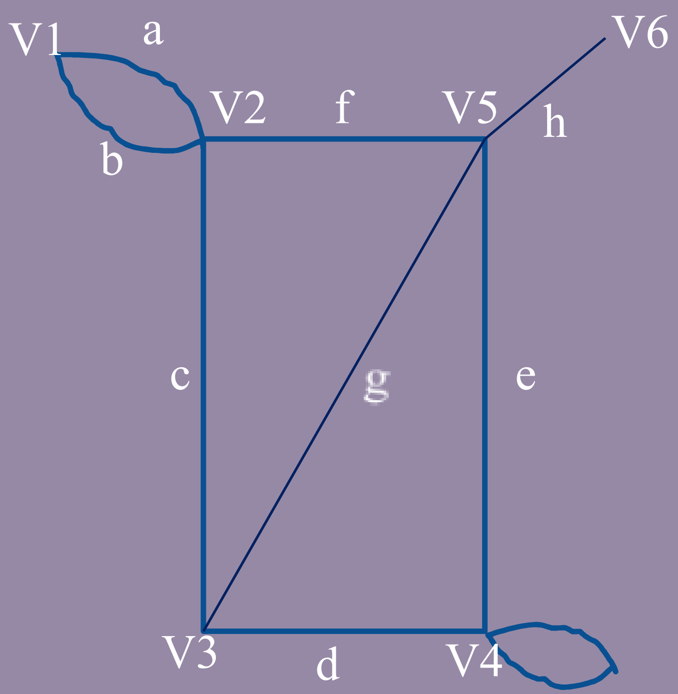
Circuit Matrix

- Circuit can be defined as “A close walk in which no vertex/edge can appear twice”.

- If edge of graph is a part of given circuit then put 1 else 0.

$$C_{ij} = \begin{cases} 1, & \text{if circuit contains edge} \\ 0, & \text{otherwise} \end{cases}$$

Circuit	**Edges**
1	d, e, g
2	c, f, g
3	c, d, e, f
4	a, b



Edges

$$\text{Circuits}\quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

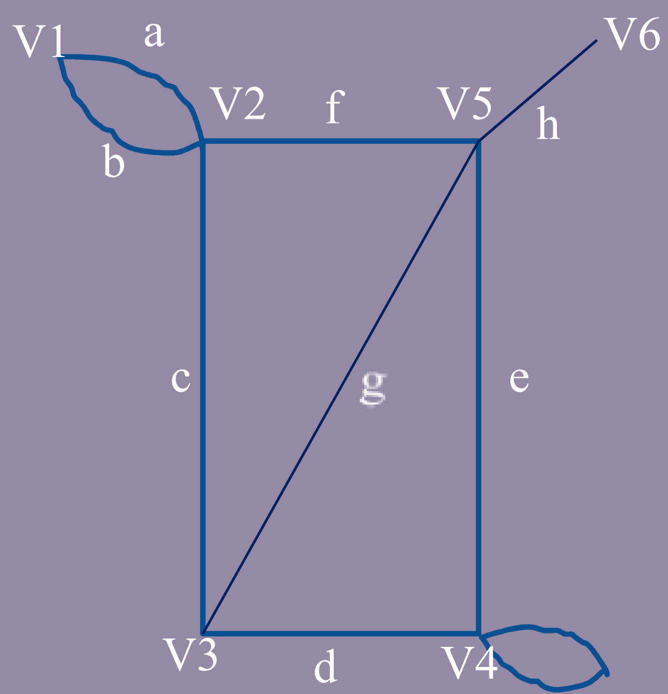
Path Matrix

- Path can be defined as “A open walk in which no vertex/edge can appear twice”.
- If edge of graph is a part of given path then put 1 else 0.

$$P(V_{j,} V_i) = \begin{cases} 1, & \text{if edge is on path} \\ 0, & \text{otherwise} \end{cases}$$

Ex: $P(V_{1,} V_6)$

Path	**Edges**
1	a, f, h
2	a, c, g, h
3	a, c, d, e, h
4	b, f, h
5	b, c, g, h
6	b,c, d, e, h



$$\text{Edges}$$

$$\text{Paths}\quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

INCIDENCE MATRIX

Incidence Matrix

Let G be a graph with n vertices, m edges and without self-loops. The incidence matrix A of G is an $n \times m$ matrix $A = [a_{ij}]$ whose n rows correspond to the n vertices and the m columns correspond to m edges such that

$$a_{ij} = \begin{cases} 1, & \textit{if jth edge } m_j \textit{ is incident on the ith vertex} \\ 0, & \textit{otherwise.} \end{cases}$$

It is also called *vertex-edge incidence matrix* and is denoted by $A(G)$.

Example Consider the graphs given in Figure 10.1. The incidence matrix of G_1 is

$$A(G_1) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

The incidence matrix of G_2 is

$$
\begin{array}{c c}
 & \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\
A(G_2) = \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.
\end{array}
$$

The incidence matrix of G_3 is

$$
\begin{array}{c c}
 & \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\
A(G_3) = \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}.
\end{array}
$$

