

## MODULE 2

### APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

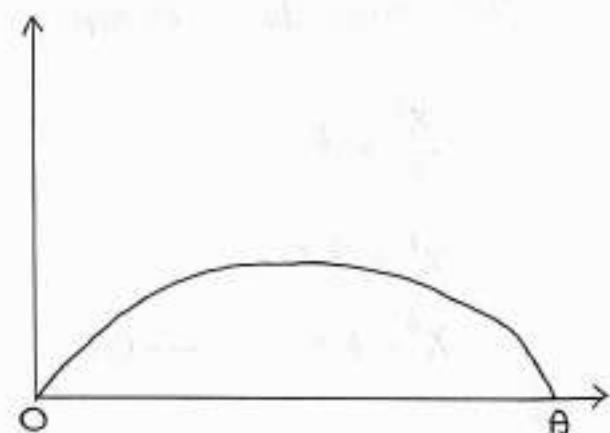
(Text 1 : Relevant portions of sections 18.3, 18.4, 18.5)

One dimensional wave equation - Vibrations of a stretched string derivation - Solution of the wave equation using method of separation of variables - D'Alembert's solution of the wave equation, One dimensional heat equation, derivation solution of the heat equation.

#### One Dimensional wave equation

Consider a uniform elastic string stretched tightly to length  $L$  and fixed at the end points  $O$  and  $A$ . Let the string be released from rest and allowed to vibrate.

Take the origin at one fixed end of the string say  $O$ ,  $x$  axis along the length of the string and  $y$  axis perpendicular to it. Then the equation of motion or wave equation is



$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$$

#### Solution of one dimensional wave equation by the method of separation of variables.

Wave equation is  $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$  — ①

dependent variable -  $u$

independent variable -  $x, t$

Let the solution be  $u(x, t) = X(x) T(t)$  —— (2)

where  $X$  is a function of  $x$  only and  $T$  is a function of  $t$  only

$$\frac{\partial u}{\partial x} = X' T \quad \frac{\partial^2 u}{\partial x^2} = X'' T, \quad \frac{\partial u}{\partial t} = X T' \quad \frac{\partial^2 u}{\partial t^2} = X T''$$

Substituting these values in (1) we get

$$X'' T = \frac{1}{c^2} X T''$$

$$\frac{X''}{X} = \frac{1}{c^2} \frac{T''}{T} = k$$

This gives two ordinary differential equations

$$\frac{X''}{X} = k \quad \text{and} \quad \frac{1}{c^2} \frac{T''}{T} = k$$

$$X'' = kX \quad T'' = k c^2 T$$

$$X'' - kX = 0 \quad \text{--- (3)} \quad T'' - k c^2 T = 0 \quad \text{--- (4)}$$

Case 1 ( $k < 0$ ) Suppose  $k = -p^2$

$$(3) \Rightarrow X'' + p^2 X = 0$$

$$D^2 X + p^2 X = 0$$

$$(D^2 + p^2) X = 0$$

$$m^2 + p^2 = 0$$

$$m^2 = -p^2$$

$$m = \pm pi$$

$$(4) \Rightarrow T'' + p^2 c^2 T = 0$$

$$D^2 T + p^2 c^2 T = 0$$

$$(D^2 + p^2 c^2) T = 0$$

$$m^2 + p^2 c^2 = 0$$

$$m^2 = -p^2 c^2$$

$$m = \pm cp$$

$$X = C_1 \cos px + C_2 \sin px$$

$$T = C_3 \cosh cpt + C_4 \sinh cpt$$

Case 2 ( $k = 0$ )

$$(3) \Rightarrow X'' = 0$$

$$(4) \Rightarrow T'' = 0$$

$$D^2x = 0$$

$$m^2 = 0$$

$$m = 0, 0$$

$$x = (c_5 + c_6x)e^{0x}$$

$$x = c_5 + c_6x$$

$$D^2T = 0$$

$$m^2 = 0$$

$$m = 0$$

$$T = (c_7 + c_8t)e^{0t}$$

$$T = c_7 + c_8t$$

case 3 ( $k > 0$ )

Suppose  $ic = p^2$

$$\textcircled{3} \Rightarrow x'' - p^2x = 0$$

$$\textcircled{4} \Rightarrow T'' - p^2c^2T = 0$$

$$D^2x - p^2x = 0$$

$$D^2T - p^2c^2T = 0$$

$$(D^2 - p^2)x = 0$$

$$(D^2 - p^2c^2)T = 0$$

$$m^2 - p^2 = 0$$

$$m^2 - p^2c^2 = 0$$

$$m^2 = p^2$$

$$m^2 = p^2c^2$$

$$m = \pm p$$

$$m = \pm cp$$

$$x = c_9 e^{px} + c_{10} e^{-px}$$

$$T = c_{11} e^{cpt} + c_{12} e^{-cpt}$$

Thus various possible solutions of the wave equation are

$$u(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$

$$u(x,t) = (c_5 + c_6x)(c_7 + c_8t)$$

$$u(x,t) = (c_9 e^{px} + c_{10} e^{-px})(c_{11} e^{cpt} + c_{12} e^{-cpt})$$

Now out of these three solutions we have to choose that solution which is consistent with the physical nature of the problem. As we are discussing the problem of vibrations, the displacement  $u(x,t)$  must contain periodic functions. Hence  $u(x,t)$  must contain trigonometric terms.

$\therefore u(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$  is the only suitable solution of the wave equation and it corresponds to  $k = -p^2$ .

Q Find the displacement of a finite string of length 'L' that is fixed at both ends and is released from rest with an initial displacement  $f(x)$ .

OR

Find one dimensional wave equation such that initial velocity zero and initial displacement  $f(x)$ .

Sols: The displacement  $u(x,t)$  of the string is the solution of the one dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \quad \text{--- } ①$$

The boundary conditions are  $u(0,t) = 0 \quad \text{--- } ②$   
 $u(L,t) = 0 \quad \text{--- } ③$

The initial displacement is  $f(x)$  (given)  $\therefore u(x,0) = f(x) \quad \text{--- } ④$

Since the string is released from rest, initial velocity is zero

$$\text{i.e. } \frac{\partial u(x,0)}{\partial t} = 0 \quad \text{--- } ⑤$$

using method of separation of variables, the general soln of ① is

$$u(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos ct + c_4 \sin ct) \quad \text{--- } ⑥$$

Substituting ② in ⑥

$$u(0,t) = c_1 (c_3 \cos ct + c_4 \sin ct)$$

$$0 = c_1 (c_3 \cos ct + c_4 \sin ct)$$

$$\therefore c_1 = 0$$

$$\therefore ⑥ \Rightarrow u(x,t) = c_2 \sin px (c_3 \cos ct + c_4 \sin ct) \quad \text{--- } ⑦$$

Substituting ③ in ⑦

$$u(l,t) = c_2 \sin pl (c_3 \cos ct + c_4 \sin ct)$$

$$0 = c_2 \sin pl (c_3 \cos ct + c_4 \sin ct)$$

$$\Rightarrow \sin pl = 0 \quad [\because c_2 \neq 0]$$

$$\sin pl = \sin n\pi \quad n=1, 2, 3, \dots$$

$$pl = n\pi$$

$$p = \frac{n\pi}{l}$$

$$\therefore ⑦ \Rightarrow u(x,t) = c_2 \sin \frac{n\pi x}{l} (c_3 \cos \frac{n\pi t}{l} + c_4 \sin \frac{n\pi t}{l})$$

$$= \sin \frac{n\pi x}{l} (c_2 c_3 \cos \frac{n\pi t}{l} + c_2 c_4 \sin \frac{n\pi t}{l})$$

$$u(x,t) = \left( a_n \cos \frac{n\pi t}{l} + b_n \sin \frac{n\pi t}{l} \right) \sin \frac{n\pi x}{l}$$

Adding up the solutions for different values of  $n$  we get

$$u(x,t) = \sum_{n=1}^{\infty} \left[ a_n \cos \frac{n\pi t}{l} + b_n \sin \frac{n\pi t}{l} \right] \sin \frac{n\pi x}{l} \quad ⑧$$

Substituting ④ in ⑧

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l}$$

which represent half range Fourier sine series for  $f(x)$

$$\text{where } a_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

In order to apply condition ⑤ we differentiate ⑧ partially w.r.t  $t$ .

$$\frac{\partial u(x,t)}{\partial t} = \sum_{n=1}^{\infty} \left[ -a_n \sin \frac{n\pi ct}{l} + b_n \cos \frac{n\pi ct}{l} \right] \frac{n\pi c}{l} \sin \frac{n\pi x}{l}$$

$$\frac{\partial u(x,0)}{\partial t} = \sum_{n=1}^{\infty} (b_n) \frac{n\pi c}{l} \sin \frac{n\pi x}{l}$$

$$0 = \sum_{n=1}^{\infty} (b_n) \frac{n\pi c}{l} \sin \frac{n\pi x}{l}$$

$$\Rightarrow b_n = 0$$

Hence the required solution is

$$⑧ \Rightarrow u(x,t) = \sum_{n=1}^{\infty} a_n \cos \left( \frac{n\pi ct}{l} \right) \sin \left( \frac{n\pi x}{l} \right) \text{ where}$$

$$a_n = \underline{\underline{\frac{2}{l} \int_0^l f(x) \sin \left( \frac{n\pi x}{l} \right) dx}}$$

- Q2) A lightly stretched string of length  $l$  is fixed at both ends. Find the displacement  $u(x,t)$  if the string is given an initial displacement  $f(x)$  and an initial velocity  $g(x)$ .  
OR

Finding one dimensional wave equation such that initial displacement  $f(x)$  and initial velocity  $g(x)$ .

