Oscillations

A motion that is repeated itself after regular intervals of time is called periodic motion or harmonic motion. Eg: motion of a satellite around a planet, vibrations of atoms in molecules etc.

A body or particle is said to possess oscillatory or vibratory motion if it moves back and forth repeatedly about the mean position. Eg: motion of simple pendulum, the vertical oscillations of a loaded spring etc.

All oscillatory motions are periodic.

Period (T) : Time required for one oscillation.

Frequency (ν): The number of oscillations per unit time is the frequency of the oscillation. $T = \frac{1}{\nu}$

Displacement : The distance of oscillating particle in any direction from its equilibrium position at any instant is the displacement of the particle at that instant.

Amplitude : The maximum displacement of the particle from its equilibrium position.

Phase: The phase of an oscillatory particle at any instant defines the states of the particle, i.e., its position and the direction of motion at that instant.

Restoring Force: In the equilibrium position of the oscillating particle, no net force acts on it. When a particle is displaced from its equilibrium position, a periodic force acts on it in such a direction to bring back the particle to its equilibrium position. This periodic force is called restoring force.

Simple Harmonic Motion

A particle is said to execute SHM, if acceleration at any instant is directly proportional to its displacement from the equilibrium position and is directed towards the equilibrium position. Eg: Motion of simple pendulum, vibration in tuning fork.

A particle or a system executing SHM is called harmonic oscillator.

For SHM, the restoring force is directly proportional to the displacement and acts in the direction opposite to that of displacement.

A system executing SHM is called simple harmonic oscillator.

Differential equation of motion of SHM

Consider a particle of mass, executing SHM. If the displacement of the particle at any instant *t* be *x*. Then its acceleration will be $\frac{d^2x}{dt^2}$.

According to the definition of SHM, we have, restoring force acting on the mass is $F \propto -x$.

$F = -Cx$

[Here – sign shows that the restoring force and the displacement are in opposite direction.]

According to Newton's law of motion, $F = ma$

i.e., $m\frac{d^2x}{dt^2} = -Cx$ or $m\frac{d^2x}{dt^2} + Cx = 0$

$$\frac{d^2x}{dt^2} + \frac{C}{m}x = 0.$$

Putting $\frac{C}{m} = \omega^2$, we get $\boldsymbol{\frac{d^2x}{dt^2} + \omega^2 x = 0}$ Here ω is natural angular frequency.

This is known as **differential equation of motion for a simple harmonic oscillator.**

The **solution of this differential equation** is

$x = a\sin(\omega_0 t+\phi)$ where x is displacement, a is amplitude, ω is angular frequency and ϕ is known as initial phase or phase constant.

Free Oscillations

If no frictional force or resistance is acting on an oscillating system, it will keep on oscillating with constant energy and amplitude indefinitely. These oscillations are known as free oscillations. The frequency of oscillation is called natural frequency.

Damped oscillations:

When frictional force or resistance is acting on an oscillator opposite to the direction of its motion, then a part of the energy of the oscillator is used to overcome this frictional force. As a result, its amplitude and velocity of oscillations decreases. Such oscillations are known as damped oscillations and these forces are known as damping force or retarding forces.

Practical cases of damping

(i) Damping plays a useful role in the oscillations of an automobile’s suspension system.

The shock absorbers provide a velocity depending damping force, so that when the car goes over a bump, it does not continue bouncing forever. For optimal passenger comfort, the system should be critically damped or slightly under damped.

(ii) An LCR circuit is an excellent example of a DHO with the resistance playing the role of a damping force.

Damped Harmonic Oscillator

An oscillating system which undergo damping due to retarding force are known as damped harmonic oscillator. The amplitude of vibrations of the oscillator gradually decreases to zero as the result of frictional forces arising due to viscosity of the medium in which the oscillator is moving.

This harmonic oscillator experiences two forces. i.e.,

(i) restoring force [$F = -Cx$]

(ii) damping force (proportional to velocity but opposes it) [$F = -\gamma\frac{dx}{dt}$]

According to Newton’s law of motion, F = ma

$$m\frac{d^2x}{dt^2} = -Cx - \gamma\frac{dx}{dt}$$

$$m\frac{d^2x}{dt^2} + Cx + \gamma\frac{dx}{dt} = 0$$

Dividing throughout by m, we get,

$$\frac{d^2x}{dt^2} + \frac{\gamma}{m}\frac{dx}{dt} + \frac{C}{m}x = 0.$$

Putting $\frac{\gamma}{m}$ = 2k and $\frac{C}{m} = \omega_0^2$. Here k is the damping constant.

$$\boldsymbol{\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega_0^2 x = 0} \quad \rightarrow \quad \textbf{(1)}$$

This is the differential equation of damped harmonic oscillator (DHO).