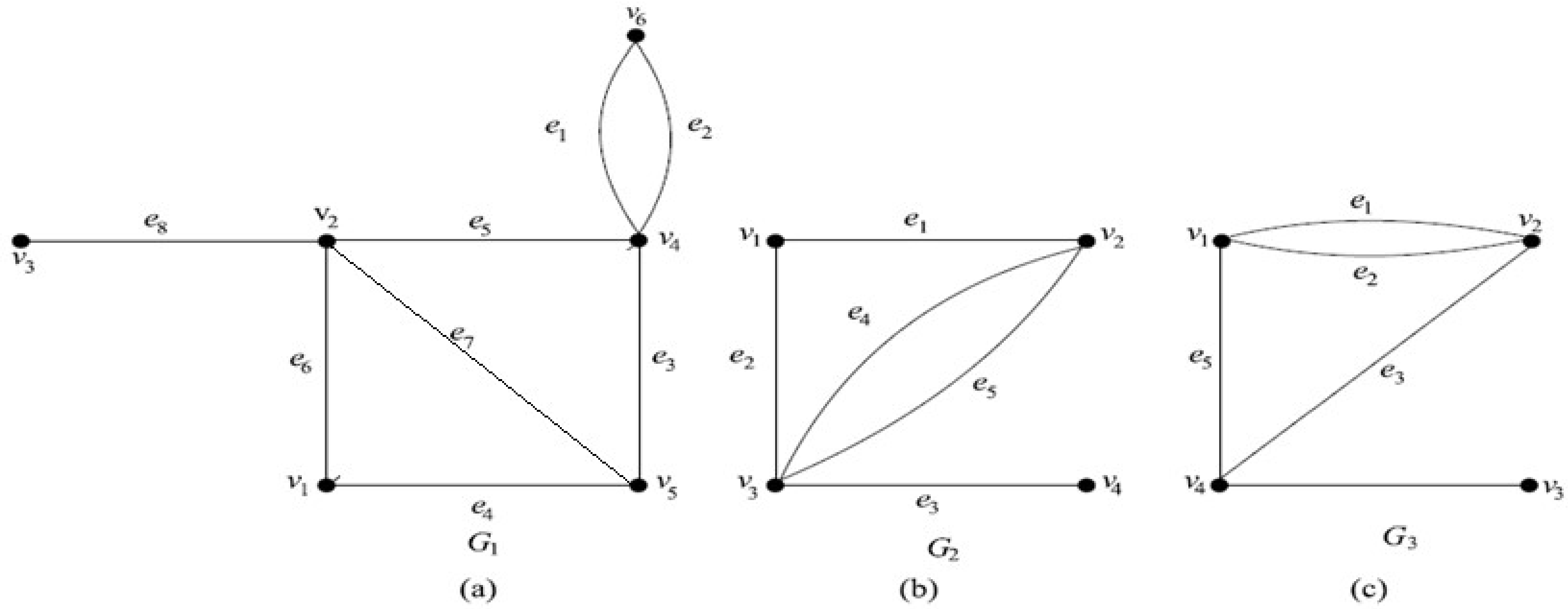


Fig. 10.1

Incidence Matrix-- Observations

The incidence matrix contains only two types of elements, 0 and 1. This clearly is a binary matrix or a (0, 1)-matrix.

We have the following observations about the incidence matrix A.

1. Since every edge is incident on exactly two vertices, each column of A has exactly two one's.

2. The number of one's in each row equals the degree of the corresponding vertex.

Incidence Matrix- Observations (Cont...)

3. A row with all zeros represents an isolated vertex.
4. Parallel edges in a graph produce identical columns in its incidence matrix.
5. If a graph is disconnected and consists of two components G_1 and G_2, the incidence matrix $A(G)$ of graph G can be written in a block diagonal form as

$$A(G) = \begin{bmatrix} A(G_1) & 0 \\ 0 & A(G_2) \end{bmatrix},$$

   where $A(G_1)$ and $A(G_2)$ are the incidence matrices of components G_1 and G_2. This observation results from the fact that no edge in G_1 is incident on vertices of G_2 and vice versa. Obviously, this is also true for a disconnected graph with any number of components.
6. Permutation of any two rows or columns in an incidence matrix simply corresponds to relabeling the vertices and edges of the same graph.

Theorem

Theorem 10.1 Two graphs G_1 and G_2 are isomorphic if and only if their incidence matrices $A(G_1)$ and $A(G_2)$ differ only by permutation of rows and columns.

Proof Let the graphs G_1 and G_2 be isomorphic. Then there is a one-one correspondence between the vertices and edges in G_1 and G_2 such that the incidence relation is preserved. Thus $A(G_1)$ and $A(G_2)$ are either same or differ only by permutation of rows and columns.

The converse follows, since permutation of any two rows or columns in an incidence matrix simply corresponds to relabeling the vertices and edges of the same graph. ❑

Rank

Rank of the incidence matrix

Let G be a graph and let $A(G)$ be its incidence matrix. Now each row in $A(G)$ is a vector over $GF(2)$ in the vector space of graph G. Let the row vectors be denoted by $A_1, A_2, \ldots, A_n$. Then,

$$A(G) = \begin{bmatrix} A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{bmatrix}.$$

Since there are exactly two ones in every column of A, the sum of all these vectors is 0 (this being a modulo 2 sum of the corresponding entries).

Thus vectors $A_1, A_2, \ldots, A_n$ are linearly dependent. Therefore, rank $A < n$.

Hence, rank $A \leq n-1$.

Remark If G is a disconnected graph with k components, then it follows from the above theorem that rank of $A(G)$ is $n-k$.

Theorem 10.2 If $A(G)$ is an incidence matrix of a connected graph G with n vertices, then rank of $A(G)$ is $n-1$.

Proof Let G be a connected graph with n vertices and let the number of edges in G be m. Let $A(G)$ be the incidence matrix and let $A_1, A_2, \ldots, A_n$ be the row vector of $A(G)$.

$$\text{Then, } A(G) = \begin{bmatrix} A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{bmatrix}. \tag{10.2.1}$$

$$\text{Clearly, rank } A(G) \leq n-1. \tag{10.2.2}$$

Consider the sum of any m of these row vectors, $m \leq n-1$. Since G is connected, $A(G)$ cannot be partitioned in the form

$$A(G) = \begin{bmatrix} A(G_1) & 0 \\ 0 & A(G_2) \end{bmatrix}$$

such that $A(G_1)$ has m rows and $A(G_2)$ has $n-m$ rows.

Thus there exists no $m \times m$ submatrix of $A(G)$ for $m \leq n-1$, such that the modulo 2 sum of these m rows is equal to zero.

As there are only two elements 0 and 1 in this field, the additions of all vectors taken m at a time for $m = 1, 2, \ldots, n-1$ gives all possible linear combinations of $n-1$ row vectors.

Thus no linear combinations of m row vectors of A, for $m \leq n-1$, is zero.

$$\text{Therefore, rank } A(G) \leq n-1. \tag{10.2.3}$$

Combining (10.2.2) and (10.2.3), it follows that rank $A(G) = n-1$. ❑

Circuit Matrix

10.3 Cycle Matrix

Let the graph G have m edges and let q be the number of different cycles in G. The cycle matrix $B = [b_{ij}]_{q\times m}$ of G is a $(0, 1)-$ matrix of order $q \times m$, with $b_{ij} = 1$, if the ith cycle includes jth edge and $b_{ij} = 0$, otherwise. The cycle matrix B of a graph G is denoted by $B(G)$.