Sols: The displacement  $u(x,t)$  of the string is the solution of the  
one dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \quad ①$$

The boundary conditions are  $u(0,t) = 0 \quad ②$   
 $u(l,t) = 0 \quad ③$

The initial displacement is  $f(x)$  [given]  $\therefore u(x,0) = f(x) \quad \text{--- (4)}$

The initial velocity is  $g(x)$  [given]  $\therefore \frac{\partial u(x,0)}{\partial t} = g(x) \quad \text{--- (5)}$

using method of separation of variables the general solution of (1) is  $u(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos ct + c_4 \sin ct) \quad \text{--- (6)}$

substituting (2) in (6)

$$u(0,t) = c_1(c_3 \cos ct + c_4 \sin ct)$$

$$0 = c_1(c_3 \cos ct + c_4 \sin ct)$$

$$\therefore c_1 = 0$$

$$\therefore (6) \Rightarrow u(x,t) = c_2 \sin px (c_3 \cos ct + c_4 \sin ct) \quad \text{--- (7)}$$

substituting (3) in (7)

$$u(t,t) = c_2 \sin pt (c_3 \cos ct + c_4 \sin ct)$$

$$0 = c_2 \sin pt (c_3 \cos ct + c_4 \sin ct)$$

$$\Rightarrow \sin pt = 0$$

$$\sin pt = \sin n\pi \quad n=1, 2, 3 \dots$$

$$pt = n\pi$$

$$p = \frac{n\pi}{t}$$

$$\therefore (7) \Rightarrow u(x,t) = c_2 \sin \frac{n\pi x}{l} \left[ c_3 \cos \frac{n\pi t}{l} + c_4 \sin \frac{n\pi t}{l} \right]$$

$$= \sin \frac{n\pi x}{l} \left[ a_n \cos \left( \frac{n\pi ct}{l} \right) + b_n \sin \left( \frac{n\pi ct}{l} \right) \right]$$

$$u(x,t) = \left[ a_n \cos \left( \frac{n\pi ct}{l} \right) + b_n \sin \left( \frac{n\pi ct}{l} \right) \right] \sin \frac{n\pi x}{l}$$

~~initial~~ Adding up the solutions for different values of  $n$

$$u(x,t) = \sum_{n=1}^{\infty} \left[ a_n \cos \left( \frac{n\pi ct}{l} \right) + b_n \sin \left( \frac{n\pi ct}{l} \right) \right] \sin \frac{n\pi x}{l} \quad \text{--- (8)}$$

Substituting ④ in ⑧

$$u(x, t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right)$$

$$\therefore f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right)$$

which represent half range Fourier sine series for  $f(x)$

where  $a_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$

In order to apply condition ⑤ we differentiate ⑧ partially w.r.t  $t$ .

$$\frac{\partial u(x, t)}{\partial t} = \sum_{n=1}^{\infty} \left[ -a_n \sin\left(\frac{n\pi x}{l}\right) + b_n \cos\left(\frac{n\pi x}{l}\right) \right] \frac{n\pi c}{l} \sin\left(\frac{n\pi x}{l}\right)$$

$$\frac{\partial u(x, 0)}{\partial t} = \sum_{n=1}^{\infty} b_n \frac{n\pi c}{l} \sin\left(\frac{n\pi x}{l}\right)$$

$$g(x) = \sum_{n=1}^{\infty} b_n \frac{n\pi c}{l} \sin\left(\frac{n\pi x}{l}\right)$$

which is a half range sine series where

$$\frac{n\pi c}{l} b_n = \frac{2}{l} \int_0^l g(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$\therefore b_n = \frac{2}{n\pi c} \int_0^l g(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

Hence the required solution is,

$$u(x, t) = \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right) \right] \sin\left(\frac{n\pi c}{l} t\right)$$

where  $a_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$

$$b_n = \frac{2}{n\pi c} \int_0^l g(x) \sin\left(\frac{n\pi x}{l}\right) dx //$$

11) Q3) write the most general form of the solution of a string equation, if the string of length  $\lambda$  is fixed at both ends and is subjected to zero initial displacement and non-zero initial velocity.

Ans:  $u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{\lambda}\right) \sin\left(\frac{n\pi x}{\lambda}\right)$  where

$$b_n = \frac{2}{n\pi c} \int_0^\lambda g(x) \sin\left(\frac{n\pi x}{\lambda}\right) dx$$

### Note

In general the displacement function,

$$u(x,t) = \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{n\pi t}{\lambda}\right) + b_n \sin\left(\frac{n\pi t}{\lambda}\right) \right] \sin\left(\frac{n\pi x}{\lambda}\right)$$

where  $a_n = \frac{2}{\lambda} \int_0^\lambda f(x) \sin\left(\frac{n\pi x}{\lambda}\right) dx$ ,  $f(x)$  - initial displacement

$$b_n = \frac{2}{n\pi c} \int_0^\lambda g(x) \sin\left(\frac{n\pi x}{\lambda}\right) dx, \quad g(x) \text{ initial velocity}$$

If initial displacement is equal to zero ie)  $f(x)=0 \Rightarrow a_n=0$

If initial velocity is equal to zero ie)  $g(x)=0 \Rightarrow b_n=0$

Q4) A uniform elastic string of length 60cm is subjected to a constant tension of 2kg. If the ends are fixed and the initial displacement  $y(x,0) = 60x - x^2$  for  $0 < x < 60$  while the initial velocity is zero, find the displacement  $u(x,t)$ .

Ans: Displacement function  $u(x,t)$  is the solution of the one

dimensional wave equation  $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \quad \text{--- (1)}$

The Boundary conditions are  $u(0, t) = 0 \quad \text{--- (2)}$

$$u(60, t) = 0 \quad \text{--- (3)}$$

The initial conditions are  $u(x, 0) = f(x) = 60x - x^2 \quad \text{--- (4)}$

$$\frac{\partial u(x, 0)}{\partial t} = g(x) = 0 \quad \text{--- (5)}$$

using method of separation of variables the general solution of (1) is

$$u(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cosh pt + c_4 \sinh pt) \quad \text{--- (6)}$$

Applying  $u(0, t) = 0$  we get  $c_1 = 0$

Applying  $u(60, t) = 0$  we get  $p = \frac{n\pi}{60}$

The general solution satisfying boundary conditions is

$$u(x, t) = \sum_{n=1}^{\infty} [a_n \cos(\frac{n\pi t}{60}) + b_n \sin(\frac{n\pi t}{60})] \sin \frac{n\pi x}{60} \quad \text{--- (7)}$$

where  $a_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$

$$= \frac{2}{60} \int_0^{60} (60x - x^2) \sin \frac{n\pi x}{60} dx$$

$$= \frac{1}{30} \left[ (60x - x^2) \left( -\frac{\cos \frac{n\pi x}{60}}{\frac{n\pi}{60}} \right) - (60 - 2x) \left( -\frac{\sin \frac{n\pi x}{60}}{\frac{n^2\pi^2}{60^2}} \right) \right]$$

$$+ (-2) \left( \frac{\cos \frac{n\pi x}{60}}{\frac{n^3\pi^3}{60^3}} \right) \Big|_{x=0}^{60}$$

$$\begin{aligned}
 &= \frac{1}{30} \left[ -(60x - x^2) \left( \frac{\cos \frac{n\pi x}{60}}{\frac{n\pi}{60}} \right) + (60 - 2x) \left( \frac{+\sin \frac{n\pi x}{60}}{\frac{n^2\pi^2}{60^2}} \right) - 2 \frac{\cos \frac{n\pi x}{60}}{\frac{n^3\pi^3}{60^3}} \right]_0^{60} \\
 &= \frac{1}{30} \left\{ \left[ 0 + 0 - \frac{2 \cos n\pi}{\frac{n^3\pi^3}{60^3}} \right] - \left[ 0 + 0 - \frac{2}{\frac{n^3\pi^3}{60^3}} \right] \right\} \\
 &= \frac{1}{30} \left[ \frac{2}{\frac{n^3\pi^3}{60^3}} - \frac{2 \cos n\pi}{\frac{n^3\pi^3}{60^3}} \right] \\
 &= \frac{1}{30} \times \frac{2}{\frac{n^3\pi^3}{60^3}} \times 60^3 [1 - \cos n\pi]
 \end{aligned}$$

$$\begin{cases} \cos 2n\pi = 1 \\ \sin 2n\pi = 0 \\ \cos n\pi = (-1)^n \\ \sin n\pi = 0 \end{cases}$$

$$b_n = \frac{2}{n\pi c} \int_0^l g(x) \sin\left(\frac{n\pi x}{l}\right) dx = 0 \quad \therefore g(x) = 0 \text{ given } \quad (5)$$

Substituting the values of  $a_n$  and  $b_n$  in (7)

$$u(x, t) = \sum_{n=1}^{\infty} \left[ \frac{14400}{n^3\pi^3} [1 - \cos n\pi] \cos\left(\frac{n\pi ct}{60}\right) \right] \sin \frac{n\pi x}{60}$$

- (5) A slightly stretched homogeneous string of length  $L$  with its fixed ends at  $x=0$  and  $x=L$  executes transverse vibrations. Motion starts with zero initial velocity by displacing the string in to the form  $f(x) = k(x^2 - x^3)$ . Find the deflection  $u(x, t)$  at any time  $t$ .

Ans: The deflection  $u(x, t)$  is the solution of the one dimensional wave equation  $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \quad \text{--- (1)}$

The Boundary conditions are  $u(0,t) = 0 \quad \text{--- (2)}$   
 $u(l,t) = 0 \quad \text{--- (3)}$

The initial conditions are  $u(x,0) = f(x) = k(x^2 - x^3) \quad \text{--- (4)}$

$$\frac{\partial u(x,0)}{\partial t} = g(x) = 0 \quad \text{--- (5)}$$

using method of separation of variables the general solution of (1) is

$$u(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cosh pt + c_4 \sinh pt) \quad \text{--- (6)}$$

Applying  $u(0,t) = 0$  we get  $c_1 = 0$

Applying  $u(l,t) = 0$  we get  $p = \frac{n\pi}{l}$

The general solution satisfying Boundary Conditions is

$$u(x,t) = \sum_{n=1}^{\infty} [a_n \cos\left(\frac{n\pi t}{l}\right) + b_n \sin\left(\frac{n\pi t}{l}\right)] \sin\left(\frac{n\pi x}{l}\right) \quad \text{--- (7)}$$

$$\text{where } a_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{l} \int_0^l k(x^2 - x^3) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2k}{l} \int_0^l (x^2 - x^3) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2k}{l} \left[ \left( x^2 - x^3 \right) \left( -\frac{\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) - (2x - 3x^2) \left( -\frac{\sin \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}} \right) \right]$$

$$+ (2 - 6x) \left( \frac{\cos \frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}} \right) - (-6) \left( \frac{\sin \frac{n\pi x}{l}}{\frac{n^4\pi^4}{l^4}} \right) \Big|_0^l$$

$$\begin{aligned}
&= \frac{2k}{\lambda} \left[ -(x^2 - x^3) \left( \frac{\cos \frac{n\pi x}{\lambda}}{\frac{n\pi}{\lambda}} \right) + (2x - 3x^2) \left( \frac{\sin \frac{n\pi x}{\lambda}}{\frac{n^2\pi^2}{\lambda^2}} \right) + (2-6x) \left( \frac{\cos \frac{n\pi x}{\lambda}}{\frac{n^3\pi^3}{\lambda^3}} \right) \right. \\
&\quad \left. + 6 \left( \frac{\sin \frac{n\pi x}{\lambda}}{\frac{n^4\pi^4}{\lambda^4}} \right) \right]_0^\lambda \\
&= \frac{2k}{\lambda} \left\{ \left[ -(x^2 - x^3) \frac{\cos n\pi}{\frac{n\pi}{\lambda}} + (2-6x) \frac{\cos n\pi}{\frac{n^3\pi^3}{\lambda^3}} \right] - \left[ 0 + 0 + \frac{2}{\frac{n^3\pi^3}{\lambda^3}} + 0 \right] \right\} \\
&= \frac{2k}{\lambda} \left[ -(x^2 - x^3) \frac{(-1)^0}{\frac{n\pi}{\lambda}} + (2-6x) \frac{(-1)^0}{\frac{n^3\pi^3}{\lambda^3}} - \frac{2}{\frac{n^3\pi^3}{\lambda^3}} \right] \\
&= \frac{2k}{\lambda} \left[ -x(x^2 - x^3) \frac{(-1)^0}{n\pi} + (2-6x)x^2 \frac{(-1)^0}{n^3\pi^3} - \frac{2x^2}{n^3\pi^3} \right] \\
&= 2k \left[ -(x^2 - x^3) \frac{(-1)^0}{n\pi} + (2-6x)x^2 \frac{(-1)^0}{n^3\pi^3} - \frac{2x^2}{n^3\pi^3} \right]
\end{aligned}$$

Required solution is

$$b_n = \frac{2}{mc} \int_0^\lambda g(x) \sin \left( \frac{n\pi x}{\lambda} \right) dx = 0 \quad \therefore g(x) = 0$$

Required solution is

$$u(x, t) = \sum_{n=1}^{\infty} a_n \cos \left( \frac{n\pi t}{\lambda} \right) \sin \left( \frac{n\pi x}{\lambda} \right)$$

$$\text{where } a_n = 2k \left[ -(x^2 - x^3) \frac{(-1)^0}{n\pi} + (2-6x)x^2 \frac{(-1)^0}{n^3\pi^3} - \frac{2x^2}{n^3\pi^3} \right]$$

- 6) A lightly stretched string with fixed end points  $x=0$  and  $x=\lambda$  is initially in a position given by

$$u(x, 0) = \begin{cases} x & \text{in } 0 < x < \frac{\lambda}{2} \\ \lambda - x & \text{in } \frac{\lambda}{2} < x < \lambda \end{cases}$$

and each of its points is given by the velocity

$(\frac{\partial u}{\partial t})_{t=0} = x(x-1)$  or  $x \cdot 1$ . Determine the displacement  $u(x,t)$  at any point of the string.

Ans: The displacement function  $u(x,t)$  is the solution of the one dimensional wave equation  $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$  — ①

The boundary conditions are  $u(0,t) = 0$  — ②

$u(l,t) = 0$  — ③

The initial conditions are  $u(x,0) = f(x) = \begin{cases} x & \text{in } 0 < x < \frac{l}{2} \\ l-x & \text{in } \frac{l}{2} < x < l \end{cases}$  — ④

$\frac{\partial u(x,0)}{\partial t} = g(x) = x(x-1) \quad 0 < x < l$  — ⑤

using method of separation of variables the general solution of ① is  $u(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pt + c_4 \sin pt)$  — ⑥

Applying  $u(0,t) = 0$  we get  $c_1 = 0$

Applying  $u(l,t) = 0$  we get  $p = \frac{n\pi}{l}$

General solution satisfying boundary conditions is

$$u(x,t) = \sum_{n=1}^{\infty} [a_n \cos(\frac{n\pi xt}{l}) + b_n \sin(\frac{n\pi xt}{l})] \sin \frac{n\pi x}{l} — ⑦$$

$$\text{where } a_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{l} \left[ \int_0^{\frac{l}{2}} x \sin\left(\frac{n\pi x}{l}\right) dx + \int_{\frac{l}{2}}^l (l-x) \sin\left(\frac{n\pi x}{l}\right) dx \right]$$

$$= \frac{2}{l} \left\{ \left[ (x) \left( -\frac{\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) - (1) \left( -\frac{\sin \frac{n\pi x}{l}}{\frac{n\pi^2}{l^2}} \right) \right] \Big|_{0}^{\frac{l}{2}} + \left[ (l-x) \left( -\frac{\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) - (-1) \left( -\frac{\sin \frac{n\pi x}{l}}{\frac{n\pi^2}{l^2}} \right) \right] \Big|_{\frac{l}{2}}^l \right\}$$

$$\begin{aligned}
&= \frac{2}{l} \left\{ \left[ -\frac{x \cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} + \frac{\sin \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}} \right]_0^l + \left[ -(1-x) \frac{\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} - \frac{\sin \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}} \right]_l^{l/2} \right\} \\
&= \frac{2}{l} \left\{ \left[ -\frac{l}{2} \frac{\cos \frac{n\pi}{2}}{\frac{n\pi}{l}} + \frac{\sin \frac{n\pi}{2}}{\frac{n^2\pi^2}{l^2}} \right] - [0+0] + [0-0] - \left[ -\frac{l}{2} \frac{\cos \frac{n\pi}{2}}{\frac{n\pi}{l}} - \frac{\sin \frac{n\pi}{2}}{\frac{n^2\pi^2}{l^2}} \right] \right\} \\
&= \frac{2}{l} \left[ 2 \frac{\sin \frac{n\pi}{2}}{\frac{n^2\pi^2}{l^2}} \right] \\
&= \frac{4l}{n^2\pi^2} \sin \frac{n\pi}{2}
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{2}{n\pi c} \int_0^l g(x) \sin \frac{n\pi x}{l} dx \\
&= \frac{2}{n\pi c} \int_0^l (x^2 - xl) \sin \frac{n\pi x}{l} dx \\
&= \frac{2}{n\pi c} \left[ (x^2 - xl) \left( -\frac{\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) - (2x - l) \left( -\frac{\sin \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}} \right) + (2) \frac{\cos \frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}} \right]_0^l \\
&= \frac{2}{n\pi c} \left\{ [0 - 0 + \frac{2\cos n\pi}{n^3\pi^3}] - \left[ \frac{2}{n^3\pi^3} \right] \right\} \\
&= \frac{2l^3}{n\pi c \times n^3\pi^3} [2\cos n\pi - 2] \\
&= \frac{4l^3}{n^4\pi^4 c} [(-1)^n - 1]
\end{aligned}$$

∴ The required solution is

$$u(x,t) = \sum_{n=1}^{\infty} \left[ \frac{4L}{n^2\pi^2} \sin \frac{n\pi}{2} \cos \frac{n\pi ct}{L} + \frac{4L^3}{n^4\pi^4 c} [(-1)^n - 1] \sin \frac{n\pi ct}{L} \right] \sin \frac{n\pi x}{L}$$

Q7) If a string of length  $4L$  is initially at rest in its equilibrium position and each of its points is given initial velocity  $v$  where

$$v = \begin{cases} \frac{kx}{L} & \text{in } 0 < x < 2L \\ \frac{k(4L-x)}{L} & \text{in } 2L < x < 4L \end{cases}$$

Find the displacement of the string at any time.

Ans: The displacement function  $u(x,t)$  is the solution of the one dimensional wave equation  $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$  — ①

Boundary conditions are  $u(0,t) = 0$  — ②

$$u(4L,t) = 0$$
 — ③

The initial conditions are  $u(x,0) = f(x) = 0$  — ④

$$\frac{\partial}{\partial t} u(x,0) = g(x) = \begin{cases} \frac{kx}{L} & \text{in } 0 < x < 2L \\ \frac{k(4L-x)}{L} & \text{in } 2L < x < 4L \end{cases}$$
 — ⑤

using method of separation of variables the general solution is

$$u(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pt + c_4 \sin pt)$$
 — ⑥