Example Consider the graph G_1 given in Figure 10.3.

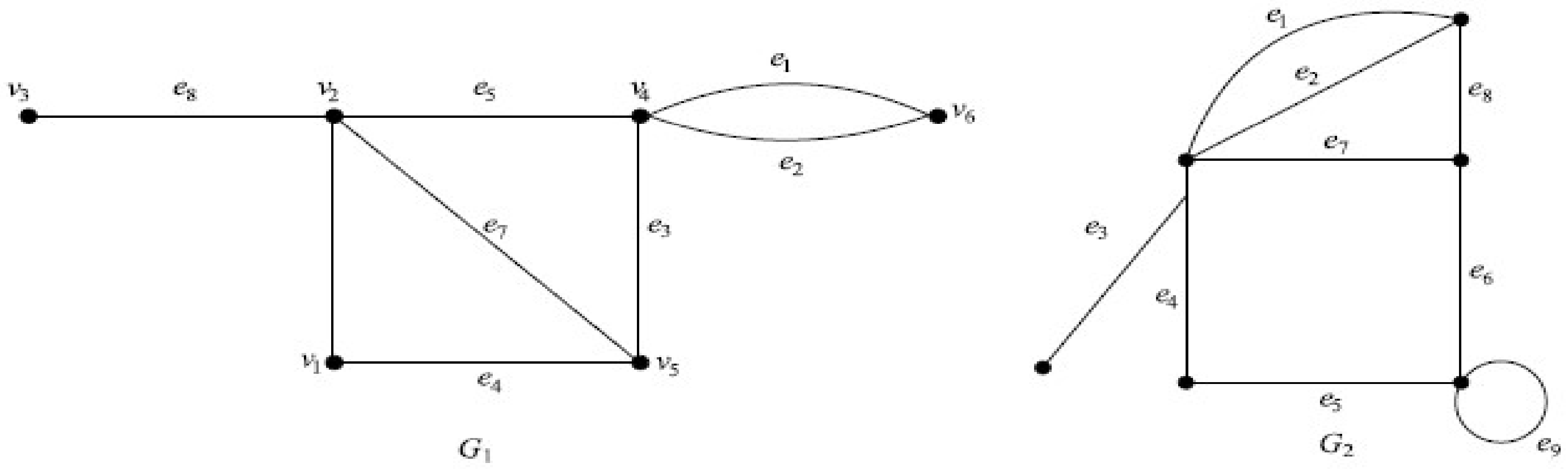


Fig. 10.3

The graph G_1 has four different cycles $Z_1 = \{e_1, e_2\}$, $Z_2 = \{e_3, e_5, e_7\}$, $Z_3 = \{e_4, e_6, e_7\}$ and $Z_4 = \{e_3, e_4, e_6, e_5\}$.

The cycle matrix is

$$B(G_1) = \begin{array}{c} \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}.$$

The graph G_2 of Figure 10.3 has seven different cycles, namely, $Z_1 = \{e_1, e_2\}$, $Z_2 = \{e_2, e_7, e_8\}$, $Z_3 = \{e_1, e_7, e_8\}$, $Z_4 = \{e_4, e_5, e_6, e_7\}$, $Z_5 = \{e_2, e_4, e_5, e_6, e_8\}$, $Z_6 = \{e_1, e_4, e_5, e_6, e_8\}$ and $Z_7 = \{e_9\}$. The cycle matrix is given by

$$B(G_2) = \begin{array}{c} \\ z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{array} \begin{array}{c} \begin{array}{ccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}.$$

Circuit Matrix--Observations

1. A column of all zeros corresponds to a non cycle edge, that is, an edge which does not belong to any cycle.

2. Each row of $B(G)$ is a cycle vector.

3. A cycle matrix has the property of representing a self-loop and the corresponding row has a single one.

4. The number of ones in a row is equal to the number of edges in the corresponding cycle.

5. If the graph G is separable (or disconnected) and consists of two blocks (or components) H_1 and H_2, then the cycle matrix $B(G)$ can be written in a block-diagonal form as

$$B(G) = \begin{bmatrix} B(H_1) & 0 \\ 0 & B(H_2) \end{bmatrix},$$

   where $B(H_1)$ and $B(H_2)$ are the cycle matrices of H_1 and H_2. This follows from the fact that cycles in H_1 have no edges belonging to H_2 and vice versa.

6. Permutation of any two rows or columns in a cycle matrix corresponds to relabeling the cycles and the edges.

Circuit Matrix—Observations (Conti...)

7. We know two graphs G_1 and G_2 are 2-isomorphic if and only if they have cycle correspondence. Thus two graphs G_1 and G_2 have the same cycle matrix if and only if G_1 and G_2 are 2-isomorphic. This implies that the cycle matrix does not specify a graph completely, but only specifies the graph within 2-isomorphism.

For example, the two graphs given in Figure 10.4 have the same cycle matrix. They are 2-isomorphic, but are not isomorphic.

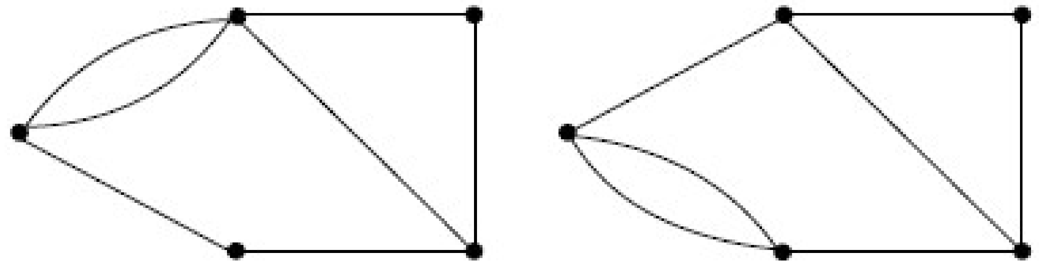


Fig. 10.4

Theorem 10.9 If G is a graph without self-loops, with incidence matrix A and cycle matrix B whose columns are arranged using the same order of edges, then every row of B is orthogonal to every row of A, that is $AB^T = BA^T \equiv 0 \pmod 2$, where A^T and B^T are the transposes of A and B respectively.