Applying  $u(0,t) = 0$  we get  $c_1 = 0$

Applying  $u(L,t) = 0$  we get  $p = \frac{n\pi}{4L}$

General solution satisfying boundary condition is

$$u(x,t) = \sum_{n=1}^{\infty} \left[ a_n \cos \frac{n\pi xt}{4L} + b_n \sin \frac{n\pi xt}{4L} \right] \sin \frac{n\pi x}{4L}$$
 — ⑦

$$\text{where } a_n = \frac{2}{4L} \int_0^{4L} f(x) \sin \frac{n\pi x}{4L} dx = 0 \quad \because f(x)=0$$

$$\begin{aligned}
 b_n &= \frac{2}{n\pi c} \int_0^{4L} g(x) \sin \frac{n\pi x}{4L} dx \\
 &= \frac{2}{n\pi c} \left[ \int_0^{2L} \frac{kx}{L} \sin \frac{n\pi x}{4L} dx + \int_{2L}^{4L} \frac{k(4L-x)}{L} \sin \frac{n\pi x}{4L} dx \right] \\
 &= \frac{2k}{n\pi c L} \left[ \int_0^{2L} x \sin \frac{n\pi x}{4L} dx + \int_{2L}^{4L} (4L-x) \sin \frac{n\pi x}{4L} dx \right] \\
 &= \frac{2k}{n\pi c L} \left\{ \left[ (0) \left( -\frac{\cos \frac{n\pi x}{4L}}{\frac{n\pi}{4L}} \right) - (0) \left( -\frac{\sin \frac{n\pi x}{4L}}{\frac{n^2\pi^2}{16L^2}} \right) \right]_0^{2L} + \right. \\
 &\quad \left. (4L-x) \left( -\frac{\cos \frac{n\pi x}{4L}}{\frac{n\pi}{4L}} \right) - (-1) \left( -\frac{\sin \frac{n\pi x}{4L}}{\frac{n^2\pi^2}{16L^2}} \right) \right]_{2L}^{4L} \\
 &= \frac{2k}{n\pi c L} \left\{ \left[ -2L \frac{\cos \frac{n\pi}{2}}{\frac{n\pi}{4L}} + \frac{\sin \frac{n\pi}{2}}{\frac{n^2\pi^2}{16L^2}} \right] - [0+0] + [0-0] \right. \\
 &\quad \left. - \left[ -2L \frac{\cos \frac{n\pi}{2}}{\frac{n\pi}{4L}} - \frac{\sin \frac{n\pi}{2}}{\frac{n^2\pi^2}{16L^2}} \right] \right\} \\
 &= \frac{2k}{n\pi c L} \left[ \frac{2 \sin \frac{n\pi}{2}}{\frac{n^2\pi^2}{16L^2}} \right]
 \end{aligned}$$

$$= \frac{64 k L}{n^3 \pi^3 c} \sin \frac{n\pi}{2}$$

$$\therefore \text{Soln is } u(x,t) = \sum_{n=1}^{\infty} \left[ \frac{64 k L}{n^3 \pi^3 c} \sin \left( \frac{n\pi}{2} \right) \sin \left( \frac{n\pi t}{4L} \right) \right] \sin \left( \frac{n\pi x}{4L} \right)$$

8) Find the solution of the vibrating string of unit length having wave velocity  $c=1$ , The end points of the string are fixed. The initial velocity is zero and the initial deflection is given by

$$u(x, 0) = \begin{cases} 1 & 0 \leq x \leq \frac{1}{2} \\ -1 & \frac{1}{2} \leq x \leq 1 \end{cases}$$

Ans: The deflection  $u(x, t)$  is the solution of the one dimensional wave equation  $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \quad \text{--- (1)} \quad [\text{given } c=1]$

Boundary conditions are  $u(0, t) = 0 \quad \text{--- (2)}$

$$u(1, t) = 0 \quad \text{--- (3)}$$

Initial conditions are  $u(x, 0) = f(x) = \begin{cases} 1 & 0 \leq x \leq \frac{1}{2} \\ -1 & \frac{1}{2} \leq x \leq 1 \end{cases} \quad \text{--- (4)}$

$$\frac{\partial u(x, 0)}{\partial t} = g(x) = 0 \quad \text{--- (5)}$$

By the method of separation of variables solution of (1) is

$$u(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pt + c_4 \sin pt) \quad \text{--- (6)}$$

Applying  $u(0, t) = 0$  we get  $c_1 = 0$

Applying  $u(1, t) = 0$  we get  $p = n\pi$

$\therefore$  Solution satisfying boundary condition is

$$u(x, t) = \sum_{n=1}^{\infty} [a_n \cos n\pi t + b_n \sin n\pi t] \sin n\pi x \quad \text{--- (7)}$$

$$\text{where } a_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= 2 \int_0^1 f(x) \sin(n\pi x) dx$$

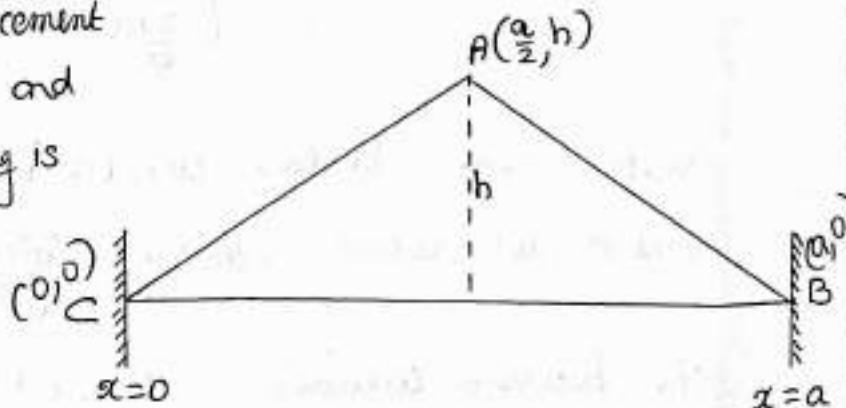
$$\begin{aligned}
 &= 2 \left[ \int_0^{\frac{1}{2}} \sin n\pi x dx + \int_{\frac{1}{2}}^1 -\sin n\pi x dx \right] \\
 &= 2 \left\{ \left[ -\frac{\cos n\pi x}{n\pi} \right]_0^{\frac{1}{2}} + \left[ +\frac{\cos n\pi x}{n\pi} \right]_{\frac{1}{2}}^1 \right\} \\
 &= 2 \left\{ \left[ -\frac{\cos \frac{n\pi}{2}}{n\pi} - \frac{-1}{n\pi} \right] + \left[ \frac{\cos n\pi}{n\pi} - \frac{\cos \frac{n\pi}{2}}{n\pi} \right] \right\} \\
 &= \frac{2}{n\pi} \left[ 1 + \cos n\pi - 2 \cos \frac{n\pi}{2} \right]
 \end{aligned}$$

$$b_n = \frac{2}{n\pi c} \int_0^c g(x) \sin \left( \frac{n\pi x}{c} \right) dx = 0 \quad \therefore g(x) = 0$$

$\therefore$  So  $g(x)$  is

$$u(x,t) = \sum_{n=1}^{\infty} \left[ \frac{2}{n\pi} \left( 1 + \cos n\pi - 2 \cos \frac{n\pi}{2} \right) \cos n\pi t \right] \underline{\sin(n\pi x)}$$

- q) A tightly - stretched violin string of lengths  $a$  and fixed at both ends is plucked at its midpoint and assume initially the shape of a triangle of height  $h$ . as shown in figure - Find the displacement  $u(x,t)$  at any distance  $x$  and any time  $t$  after the string is released from rest.



Solution: we first derive the equation of the line segment CA with end points  $C(0,0)$  and  $A(\frac{a}{2}, h)$

$$\frac{x-0}{\frac{a}{2}-0} = \frac{y-0}{h-0}$$

$$\frac{y}{\frac{a}{2}} = \frac{y}{b}$$

$$\frac{2xh}{a} = y$$

$$y = \frac{2xb}{a} \quad 0 \leq x \leq \frac{a}{2}$$

equation of AB  $\Rightarrow \frac{x - \frac{a}{2}}{\frac{a}{2} - \frac{a}{2}} = \frac{y - b}{-b}$

$$\frac{2x-a}{a} = \frac{y-b}{-b}$$

$$-b(2x-a) = a(y-b)$$

$$-2xh + ah + ah = ay$$

$$ay = 2ah - 2xb$$

$$y = \frac{2h(a-x)}{a} \quad \frac{a}{2} \leq x \leq a$$

$\therefore$  The initial displacement  $u(x, 0)$  is given by

$$u(x, 0) = f(x) = \begin{cases} \frac{2hx}{a} & 0 \leq x \leq \frac{a}{2} \\ \frac{2h(a-x)}{a} & \frac{a}{2} \leq x \leq a \end{cases}$$

displacement function  $u(x, t)$  is the solution of the one dimensional wave equation  $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \quad \text{--- (1)}$

The boundary conditions  $u(0, t) = 0 \quad \text{--- (2)}$

$u(a, t) = 0 \quad \text{--- (3)}$

The initial conditions

$$u(x, 0) = f(x) = \begin{cases} \frac{2hx}{a} & 0 \leq x \leq \frac{a}{2} \\ \frac{2h(a-x)}{a} & \frac{a}{2} \leq x \leq a \end{cases} \quad \text{--- (4)}$$

$$\frac{\partial u(x, 0)}{\partial t} = g(x) = 0 \quad \text{--- (5)}$$

By the method of separation of variables solution of the wave equation

$$u(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cosec pt + c_4 \sin cp t) \quad (6)$$

Applying  $u(0,t) = 0$  we get  $c_1 = 0$

Applying  $u(a,t) = 0$  we get  $p = \frac{n\pi}{a}$

∴ Solution satisfying boundary conditions is

$$u(x,t) = \sum_{n=1}^{\infty} [a_n \cos \frac{n\pi t}{a} + b_n \sin \frac{n\pi t}{a}] \sin \frac{n\pi x}{a} \quad (7)$$

where

$$a_n = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a} dx$$

$$= \frac{2}{a} \left[ \int_0^{a/2} \frac{2bx}{a} \sin \frac{n\pi x}{a} dx + \int_{a/2}^a \frac{2b(a-x)}{a} \sin \frac{n\pi x}{a} dx \right]$$

$$= \frac{4b}{a^2} \left[ \int_0^{a/2} x \sin \frac{n\pi x}{a} dx + \int_{a/2}^a (a-x) \sin \left( \frac{n\pi x}{a} \right) dx \right]$$

$$= \frac{4b}{a^2} \left\{ \left[ (x) \left( -\frac{\cos \frac{n\pi x}{a}}{\frac{n\pi}{a}} \right) - (-1) \left( -\frac{\sin \frac{n\pi x}{a}}{\frac{n^2\pi^2}{a^2}} \right) \right] \Big|_0^{a/2} + \right.$$

$$\left. \left[ (a-x) \left( -\frac{\cos \frac{n\pi x}{a}}{\frac{n\pi}{a}} \right) - (-1) \left( -\frac{\sin \frac{n\pi x}{a}}{\frac{n^2\pi^2}{a^2}} \right) \right] \Big|_{a/2}^a \right]$$

$$= \frac{4b}{a^2} \left\{ -\frac{a}{2} \frac{\cos \frac{n\pi}{2}}{\frac{n\pi}{a}} + \frac{\sin \frac{n\pi}{2}}{\frac{n^2\pi^2}{a^2}} \right\} - \left[ 0+0 \right] + \left[ 0-0 \right]$$

$$- \left[ -\frac{a}{2} \frac{\cos \frac{n\pi}{2}}{\frac{n\pi}{a}} - \frac{\sin \frac{n\pi}{2}}{\frac{n^2\pi^2}{a^2}} \right] \}$$

$$= \frac{4b}{a^2} \left[ \frac{2 \sin \frac{n\pi}{2}}{\frac{n^2\pi^2}{a^2}} \right]$$

$$= \frac{8b}{n^2\pi^2} \sin \frac{n\pi}{2}$$

$$b_n = \frac{2}{n\pi c} \int_0^a g(x) \sin \frac{n\pi x}{a} dx = 0 \quad \therefore g(x) = 0$$