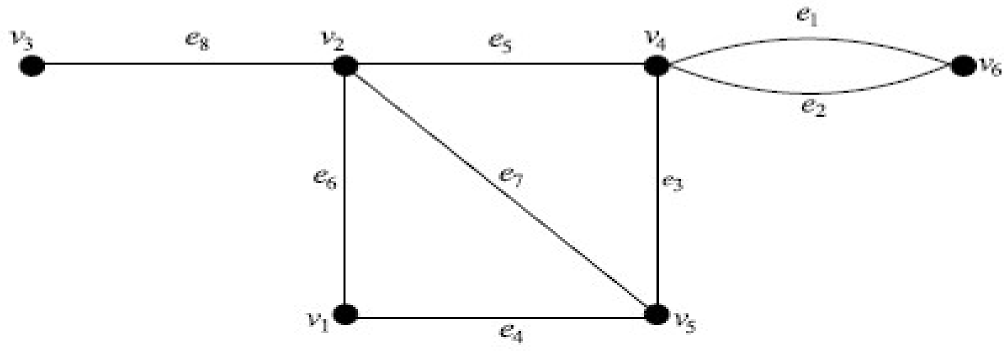


Fig. 10.5

Clearly,

$$AB^T = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 2 \\ 0 & 2 & 2 & 2 \\ 2 & 0 & 0 & 0 \end{bmatrix} \equiv 0(\text{mod}2).$$

Proof: Consider a vertex v and a circuit Γ in the graph G. Either v is in Γ or it is not. If v is not in Γ, there is no edge in the circuit Γ that is incident on v. On the other hand, if v is in Γ, the number of those edges in the circuit Γ that are incident on v is exactly two.

With this remark in mind, consider the ith row in A and the jth row in B. Since the edges are arranged in the same order, the nonzero entries in the corresponding positions occur only if the particular edge is incident on the ith vertex and is also in the jth circuit.

If the ith vertex is not in the jth circuit, there is no such nonzero entry, and the dot product of the two rows is zero. If the ith vertex is in the jth circuit, there will be exactly two 1's in the sum of the products of individual entries. Since $1 + 1 = 0 \pmod 2$, the dot product of the two arbitrary rows—one from A and the other from B—is zero. Hence the theorem. ∎

Fundamental Circuit Matrix

- In a cycle matrix, if we take only those rows that correspond to a set of fundamental cycles and remove all other rows, we do not lose any information.
- The removed rows can be formed from the rows corresponding to the set of fundamental cycles.
- For example, in the cycle matrix of the graph given in Figure 10.6, the fourth row is simply the mod 2 sum of the second and the third rows. Fundamental cycles are

$Z_1 = \{e_1, e_2, e_4, e_7\}$
$Z_2 = \{e_3, e_4, e_7\}$
$Z_3 = \{e_5, e_6, e_7\}$

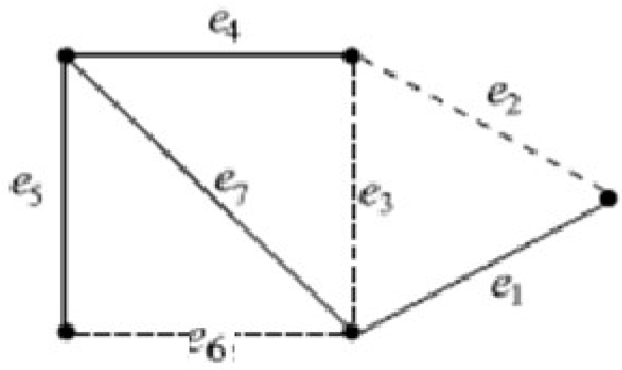


Fig. 10.6

$$\begin{array}{c} \\ Z_1 \\ Z_2 \\ Z_3 \end{array} \begin{array}{c} \begin{array}{ccccccccc} e_2 & e_3 & e_6 & & e_1 & e_4 & e_5 & e_7 \end{array} \\ \left[\begin{array}{ccc:cccc} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right] \end{array}.$$

A submatrix of a cycle matrix in which all rows correspond to a set of fundamental cycles is called a *fundamental cycle matrix* B_f.

The permutation of rows and/or columns do not affect B_f. If n is the number of vertices, m the number of edges in a connected graph G, then B_f is an $(m-n+1) \times m$ matrix because the number of fundamental cycles is $m-n+1$, each fundamental cycle being produced by one chord.

Now, arranging the columns in B_f such that all the $m-n+1$ chords correspond to the first $m-n+1$ columns and rearranging the rows such that the first row corresponds to the fundamental cycle made by the chord in the first column, the second row to the fundamental cycle made by the second, and so on. This arrangement is done for the above fundamental cycle matrix.

A matrix B_f thus arranged has the form

$$B_f = [I_\mu : B_t],$$

where I_μ is an identity matrix of order $\mu = m-n+1$ and B_t is the remaining $\mu \times (n-1)$ submatrix, corresponding to the branches of the spanning tree.

From equation $B_f = [I_\mu : B_t]$, we have rank $B_f = \mu = m-n+1$.

Since B_f is a submatrix of the cycle matrix B, therefore, rank $B \geq$ rank B_f and thus,

$$\text{rank } B \geq m-n+1.$$

The following result gives the rank of the cycle matrix.

Theorem 10.10 If B is a cycle matrix of a connected graph G with n vertices and m edges, then rank $B = m - n + 1$.

Proof Let A be the incidence matrix of the connected graph G.

Then $AB^T \equiv 0 (\text{mod} 2)$.

Using Sylvester's theorem (Theorem 10.13), we have rank $A+$ rank $B^T \leq m$ so that rank $A+$ rank $B \leq m$.

Therefore, rank $B \leq m-$ rank A.

As rank $A = n - 1$, we get rank $B \leq m - (n-1) = m - n + 1$.

But, rank $B \geq m - n + 1$.

Combining, we get rank $B = m - n + 1$. ❏

Theorem 10.10 can be generalised in the following form.

Theorem 10.11 If B is a cycle matrix of a disconnected graph G with n vertices, m edges and k components, then rank $B = m - n + k$.

Cut Set Matrix

10.4 Cut-Set Matrix

Let G be a graph with m edges and q cutsets. The cut-set matrix $C = [c_{ij}]_{q \times m}$ of G is a (0, 1)-matrix with

$$c_{ij} = \begin{cases} 1, & if\ ith\ cutset\ contains\ jth\ edge, \\ 0, & otherwise. \end{cases}$$

Example Consider the graphs shown in Figure 10. 7.

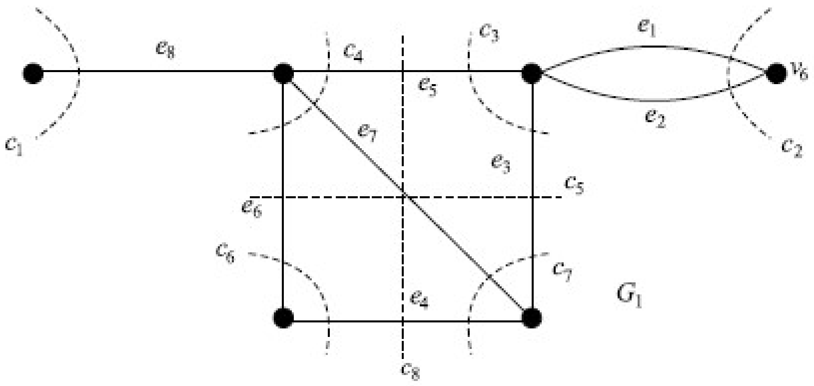


Fig. 10.7(a)

In the graph G_1, $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$.

The cut-sets are $c_1 = \{e_8\}$, $c_2 = \{e_1, e_2\}$, $c_3 = \{e_3, e_5\}$, $c_4 = \{e_5, e_6, e_7\}$, $c_5 = \{e_3, e_6, e_7\}$, $c_6 = \{e_4, e_6\}$, $c_7 = \{e_3, e_4, e_7\}$ and $c_8 = \{e_4, e_5, e_7\}$.

The cut-sets for the graph G_2 are $c_1 = \{e_1, e_2\}$, $c_2 = \{e_3, e_4\}$, $c_3 = \{e_4, e_5\}$, $c_4 = \{e_1, e_6\}$, $c_5 = \{e_2, e_6\}$, $c_6 = \{e_3, e_5\}$, $c_7 = \{e_1, e_4, c_7\}$, $c_8 = \{e_2, e_3, e_7\}$ and $c_9 = \{e_5, e_6, e_7\}$.

Thus the cut-set matrices are given by

$$C(G_1) = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}, \text{ and}$$

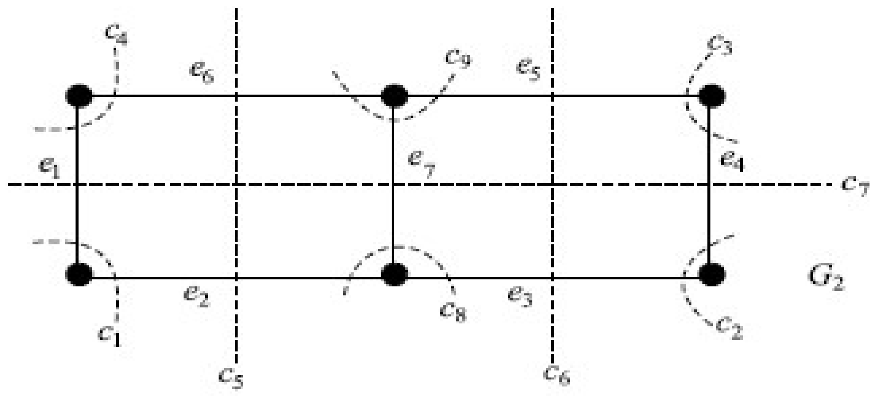


Fig. 10.7(b)

$$C(G_2) = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \\ c_9 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}.$$

Cut set Matrix -- Observations

1. The permutation of rows or columns in a cut-set matrix corresponds simply to re-naming of the cut-sets and edges respectively.
2. Each row in $C(G)$ is a cut-set vector.
3. A column with all zeros corresponds to an edge forming a self-loop.
4. Parallel edges form identical columns in the cut-set matrix.
5. In a non-separable graph, since every set of edges incident on a vertex is a cut-set, therefore every row of incidence matrix $A(G)$ is included as a row in the cut-set matrix $C(G)$. That is, for a non-separable graph G, $C(G)$ contains $A(G)$. For a separable graph, the incidence matrix of each block is contained in the cut-set matrix. For example, in the graph G_1 of Figure 10.7, the incidence matrix of the block $\{e_3, e_4, e_5, e_6, e_7\}$ is the 4×5 submatrix of C, left after deleting rows c_1, c_2, c_5, c_8 and columns e_1, e_2, e_8.
6. It follows from observation 5, that rank $C(G) \geq$ rank $A(G)$. Therefore, for a connected graph with n vertices, rank $C(G) \geq n - 1$.

Theorem 10.14 If G is a connected graph, then the rank of a cut-set matrix $C(G)$ is equal to the rank of incidence matrix $A(G)$, which equals the rank of graph G.