∴ required solution is

$$u(x,t) = \sum_{n=1}^{\infty} \left[ \frac{8b}{n^2\pi^2} \sin \frac{n\pi}{2} \cos \frac{n\pi ct}{a} \right] \sin \frac{n\pi x}{a}$$

- 10) A string is stretched and fastened to 600 points  $L$  apart. Motion is started by displacing the string in to the form of the curve  $u = a \sin(\frac{\pi x}{L})$  from which it is released at time  $t=0$ . Show that the displacement at any point at a distance  $x$  from one end at time  $t$  is given by  $u = a \sin\left(\frac{\pi x}{L}\right) \cos\left(\frac{\pi ct}{L}\right)$ .

Ans: given  $u(x,0) = f(x) = a \sin\left(\frac{\pi x}{L}\right) \quad \text{--- (4)}$

$$\frac{\partial u(x,0)}{\partial t} = g(x) = 0 \quad \text{--- (5)}$$

Solution satisfying the boundary conditions is

$$u(x,t) = \sum_{n=1}^{\infty} [a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right)] \sin\left(\frac{n\pi x}{L}\right) \quad \text{--- (6)}$$

where  $a_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$

$$= \frac{2}{\lambda} \int_0^l a \sin\left(\frac{\pi x}{\lambda}\right) \sin\left(\frac{n\pi x}{\lambda}\right) dx$$

$$a_n = \frac{2a}{\lambda} \int_0^l \sin\left(\frac{\pi x}{\lambda}\right) \sin\left(\frac{n\pi x}{\lambda}\right) dx$$

$\int_0^l \sin\left(\frac{n\pi x}{\lambda}\right) \sin\left(\frac{m\pi x}{\lambda}\right) dx = \frac{1}{2}$ if $m=0$
$= 0$ if $m \neq 0$

$$\therefore a_1 = \frac{2a}{\lambda} \int_0^l \sin\left(\frac{\pi x}{\lambda}\right) \sin\left(\frac{\pi x}{\lambda}\right) dx = \frac{2a}{\lambda} \times \frac{1}{2} = \underline{\underline{a}}$$

$$a_2 = \frac{2a}{\lambda} \int_0^l \sin\left(\frac{\pi x}{\lambda}\right) \sin\left(\frac{2\pi x}{\lambda}\right) = \frac{2a}{\lambda} \times 0 = \underline{\underline{0}}$$

$$a_3, a_4, a_5 \dots = 0$$

$$b_0 = \frac{2}{n\pi c} \int_0^l g(x) \sin\left(\frac{n\pi x}{\lambda}\right) dx = 0 \quad \because g(x)=0$$

∴ required solution is

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{\lambda}\right) \sin\left(\frac{n\pi x}{\lambda}\right) \\ &= a_1 \cos\left(\frac{\pi t}{\lambda}\right) \sin\left(\frac{\pi x}{\lambda}\right) + 0 + 0 + \dots \\ &= a \cos\left(\frac{\pi t}{\lambda}\right) \underline{\underline{\sin\left(\frac{\pi x}{\lambda}\right)}} \end{aligned}$$

- II) If a string of length  $\lambda$  is initially at rest in the equilibrium position and each of its points is given the velocity  $(\frac{\partial u}{\partial t})_{t=0} = v_0 \sin^3\left(\frac{\pi x}{\lambda}\right)$  over determine the displacement  $u(x,t)$ .

Ans:

$$u(x, 0) = f(x) = 0 \quad \text{--- (4)}$$

$$\frac{\partial u(x, 0)}{\partial t} = g(x) = V_0 \sin^3 \frac{\pi x}{\lambda} \quad \text{--- (5)}$$

general solution satisfying the initial condition is

$$u(x, t) = \sum_{n=1}^{\infty} [a_n \cos(\frac{n\pi nt}{\lambda}) + b_n \sin(\frac{n\pi nt}{\lambda})] \sin(\frac{n\pi x}{\lambda}) \quad \text{--- (6)}$$

where  $a_n = \frac{2}{\lambda} \int_0^L f(x) \sin\left(\frac{n\pi x}{\lambda}\right) dx = 0 \quad \because f(x) = 0$

$$b_n = \frac{2}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi x}{\lambda}\right) dx$$

$$= \frac{2V_0}{n\pi c} \int_0^L \sin^3\left(\frac{\pi x}{\lambda}\right) \sin\left(\frac{n\pi x}{\lambda}\right) dx \quad \begin{cases} \sin 3\theta = 3\sin \theta - 4\sin^3 \theta \\ 4\sin^3 \theta = 3\sin \theta - \sin 3\theta \end{cases}$$

$$= \frac{2V_0}{n\pi c} \int_0^L \frac{1}{4} \left[ 3\sin \frac{\pi x}{\lambda} - \sin \frac{3\pi x}{\lambda} \right] \sin \frac{n\pi x}{\lambda} dx$$

$$= \frac{V_0}{2n\pi c} \left[ 3 \int_0^L \sin\left(\frac{\pi x}{\lambda}\right) \sin\left(\frac{n\pi x}{\lambda}\right) dx - \int_0^L \sin\left(\frac{3\pi x}{\lambda}\right) \sin\left(\frac{n\pi x}{\lambda}\right) dx \right]$$

$$b_1 = \frac{V_0}{2\pi c} \left[ 3 \cdot \frac{\lambda}{2} - 0 \right] = \frac{3V_0\lambda}{4\pi c}$$

$$b_2 = \frac{V_0}{4\pi c} \left[ 3 \times 0 - 0 \right] = 0$$

$$b_3 = \frac{V_0}{6\pi c} \left[ 3 \times 0 - \frac{\lambda}{2} \right] = -\frac{V_0\lambda}{12\pi c}$$

$$b_4 = b_5 = b_6 = \dots = 0$$

Substituting these values in ⑦

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \\ &= b_1 \sin\left(\frac{\pi t}{L}\right) \sin\left(\frac{\pi x}{L}\right) + b_3 \sin\left(\frac{3\pi t}{L}\right) \sin\left(\frac{3\pi x}{L}\right) \\ &= \frac{3V_0L}{4\pi c} \sin\left(\frac{\pi t}{L}\right) \sin\left(\frac{\pi x}{L}\right) - \frac{V_0L}{12\pi c} \sin\left(\frac{3\pi t}{L}\right) \sin\left(\frac{3\pi x}{L}\right) \\ &= \end{aligned}$$

## ONE DIMENSIONAL HEAT EQUATION

Consider the flow of heat conduction in a uniform bar. It is assumed that the sides of the bar are insulated and loss of heat from the sides by conduction or radiation is negligible. Take one end of the bar as origin and the direction of flow as the positive  $x$  axis. The temperature ' $u$ ' at any point of the bar depends on the distance  $x$  of the point from one end and the time  $t$ . One dimensional heat transfer equation or diffusion equation is

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{h} \frac{\partial u}{\partial t}$$

where  $h$  is called thermal diffusivity and is assumed to be a constant.

## Solution of one dimensional heat transfer equation

One dimensional heat transfer equation is

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{h} \frac{\partial u}{\partial t} \quad \text{--- ①}$$

Solution is  $u(x,t) = X(x)T(t)$  --- ②

Differentiating partially w.r.t  $x$  and  $t$

$$\frac{\partial u}{\partial x} = X' T$$

$$\frac{\partial^2 u}{\partial x^2} = X'' T$$

$$\frac{\partial u}{\partial t} = X T'$$

Substituting these values in ①

$$X'' T = \frac{1}{h} X T'$$

$$\frac{x''}{x} = \frac{1}{b} \frac{T'}{T} \rightarrow \textcircled{3}$$

LHS is a function of  $x$  only RHS is a function of  $t$  only.  
This can be true only if both sides equal to a constant  $K$

$$\therefore \frac{x''}{x} = \frac{1}{b} \frac{T'}{T} = k \rightarrow \textcircled{4}$$