Proof Let $A(G)$, $B(G)$ and $C(G)$ be the incidence, cycle and cut-set matrix of the connected graph G. Then we have

$$\text{rank } C(G) \geq n-1. \tag{10.14.1}$$

Since the number of edges common to a cut-set and a cycle is always even, every row in C is orthogonal to every row in B, provided the edges in both B and C are arranged in the same order.

$$\text{Thus, } BC^T = CB^T \equiv 0 \pmod 2). \tag{10.14.2}$$

Now, applying Sylvester's theorem to equation (10.14.2), we have

$$\text{rank } B + \text{rank } C \leq m.$$

For a connected graph, we have rank $B = m - n + 1$.

$$\text{Therefore, rank } C \leq m - \text{rank } B = m-(m-n+1) = n-1.$$

$$\text{So, rank } C \leq n-1. \tag{10.14.3}$$

It follows from (10.14.1) and (10.14.3) that rank $C = n-1$. ❑

Fundamental Cut set Matrix

10.5 Fundamental Cut-Set Matrix

Let G be a connected graph with n vertices and m edges. The fundamental cut-set matrix C_f of G is an $(n-1)\times m$ submatrix of C such that the rows correspond to the set of fundamental cut-sets with respect to some spanning tree. Clearly, a fundamental cut-set matrix C_f can be partitioned into two submatrices, one of which is an identity matrix I_{n-1} of order $n-1$. We have

$$C_f = [C_c : I_{n-1}],$$

where the last $n-1$ columns forming the identity matrix correspond to the $n-1$ branches of the spanning tree and the first $m-n+1$ columns forming C_c correspond to the chords.

Example Consider the connected graphs G_1 and G_2 given in Figure 10.8. The spanning tree is shown with bold lines. The fundamental cut-sets of G_1 are c_1, c_2, c_3, c_6 and c_7 while the fundamental cut-sets of G_2 are c_1, c_2, c_3, c_4 and c_7.

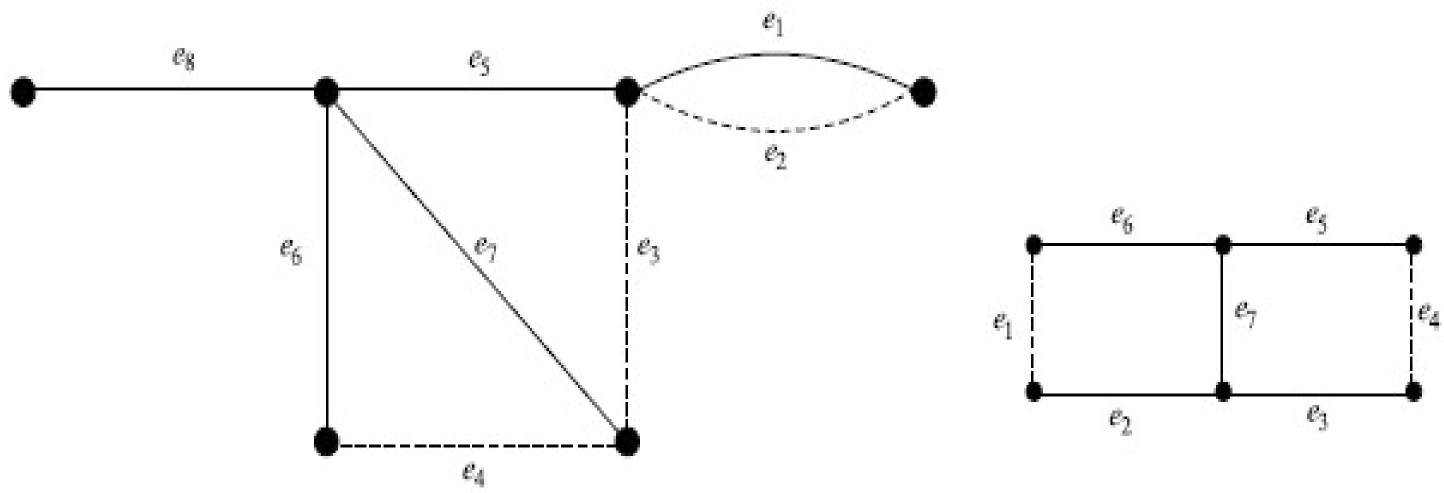


Fig. 10.8

The fundamental cut-set matrix of G_1 and G_2, respectively are given by

$$C_f = \begin{array}{c} \begin{array}{cccccccccc} e_2 & e_3 & e_4 & & e_1 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \left[\begin{array}{cccccccccc} 1 & 0 & 0 & \vdots & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \vdots & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \vdots & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & \vdots & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array} \text{ and}$$

$$C_f = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_7 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_4 & & e_2 & e_3 & e_5 & e_6 & e_7 \end{array} \\ \left[\begin{array}{cccccccc} 1 & 0 & \vdots & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & \vdots & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & \vdots & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & \vdots & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & \vdots & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

10.6 Relations between A_f, B_f and C_f

Let G be a connected graph and A_f, B_f and C_f be respectively the reduced incidence matrix, the fundamental cycle matrix, and the fundamental cut-set matrix of G.

We have shown that

$$B_f = \left[I_\mu \vdots B_t \right] \tag{10.6.i}$$

and $$C_f = \left[C_c \vdots I_{n-1} \right], \tag{10.6.ii}$$

where B_t denotes the submatrix corresponding to the branches of a spanning tree and C_c denotes the submatrix corresponding to the chords.

Let the spanning tree T in Equations (10.6.i) and (10.6.ii) be the same and let the order of the edges in both equations be same. Also, in the reduced incidence matrix A_f of size $(n-1) \times m$, let the edges (i.e., the columns) be arranged in the same order as in B_f and C_f.

Partition A_f into two submatrices given by

$$A_f = \left[A_c \vdots A_t \right], \tag{10.6.iii}$$

where A_t consists of $n-1$ columns corresponding to the branches of the spanning tree T and A_c is the spanning submatrix corresponding to the $m-n+1$ chords.

Since the columns in A_f and B_f are arranged in the same order, the equation $AB^T = BA^T = 0 (\text{mod } 2)$ gives

$$A_f B_f^T \equiv 0(\text{mod } 2),$$

or $$\begin{bmatrix} A_c \vdots A_t \end{bmatrix} \begin{bmatrix} I_\mu \\ \vdots \\ B_t^T \end{bmatrix} \equiv 0(\text{mod } 2),$$

or $$A_c + A_t B_f^T \equiv 0(\text{mod } 2). \qquad (10.6.\text{iv})$$

Since A_t is non singular, A_t^{-1} exists. Now, premultiplying both sides of equation (10.6.iv) by A_t^{-1}, we have

$$A_t^{-1} A_c + A_t^{-1} A_t B_t^T \equiv 0(\text{mod } 2),$$

or $$A_t^{-1} A_c + B_t^T \equiv 0(\text{mod } 2).$$

Therefore, $A_t^{-1} A_c = -B_t^T$.

Since in mod 2 arithmetic $-1 = 1$,

$$B_t^T = A_t^{-1} A_c. \qquad (10.6.\text{v})$$

Now as the columns in B_f and C_f are arranged in the same order, therefore (in mod 2 arithmetic) $C_f . B_f^T \equiv 0(\text{mod } 2)$ in mod 2 arithmetic gives $C_f . B_f^T = 0$.

Therefore, $\begin{bmatrix} C_c \vdots I_{n-1} \end{bmatrix} \begin{bmatrix} I_\mu \\ \vdots \\ B_t^T \end{bmatrix} = 0$, so that $C_c + B_t^T = 0$, that is, $C_c = -B_t^T$.

Thus, $C_c = B_t^T$ (as $-1 = 1$ in mod 2 arithmetic).

Hence, $C_c = A_t^{-1} A_c$ from (10.6.v).

Example Consider the graph G of Figure 10.9.

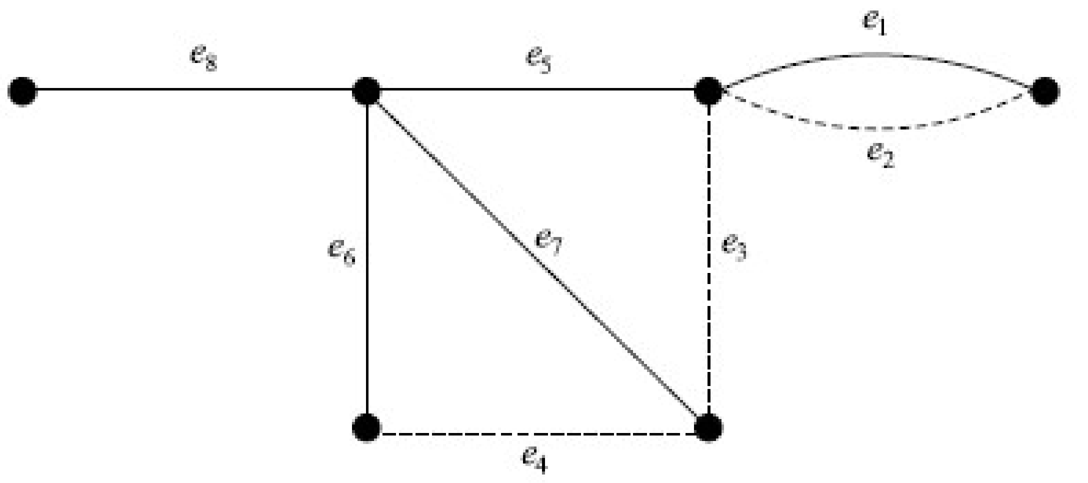


Fig. 10.9

Let $\{e_1, e_5, e_6, e_7, e_8\}$ be the spanning tree.

$$\text{We have, } A = \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

Dropping the sixth row in A, we get

$$A_f = \begin{array}{c} \begin{array}{ccccccccc} e_2 & e_3 & e_4 & & e_1 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \left[\begin{array}{ccccccccc} 0 & 0 & 1 & : & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & : & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & : & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & : & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & : & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} = [A_c : A_t].$$

$$B_f = \begin{array}{c} \begin{array}{ccccccccc} e_2 & e_3 & e_4 & & e_1 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \left[\begin{array}{ccccccccc} 1 & 0 & 0 & : & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & : & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & : & 0 & 0 & 1 & 1 & 0 \end{array}\right] \end{array} = [I_3 : B_t] \text{ and}$$

$$C_f = \begin{array}{c} \begin{matrix} e_2 & e_3 & e_4 & e_1 & e_5 & e_6 & e_7 & e_8 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} = [C_c : \ I_5] .$$

Clearly, $B_t^T = C_c$.

We verify $A_t^{-1}A_c = B_t^T$.

Now,

$$A_t = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, B_t = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Therefore, $A_t^{-1}Ac = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Hence, $A_t^{-1}A_c = B_t^T$.

Remarks We make the following observations from the above relations.

1. If A or A_f is given, we can construct B_f and C_f starting from an arbitrary spanning tree and its submatrix A_t in A_f.
2. If either B_f or C_f is given, we can construct the other. Therefore, since B_f determines a graph within 2-isomorphism, so does C_f.
3. If either B_f and C_f is given, then A_f in general cannot be determined completely.

Path Matrix

10.7 Path Matrix

Let G be a graph with m edges, and u and v be any two vertices in G. The path matrix for vertices u and v denoted by $P(u, v) = [p_{ij}]_{q \times m}$, where q is the number of different paths between u and v, is defined as

$$p_{ij} = \begin{cases} 1, & \textit{if jth edge lies in the ith path}, \\ 0, & \textit{otherwise}. \end{cases}$$

Clearly, a path matrix is defined for a particular pair of vertices, the rows in $P(u, v)$ correspond to different paths between u and v, and the columns correspond to different edges in G. For example, consider the graph in Figure 10.10.

Example

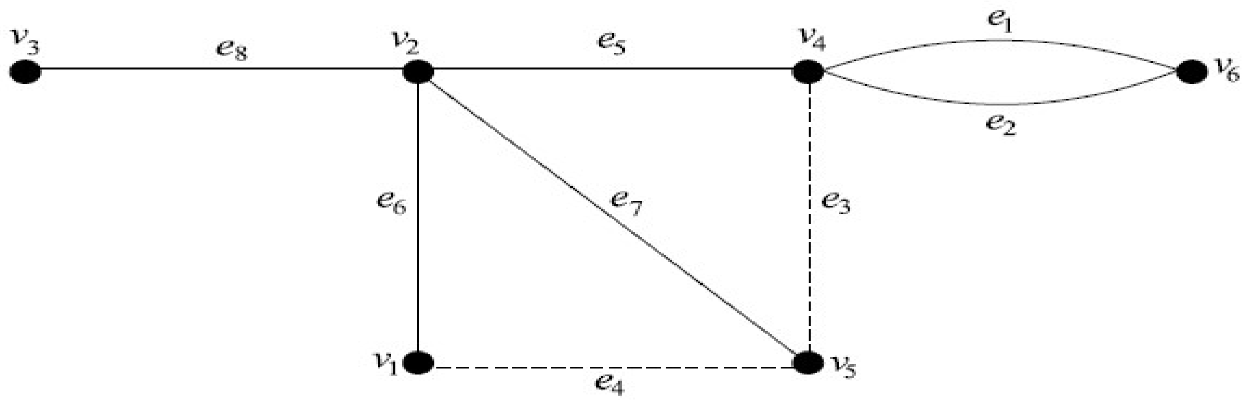


Fig. 10.10

The different paths between the vertices v_3 and v_4 are

$$p_1 = \{e_8, e_5\},\ p_2 = \{e_8, e_7, e_3\} \text{ and } p_3 = \{e_8, e_6, e_4, e_3\}.$$