This gives two ordinary differential equations

$$\frac{x''}{x} = k \quad \text{and} \quad \frac{1}{b} \frac{T'}{T} = k$$

$$\textcircled{5) } x'' - kx = 0 \quad \text{and} \quad \textcircled{6) } T' - kbT = 0$$

case 1 :  $k$  is negative ( $k = -p^2$ )

$$\textcircled{5) } \Rightarrow x'' + p^2x = 0 \quad \text{and} \quad \textcircled{6) } \Rightarrow T' + bp^2T = 0$$

$$D^2x + p^2x = 0 \quad DT + bp^2T = 0$$

$$(D^2 + p^2)x = 0 \quad (D + bp^2)T = 0$$

$$AE \Rightarrow m^2 + p^2 = 0 \quad AE \Rightarrow m + bp^2 = 0$$

$$m^2 = -p^2 \quad m = -bp^2$$

$$m = \pm pi$$

$$x = (c_1 \cos px + c_2 \sin px) \quad T = c_3 e^{-bp^2 t}$$

case 2 :  $k = 0$

$$\textcircled{5) } \Rightarrow x'' = 0$$

$$D^2x = 0$$

$$AE \Rightarrow m^2 = 0$$

$$m = 0, 0$$

$$x = (c_4 + c_5 x)$$

$$\textcircled{6) } \Rightarrow T' = 0$$

$$DT = 0$$

$$AE \Rightarrow m = 0$$

$$\therefore m = 0$$

$$T = c_6 e^{ob} = c_6$$

case 3:  $\kappa$  is positive ( $\kappa = p^2$ )

$$\textcircled{5} \Rightarrow x'' - p^2 x = 0$$

$$D^2 x - p^2 x = 0$$

$$(D^2 - p^2)x = 0$$

$$\textcircled{AE} \Rightarrow m^2 - p^2 = 0$$

$$m^2 = p^2$$

$$m = \pm p$$

$$x = C_1 e^{px} + C_2 e^{-px}$$

$$\textcircled{6} \Rightarrow T' - p^2 b T = 0$$

$$DT - bp^2 T = 0$$

$$(D - bp^2)T = 0$$

$$\textcircled{AE} \Rightarrow m - bp^2 = 0$$

$$m = bp^2$$

$$T = C_3 e^{bp^2 t}$$

∴ The three possible solutions of the one dimensional heat equation are

$$u(x,t) = (C_1 \cos px + C_2 \sin px) C_3 e^{-hp^2 t}$$

$$u(x,t) = (C_4 + C_5 x) C_6$$

$$u(x,t) = (C_7 e^{px} + C_8 e^{-px}) C_9 e^{hp^2 t}$$

Of these three solutions we have to choose that solution which is consistent with the physical nature of the problem. Since  $u$  decreases as time  $t$  increases, the only suitable solution of the heat equation is

$$u(x,t) = (C_1 \cos px + C_2 \sin px) \underline{\underline{C_3 e^{-hp^2 t}}}$$

Note

If we compare the heat transfer equation with vibrating string problems, we find that heat transfer equation contains only a first derivative  $\frac{\partial u}{\partial t}$  w.r.t time,

while the wave equation contains the second derivative  $\frac{\partial^2 u}{\partial t^2}$ . This means that the heat transfer equation requires only one initial temperature condition  $u(x, 0)$  to determine the temperature  $u(x, t)$ .

Solution of heat transfer equations when the end points are kept at zero temperature.

- (1) Find the temperature distribution in a rod of length  $l$  whose end points are fixed at temperature zero and the initial temperature distribution is  $f(x)$ .

Ans: The temperature distribution in a rod  $u(x, t)$  is given by the solution of the heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{h} \frac{\partial u}{\partial t} \quad \text{--- (1)}$$

If we take one end of the rod at the origin, the boundary conditions are,

$$u(0, t) = 0 \quad \text{--- (2)}$$

$$u(l, t) = 0 \quad \text{--- (3)}$$

The initial temperature is  $u(x, 0) = f(x) \quad \text{--- (4)}$

using method of separation of variables the general solution of (1) is

$$u(x, t) = (c_1 \cos px + c_2 \sin px) c_3 e^{-hp^2 t} \quad \text{--- (5)}$$

Applying  $u(0, t) = 0$  in (5)

$$u(0, t) = (c_1) c_3 e^{-hp^2 t}$$

$$\text{ie) } 0 = (c_1) c_3 e^{-hp^2 t} \quad \therefore c_3 e^{-hp^2 t} \neq 0$$

$$\therefore c_1 = 0$$

$$\textcircled{5} \Rightarrow (c_2 \sin px) c_3 e^{-\frac{hp^2 t}{l^2}}$$

$$\textcircled{5} \Rightarrow u(x,t) = (c_2 \sin px) c_3 e^{-\frac{hp^2 t}{l^2}}$$

$$u(x,t) = a_n \sin px e^{-\frac{hp^2 t}{l^2}} \quad \text{--- (6)} \quad [a_n = c_2 c_3]$$

Applying  $u(l,t) = 0$  in (6)

$$u(l,t) = a_n \sin pl e^{-\frac{hp^2 t}{l^2}}$$

$$0 = a_n \sin pl e^{-\frac{hp^2 t}{l^2}}$$

$$\Rightarrow \sin pl = 0$$

$$\sin pl = \sin n\pi l \quad n=1, 2, 3, \dots$$

$$pl = n\pi l$$

$$P = \frac{n\pi}{l}$$

$$\therefore \textcircled{6} \Rightarrow u(x,t) = a_n \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{hn^2\pi^2 t}{l^2}}$$

using principle of superposition solution of (1) satisfying boundary conditions is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{hn^2\pi^2 t}{l^2}} \quad \text{--- (7)}$$

Applying  $u(x,0) = f(x)$  in (7)

$$\text{ie) } u(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right)$$

which represents a half range fourier sine series for  $f(x)$

$$\text{where } a_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$\therefore$  required temperature distribution of the rod is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{h\pi^2 n^2 t}{l^2}}$$

where  $a_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$

Q2) A homogeneous laterally insulated bar of length 100cm has its ends kept at zero temperature. Find the temperature distribution  $u(x,t)$  if the initial temperature is

$$f(x) = \begin{cases} x & 0 \leq x \leq 50 \\ 100-x, & 50 \leq x \leq 100 \end{cases}$$

Ans: The temperature distribution  $u(x,t)$  is the soln of the one dimensional heat transfer equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{h} \frac{\partial u}{\partial t} \quad \text{--- (1)}$$

The boundary conditions  $u(0,t) = 0 \quad \text{--- (2)}$

$$u(100,t) = 0 \quad \text{--- (3)}$$

initial temperature  $u(x,0) = f(x) = \begin{cases} x, & 0 \leq x \leq 50 \\ 100-x, & 50 \leq x \leq 100 \end{cases} \quad \text{--- (4)}$

using method of separation of variables soln of (1) is

$$u(x,t) = (c_1 \cos px + c_2 \sin px) c_3 e^{-\frac{h p^2 t}{l^2}} \quad \text{--- (5)}$$

Applying  $u(0,t) = 0$  we get  $c_1 = 0$

Applying  $u(100,t) = 0$  we get  $P = \frac{n\pi}{100}$

$\therefore$  solution satisfying boundary conditions is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{100}\right) e^{-\frac{n^2\pi^2 t}{100^2}} \quad \text{--- (6)}$$

where  $a_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$

$$= \frac{2}{100} \left[ \int_0^{100} f(x) \sin\left(\frac{n\pi x}{100}\right) dx \right]$$

$$= \frac{1}{50} \left[ \int_0^{50} x \sin\left(\frac{n\pi x}{100}\right) dx + \int_{50}^{100} (100-x) \sin\left(\frac{n\pi x}{100}\right) dx \right]$$

$$= \frac{1}{50} \left\{ \left[ (0) \left( -\frac{\cos\left(\frac{n\pi x}{100}\right)}{\frac{n\pi}{100}} \right) - (1) \left( -\frac{\sin\left(\frac{n\pi x}{100}\right)}{\frac{n^2\pi^2}{100^2}} \right) \right]_0^{50} + \right.$$

$$\left. \left[ (100-x) \left( -\frac{\cos\left(\frac{n\pi x}{100}\right)}{\frac{n\pi}{100}} \right) - (-1) \left( -\frac{\sin\left(\frac{n\pi x}{100}\right)}{\frac{n^2\pi^2}{100^2}} \right) \right]_{50}^{100} \right\}$$

$$= \frac{1}{50} \left\{ \left[ -x \frac{\cos\left(\frac{n\pi x}{100}\right)}{\frac{n\pi}{100}} + \frac{\sin\left(\frac{n\pi x}{100}\right)}{\frac{n^2\pi^2}{100^2}} \right]_0^{50} + \left[ (100-x) \frac{\cos\left(\frac{n\pi x}{100}\right)}{\frac{n\pi}{100}} - \frac{\sin\left(\frac{n\pi x}{100}\right)}{\frac{n^2\pi^2}{100^2}} \right]_{50}^{100} \right\}$$

$$= \frac{1}{50} \left\{ \left[ -\frac{50 \cos \frac{n\pi}{2}}{\frac{n\pi}{100}} + \frac{\sin \frac{n\pi}{2}}{\frac{n^2\pi^2}{100^2}} \right] - [0+0] + [0-0] - \right.$$

$$\left. \left[ -\frac{50 \cos \frac{n\pi}{2}}{\frac{n\pi}{100}} - \frac{\sin \frac{n\pi}{2}}{\frac{n^2\pi^2}{100^2}} \right] \right\}$$

$$= \frac{1}{50} \left[ \frac{2 \sin \frac{n\pi}{2}}{\frac{n^2\pi^2}{100^2}} \right]$$

$$= \frac{100^2 \times 2}{50} \frac{\sin \frac{n\pi}{2}}{n^2 \pi^2}$$

$$a_n = \frac{400}{n^2 \pi^2} \sin\left(\frac{n\pi}{2}\right)$$

Substituting this value in ⑥ the required solution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{400}{n^2 \pi^2} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi x}{100}\right) e^{-\frac{h n^2 \pi^2 t}{100^2}}$$

- Q3) Find the temperature distribution in a rod of length 2m whose end points are maintained at temperature zero and the initial temperature is  $f(x) = 100(2x - x^2)$

Ans: The temperature distribution  $u(x,t)$  is the solution of the one dimensional heat transfer equation  $\frac{\partial u}{\partial x^2} = \frac{1}{h} \frac{\partial u}{\partial t}$  — ①

The boundary conditions  $u(0,t) = 0$  — ②

$u(2,t) = 0$  — ③

Initial temperature  $u(x,0) = f(x) = 100(2x - x^2)$  — ④

using method of separation of variables solution of ① is

$$u(x,t) = (c_1 \cos px + c_2 \sin px) c_3 e^{-\frac{h p^2 t}{100^2}}$$

Applying  $u(0,t) = 0$  we get  $c_1 = 0$

Applying  $u(2,t) = 0$  we get  $p = \frac{n\pi}{2}$

∴ Solution satisfying boundary conditions is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{2}\right) e^{-\frac{h n^2 \pi^2 t}{100^2}}$$

$$\begin{aligned}
 \text{where } a_n &= \frac{2}{\pi} \int_0^\pi f(x) \sin\left(\frac{n\pi x}{\pi}\right) dx \\
 &= \frac{2}{\pi} \int_0^2 100(2x-x^2) \sin\left(\frac{n\pi x}{2}\right) dx \\
 &= 100 \int_0^2 (2x-x^2) \sin\left(\frac{n\pi x}{2}\right) dx \\
 &= 100 \left[ \left(2x - x^2\right) \left(-\frac{\cos \frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (2-2x) \left(-\frac{\sin \frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right) + (-2) \left(\frac{\cos \frac{n\pi x}{2}}{\frac{n^3\pi^3}{8}}\right) \right]_0 \\
 &= 100 \left\{ \left[ 0 + 0 - \frac{2 \cos n\pi}{n^3\pi^3} \right] - \left[ 0 + 0 - \frac{2}{n^3\pi^3} \right] \right\} \\
 &= \frac{100 \times 8}{n^3\pi^3} [2 - 2 \cos n\pi]
 \end{aligned}$$

$$a_n = \frac{1600}{n^3\pi^3} [1 - \cos n\pi]$$

$\therefore$  Soln is  $\textcircled{6} \Rightarrow u(x,t) = \sum_{n=1}^{\infty} \frac{1600}{n^3\pi^3} [1 - \cos n\pi] \sin\left(\frac{n\pi x}{2}\right) e^{-\frac{h^2\pi^2 t}{4}}$

Q4) Find the temperature distribution in a bar of length  $\pi$  whose surface is thermally insulated with end points maintained at 0°C. The initial temperature distribution in the rod is

$$u(x,0) = f(x) = \begin{cases} x & 0 \leq x \leq \pi/2 \\ \pi - x & \pi/2 \leq x \leq \pi \end{cases}$$

OR

Solve the one dimensional diffusion equation for the function  $u(x, t)$  in the region  $0 \leq x \leq \pi$ ,  $t \geq 0$  when (i)  $u$  remains finite as  $t \rightarrow \infty$   
(ii)  $u=0$  if  $x=0$  or  $\pi$  for all values of  $t > 0$ ; and at  $t=0$ .

$$u = \begin{cases} x & 0 \leq x \leq \pi/2 \\ \pi-x & \pi/2 \leq x \leq \pi \end{cases}$$

Soln: Temperature distribution  $u(x, t)$  is the solution of the one dimensional heat equation  $\frac{\partial u}{\partial x^2} = \frac{1}{h} \frac{\partial u}{\partial t}$  — ①

Boundary conditions  $u(0, t) = 0$  — ②

$u(\pi, t) = 0$  — ③

Initial condition  $u(x, 0) = f(x) = \begin{cases} x & 0 \leq x \leq \pi/2 \\ \pi-x & \pi/2 \leq x \leq \pi \end{cases}$  — ④

By the method of separation of variables soln

$$u(x, t) = (c_1 \cos px + c_2 \sin px) c_3 e^{-hp^2 t}$$
 — ⑤

Applying  $u(0, t) = 0$  we get  $c_1 = 0$

$$\text{Applying } u(\pi, t) = 0 \text{ we get } p = \frac{n\pi}{\pi} = n$$

∴ Solution satisfying boundary conditions is

$$u(x, t) = \sum_{n=1}^{\infty} a_n \sin(nx) e^{-hn^2 t}$$
 — ⑥

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin\left(\frac{n\pi x}{\pi}\right) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx$$

$$= \frac{2}{\pi} \left[ \int_0^{\pi/2} x \sin(nx) dx + \int_{\pi/2}^{\pi} (\pi-x) \sin nx dx \right]$$

$$\begin{aligned}
 &= \frac{2}{\pi} \left\{ \left[ (\alpha) \left( -\frac{\cos nx}{n} \right) - (-1) \left( -\frac{\sin nx}{n^2} \right) \right]_0^{\frac{\pi}{2}} + \left[ (\pi - x) \left( -\frac{\cos nx}{n} \right) - (-1) \left( -\frac{\sin nx}{n^2} \right) \right]_0^{\frac{\pi}{2}} \right\} \\
 &= \frac{2}{\pi} \left\{ \left[ -x \frac{\cos nx}{n} + \frac{\sin nx}{n^2} \right]_0^{\frac{\pi}{2}} + \left[ -(\pi - x) \left( \frac{\cos nx}{n} \right) - \left( \frac{\sin nx}{n^2} \right) \right]_0^{\frac{\pi}{2}} \right\} \\
 &= \frac{2}{\pi} \left[ -\frac{\pi}{2} \frac{\cos \frac{n\pi}{2}}{n} + \frac{\sin \frac{n\pi}{2}}{n^2} \right] - [0+0] + [0-0] - \left[ -\frac{\pi}{2} \frac{\cos \frac{n\pi}{2}}{n} - \frac{\sin \frac{n\pi}{2}}{n^2} \right] \\
 &= \frac{2}{\pi} \left[ \frac{2 \sin \frac{n\pi}{2}}{n^2} \right] \\
 &= \frac{4}{\pi n^2} \sin\left(\frac{n\pi}{2}\right) \\
 \therefore \text{Soln is } u(x,t) = \sum_{n=1}^{\infty} \frac{4}{\pi n^2} \sin\left(\frac{n\pi}{2}\right) \sin(nx) e^{-bn^2 t}
 \end{aligned}$$

- Q5) Solve one dimensional diffusion equation  $\frac{\partial^2 u}{\partial x^2} = \frac{1}{h} \frac{\partial u}{\partial t}$  subject to the condition  $u(x,t) \neq \infty$  if  $t \rightarrow \infty$
- $$u(0,t) = 0 = u(\pi,t)$$
- $$u(x,0) = \pi x - x^2$$

Ans:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{h} \frac{\partial u}{\partial t} \quad \text{--- (1)}$$

$$u(0,t) = 0 \quad \text{--- (2)}$$

$$u(\pi,t) = 0 \quad \text{--- (3)}$$

$$u(x,0) = f(x) = \pi x - x^2 \quad \text{--- (4)}$$

By the method of separation of variables soln is

$$u(x, t) = (c_1 \cos px + c_2 \sin px) c_3 e^{-hp^2 t} \quad \text{--- (5)}$$

Applying  $u(0, t) = 0$  we get  $c_1 = 0$

Applying  $u(\pi, t) = 0$  we get  $p = \frac{n\pi}{L} = \frac{n\pi}{\pi} = n$

Solution satisfying Boundary condition is

$$u(x, t) = \sum_{n=1}^{\infty} a_n \sin nx e^{-hn^2 t} \quad \text{--- (6)}$$

$$\text{where } a_n = \frac{2}{\pi} \int_0^\pi f(x) \sin\left(\frac{n\pi x}{\pi}\right) dx$$

$$= \frac{2}{\pi} \int_0^\pi (\pi x - x^2) \sin(nx) dx$$

$$= \frac{2}{\pi} \left\{ \left[ (\pi x - x^2) \left( \frac{\cos nx}{n} \right) - (\pi - 2x) \left( -\frac{\sin nx}{n^2} \right) + (-2) \left( \frac{\cos nx}{n^3} \right) \right] \right|_0^\pi$$

$$= \frac{2}{\pi} \left\{ \left[ 0 + 0 - \frac{2 \cos n\pi}{n^3} \right] - \left[ 0 + 0 - \frac{2}{n^3} \right] \right\}$$

$$= \frac{2}{\pi} \left[ \frac{2}{n^3} - \frac{2 \cos n\pi}{n^3} \right]$$

$$= \frac{4}{\pi n^3} [1 - \cos n\pi]$$

$$\therefore \text{Soln is } u(x, t) = \sum_{n=1}^{\infty} \frac{4}{\pi n^3} [1 - \cos n\pi] \sin(nx) e^{-hn^2 t}$$

H.W

Q6) A rod of length  $L$  is heated so that its ends A and B are at zero temperature. if initially its temperature is given by

$u = \frac{cx(L-x)}{x^2}$ . find the temperature at time  $t$ .

$$\text{Ans: } u(x, t) = \sum_{n=1}^{\infty} \frac{4c}{n^3 \pi^3} [1 - \epsilon(n)] \sin\left(\frac{n\pi x}{L}\right) e^{-hn^2 \pi^2 t}$$

Q7) A uniform bar of length  $l$  through which heat flows is insulated at its sides. The ends are kept at zero temperature. If the initial temperature at the interior points of the bar is given by  $f(x) = k \sin^3\left(\frac{\pi x}{l}\right)$ . Find the temperature distribution in the bar after time  $t$ .