The path matrix for v_3, v_4 is given by

$$P(v_3, v_4) = \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}.$$

Path Matrix-- Observations

1. A column of all zeros corresponds to an edge that does not lie in any path between u and v.
2. A column of all ones corresponds to an edge that lies in every path between u and v.
3. There is no row with all zeros.
4. The ring sum of any two rows in $P(u, v)$ corresponds to a cycle or an edge-disjoint union of cycles.

THEOREM 7-7

If the edges of a connected graph are arranged in the same order for the columns of the incidence matrix A and the path matrix $P(x, y)$, then the product (mod 2)

$$A \cdot P^T(x, y) = M,$$

where the matrix M has 1's in two rows x and y, and the rest of the $n - 2$ rows are all 0's.

Proof: The proof is left as an exercise for the reader (Problem 7-14).

As an example, multiply the incidence matrix in Fig. 7-1 to the transposed $P(v_3, v_4)$, just discussed.

$$A \cdot P^T(v_3, v_4) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \quad \text{(mod 2)}.$$

Other properties of the path matrix, such as the rank, are left for the reader to investigate on his own. It should be noted that a path matrix contains less information about the graph in general than any of the matrices A, B, or C does.

Adjacency Matrix

10.8 Adjacency Matrix

Let $V = (V, E)$ be a graph with $V = \{v_1, v_2, \ldots, v_n\}$, $E = \{e_1, e_2, \ldots, e_m\}$ and without parallel edges. The adjacency matrix of G is an $n \times n$ symmetric binary matrix $X = [x_{ij}]$ defined over the ring of integers such that

$$x_{ij} = \begin{cases} 1, & if\ v_i v_j \in E, \\ 0, & otherwise. \end{cases}$$

Example Consider the graph G given in Figure 10.12.

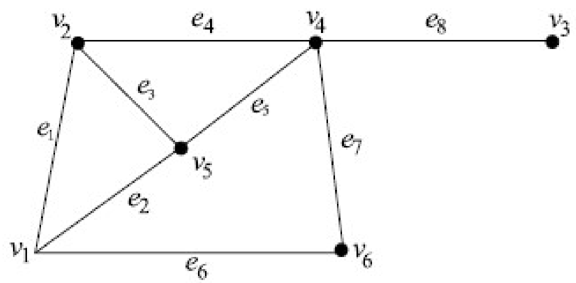


Fig. 10.12

Example

The adjacency matrix of G is given by

$$X = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}.$$

Adjacency Matrix-- Observations

1. The entries along the principal diagonal of X are all zeros if and only if the graph has no self-loops. However, a self-loop at the ith vertex corresponds to $x_{ii} = 1$.

2. If the graph has no self-loops, the degree of a vertex equals the number of ones in the corresponding row or column of X.

3. Permutation of rows and the corresponding columns imply reordering the vertices. We note that the rows and columns are arranged in the same order. Therefore, when two rows are interchanged in X, the corresponding columns are also interchanged. Thus two graphs G_1 and G_2 without parallel edges are isomorphic if and only if their adjacency matrices $X(G_1)$ and $X(G_2)$ are related by

$$X(G_2) = R^{-1}X(G_1)R,$$

   where R is a permutation matrix.

4. A graph G is disconnected having components G_1 and G_2 if and only if the adjacency matrix $X(G)$ is partitioned as

$$X(G) = \begin{bmatrix} X(G_1) & : & O \\ .. & : & .. \\ O & : & X(G_2) \end{bmatrix},$$

   where $X(G_1)$ and $X(G_2)$ are respectively the adjacency matrices of the components G_1 and G_2. Obviously, the above partitioning implies that there are no edges between vertices in G_1 and vertices in G_2.

5. If any square, symmetric and binary matrix Q of order n is given, then there exists a graph G with n vertices and without parallel edges whose adjacency matrix is Q.

Power of X

Powers of X: Let us multiply by itself the 6 by 6 adjacency matrix of the simple graph in Fig. 7-7. The result, another 6 by 6 symmetric matrix $\mathbf{X}^2$, is shown below (note that this is ordinary matrix multiplication in the ring of integers and *not* mod 2 multiplication):

$$\mathbf{X}^2 = \begin{bmatrix} 3 & 1 & 0 & 3 & 1 & 0 \\ 1 & 3 & 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 4 & 1 & 0 \\ 1 & 2 & 1 & 1 & 3 & 2 \\ 0 & 2 & 1 & 0 & 2 & 2 \end{bmatrix}$$

The value of an off-diagonal entry in $\mathbf{X}^2$, that is, ijth entry ($i \neq j$) in $\mathbf{X}^2$,

= number of 1's in the dot product of ith row and jth column (or jth row) of $\mathbf{X}$.

= number of positions in which both ith and jth rows of $\mathbf{X}$ have 1's.

= number of vertices that are adjacent to both ith and jth vertices.

= number of different paths of length two between ith and jth vertices.

Relationship Between A(G) *and* X(G): Recall that if a graph G has no self-loops, its incidence matrix A(G) contains all the information about G. Likewise, if G has no parallel edges, its adjacency matrix X(G) contains all the information about G. Therefore, if a graph G has neither self-loops nor parallel edges (i.e., G is a simple graph), both A(G) and X(G) contain the entire information. Thus it is natural to expect that either matrix can be obtained directly from the other, in the case of a simple graph. This relationship is given in Problem 7-23.

MODULE 5
END