Ans: The temperature  $u(x,t)$  is the solution of the heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{h} \frac{\partial u}{\partial t} \quad \text{--- (1)}$$

Boundary conditions  $u(0,t) = 0 \quad \text{--- (2)}$

$u(l,t) = 0 \quad \text{--- (3)}$

Initial condition  $u(x,0) = f(x) = k \sin^3\left(\frac{\pi x}{l}\right) \quad \text{--- (4)}$

By the method of separation of variables

$$u(x,t) = (c_1 \cos px + c_2 \sin px) c_3 e^{-hp^2 t} \quad \text{--- (5)}$$

Applying  $u(0,t) = 0$  we get  $c_1 = 0$

Applying  $u(l,t) = 0$  we get  $p = \frac{n\pi}{l}$

Solution satisfying Boundary conditions is

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) e^{-hn^2 \pi^2 t/l^2} \quad \text{--- (6)}$$

where  $c_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$

$$= \frac{2k}{l} \int_0^l \sin^3\left(\frac{\pi x}{l}\right) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2k}{L} \int_0^L \frac{1}{4} \left[ 3\sin\left(\frac{\pi x}{L}\right) - \sin\left(\frac{3\pi x}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right) dx$$

$$a_n = \frac{k}{2L} \left[ 3 \int_0^L \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx - \int_0^L \sin\left(\frac{3\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx \right]$$

$$a_1 = \frac{k}{2L} \left[ 3 \cdot \frac{1}{2} - 0 \right] = \frac{3kL}{4L} = \frac{3k}{4}$$

$$a_2 = \frac{k}{2L} [3 \times 0 - 0] = 0$$

$$a_3 = \frac{k}{2L} \left[ 3 \times 0 - \frac{1}{2} \right] = -\frac{k}{4}$$

$$a_4 = a_5 = a_6 = \dots = 0$$

solution ⑥  $\Rightarrow u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{h\pi^2 n^2 t}{L^2}}$

$$= a_1 \sin\left(\frac{\pi x}{L}\right) e^{-\frac{h\pi^2 t}{L^2}} + a_3 \sin\left(\frac{3\pi x}{L}\right) e^{-\frac{9h\pi^2 t}{L^2}}$$

$$= \frac{3k}{4} \sin\left(\frac{\pi x}{L}\right) e^{-\frac{h\pi^2 t}{L^2}} - \frac{k}{4} \sin\left(\frac{3\pi x}{L}\right) e^{-\frac{9h\pi^2 t}{L^2}}$$

Q8) In steady state conditions derive the solution of one dimensional heat equation.

Ans: One dimensional heat equation is  $\frac{\partial^2 u}{\partial x^2} = \frac{1}{h} \frac{\partial u}{\partial t}$

In steady state the temperature distribution is independent of time.  $\therefore \frac{\partial u}{\partial t} = 0$

$\therefore$  heat equation becomes  $\frac{\partial^2 u}{\partial x^2} = 0$

$$\text{ie) } D^2u = 0$$

$$AE \Rightarrow m^2 = 0$$

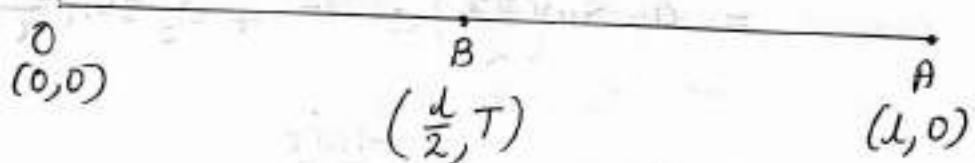
$$m = 0, 0$$

$$\therefore \text{solution } u = \underline{(c_1 + c_2 x)}$$

In steady state conditions the solution of one dimensional heat equation is  $\underline{u = (c_1 + c_2 x)}$

- Q9) A homogeneous rod of conducting material of length  $L$  has its ends kept at zero temperature. The temperature at the centre is  $T$  and falls uniformly to zero at the two ends. Find  $u(x, t)$ .

Ans:



$$\text{Equation of OB} \Rightarrow \frac{x-0}{\frac{L}{2}-0} = \frac{y-0}{T-0}$$

$$\frac{x}{\frac{L}{2}} = \frac{y}{T}$$

$$y = \frac{2xT}{L}, \quad 0 < x < \frac{L}{2}$$

$$\text{Equation of BA} \Rightarrow \frac{x-\frac{L}{2}}{L-\frac{L}{2}} = \frac{y-T}{0-T}$$

$$\frac{\frac{2x-l}{2}}{\frac{l}{2}} = \frac{y-T}{-T}$$

$$\frac{2x-l}{l} = \frac{y-T}{-T}$$

$$(2x-l)(-T) = l(y-T)$$

$$-2xT + lT = ly - lT$$

$$2lT - 2xT = ly$$

$$2T(l-x) = ly$$

$$\therefore y = \frac{2T(l-x)}{l}, \quad \frac{l}{2} < x < l$$

$$\therefore \text{initial temperature } u(x,0) = f(x) = \begin{cases} \frac{2xT}{l}, & 0 < x < \frac{l}{2} \\ \frac{2T(l-x)}{l}, & \frac{l}{2} < x < l \end{cases}$$

HW

$$u(x,t) = \sum_{n=1}^{\infty} \frac{8T}{n^2 \pi^2} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{n^2 \pi^2 k^2 t}{l^2}}$$

## D'Alembert's solution of the wave equation

one dimensional wave equation is  $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$  - ①

let  $u(x, 0) = f(x)$  - ② and

$$\frac{\partial u(x, 0)}{\partial t} = 0 \quad \text{--- ③}$$

$$\text{①} \Rightarrow \frac{\partial^2 u}{\partial x^2} c^2 = \frac{\partial^2 u}{\partial t^2}$$

$$\text{②) } \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

Take  $D = \frac{\partial}{\partial t}$  and  $D' = \frac{\partial}{\partial x}$  then the PDE takes the form

$$D^2 u - c^2 D'^2 u = 0$$

$$(D^2 - c^2 D'^2) u = 0$$

AE is obtained by substituting  $D = m$  &  $D' = 1$

$$\therefore m^2 - c^2 = 0$$

$$m^2 = c^2$$

$$m = \pm c$$

So the general solution is

$$u(x,t) = \phi(x+ct) + \psi(x-ct) \quad \text{--- (4)}$$

where  $\phi, \psi$  are arbitrary functions.

Now we shall apply the initial conditions to find the particular values of these arbitrary functions.

Applying  $u(x,0) = f(x)$  in (4) we get

$$u(x,0) = \phi(x) + \psi(x)$$

$$\text{ie) } f(x) = \phi(x) + \psi(x) \quad \text{--- (5)}$$

Differentiating (4) w.r.t  $t$  we get

$$\frac{\partial u(x,t)}{\partial t} = c\phi'(x+ct) - c\psi'(x-ct)$$

Applying  $\frac{\partial u(x,0)}{\partial t} = 0$  we get

$$\frac{\partial u(x,0)}{\partial t} = c\phi'(x) - c\psi'(x)$$

$$\text{ie) } 0 = c\phi'(x) - c\psi'(x)$$

$$\therefore c\phi'(x) = c\psi'(x)$$

$$\therefore \phi'(x) = \psi'(x)$$

integrating  $\phi(x) = \psi(x) + k \quad \text{--- (6)} \quad [k = \text{constant of integration}]$

$$\textcircled{5} \Rightarrow f(x) = \psi(x) + k + \psi(x)$$

$$f(x) - k = 2\psi(x)$$

$$\therefore \psi(x) = \frac{1}{2}[f(x) - k] \quad \text{--- (1)}$$

$$\textcircled{6} \Rightarrow \phi(x) = \frac{1}{2}[f(x) - k] + k$$

$$= \frac{1}{2}f(x) - \frac{1}{2}k + k$$

$$= \frac{1}{2}f(x) + \frac{k}{2} = \frac{1}{2}[f(x) + k] \quad \text{--- (8)}$$

Substituting ⑦ & ⑧ in ④ The general solution is,

$$u(x,t) = \frac{1}{2} [f(x+ct) + k] + \frac{1}{2} [f(x-ct) - k]$$

$u(x,t) = \frac{1}{2} [f(x+ct) + f(x-ct)]$  which is the d'Alembert's solution.

### Problem

Q) Using d'Alembert's method, find the deflection of a vibrating string of unit length having fixed ends with initial velocity zero and initial deflection

$$(i) f(x) = k[\sin x - \sin 2x]$$

$$(ii) f(x) = a(x-x^3)$$

Ans ① By d'Alembert's method, the solution is

$$u(x,t) = \frac{1}{2} [f(x+ct) + f(x-ct)]$$

$$= \frac{1}{2} [k(\sin(x+ct) - \sin 2(x+ct)) + k(\sin(x-ct) - \sin 2(x-ct))]$$

$$= \frac{k}{2} [\sin(x+ct) - \sin(2x+2ct) + \sin(x-ct) - \sin(2x-2ct)]$$

$$= \frac{k}{2} \left\{ [\sin x \cos ct + \cos x \sin ct] - [\sin 2x \cos 2ct + \cos 2x \sin 2ct] + [\sin x \cos ct - \cos x \sin ct] - [\sin 2x \cos 2ct - \cos 2x \sin 2ct] \right\}$$

$$= \frac{k}{2} [2 \sin x \cos ct - 2 \sin 2x \cos 2ct]$$

$$u(x,t) = k[\sin x \cos ct - \sin 2x \cos 2ct]$$

Also  $u(x,0) = k[\sin x - \sin 2x] = f(x)$

$$\frac{\partial u(x,t)}{\partial t} = k[\sin x \sin ct - \sin 2x \sin 2ct (2c)]$$

$$(e) \frac{\partial u(x,t)}{\partial t} = k[-c \sin x \sin ct + 2c \sin x \cos ct]$$

$$\frac{\partial u(x,0)}{\partial t} = k[0+0] = \underline{0}$$

(e) given boundary conditions are satisfied.

$$Ans 2) f(x) = a(x - x^3)$$

By D'Alembert's method the solution is,

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)]$$

$$= \frac{1}{2} [a[(x+ct) - (x+ct)^3] + a[(x-ct) - (x-ct)^3]]$$

$$= \frac{a}{2} [(x+ct) - (x+ct)^3 + (x-ct) - (x-ct)^3]$$

$$= \frac{a}{2} [2x - [x^3 + 3x^2ct + 3xct^2 + c^3t^3] - [x^3 - 3x^2ct + 3xct^2 - c^3t^3]]$$

$$= \frac{a}{2} [2x - 2x^3 - 6x^2ct^2]$$

$$u(x,t) = a[x - x^3 - \underline{3x^2ct^2}]$$

Also  $u(x,0) = a[x - x^3] = \underline{f(x)}$

$$\frac{\partial u(x,t)}{\partial t} = a[-6x^2ct]$$

$$\frac{\partial u(x,0)}{\partial t} = a \times 0 = \underline{0}$$

(e) given boundary conditions are satisfied