Solution of differential equation of DHO

Let the solution of this equation is $x = Ae^{\alpha t}$

$$\frac{dx}{dt} = A\,\alpha\, e^{\alpha t}$$

$$\frac{d^2x}{dt^2} = \alpha^2 A\, e^{\alpha t}$$

Putting these equations in equation (1), we get,

$$\alpha^2 A\, e^{\alpha t} + 2k A\,\alpha\, e^{\alpha t} + \omega_0^2 A e^{\alpha t} = 0$$

i.e., $Ae^{\alpha t}[\alpha^2 + 2k\alpha + \omega_0^2] = 0$

i.e., $[\alpha^2 + 2k\alpha + \omega_0^2] = 0$

i.e., $\alpha = \frac{-2k \pm \sqrt{4k^2 - 4\omega_0^2}}{2} = -k \pm \sqrt{k^2 - \omega_0^2}$

Solution for quadriatic equation $ax^2+bx+c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

So the solution $x = Ae^{\alpha t}$ can be written as

$$\boldsymbol{x = A_1\, e^{\left(-k + \sqrt{k^2 - \omega_0^2}\right)t} + A_2\, e^{\left(-k - \sqrt{k^2 - \omega_0^2}\right)t}} \quad \rightarrow \quad \textbf{(2)}$$

$$\textbf{or}\quad \boldsymbol{x = e^{-kt}\left[A_1\, e^{\left(\sqrt{k^2 - \omega_0^2}\right)t} + A_2\, e^{\left(-\sqrt{k^2 - \omega_0^2}\right)t}\right]}$$

where A_1 and A_2 are arbitrary constants whose value depend on the initial conditions of motion.

Depending upon the relative values of k and ω_0, three different cases will arise.

The quantity $\sqrt{k^2 - \omega_0^2} = \beta$ can be real, imaginary or zero.

(i) If $k > \omega_0$, Then $\sqrt{k^2 - \omega_0^2} = \beta$ is a real quantity and it is called over damped case, non oscillatory motion, dead beat or aperodic.

(ii) If $k < \omega_0$, Then $\sqrt{k^2 - \omega_0^2}$ is imaginary and it is called under damped case.

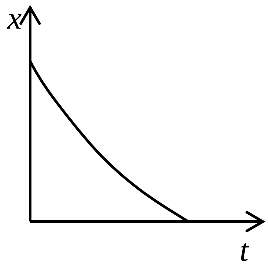
(iii) If $k = \omega_0$, Then $\sqrt{k^2 - \omega_0^2} = 0$ and it is called critically damped case.

Case I : Over damped ($k > \omega_0$)

If the damping is so high, such that $k > \omega_0$, then $\sqrt{k^2 - \omega_0^2}$ = β is a real quantity and is less than k.

Then $x = A_1 e^{(-k+\beta)t} + A_2 e^{(-k-\beta)t}$

As $k > \beta$, both terms of RHS decreases exponentially with time. After attaining the maximum value, the displacement dies off exponentially without changing direction. Time-displacement curve for over damped harmonic motion is as shown below.



Thus the motion is non oscillatory. Such a motion is called dead-beat or aperiodic.

Application: Its main application is in dead beat galvanometer.

Case II : Critically damped ($k = \omega_0$)

If $k = \omega_0$, then $\sqrt{k^2 - \omega_0^2} = 0$.

∴ Then eqn (2) becomes, $x = (A_1 + A_2)\, e^{-kt}$ or $x = Be^{-kt}$ where $B = A_1 + A_2$.

But in this equation, there is only one constant and hence does not form the solution of the second order differential equation.

Now suppose $\sqrt{k^2 - \omega_0^2}$ = h which is a very small quantity, i.e., h → 0, then

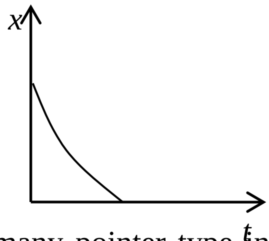
$x = e^{-kt}\,[A_1\, e^{ht} + A_2 e^{-ht}]$

$= e^{-kt}\,[A_1\,(1 + ht) + A_2(1 - ht)]$

$= e^{-kt}\,[(A_1 + A_2) + (A_1 - A_2)\, ht]$

$x = e^{-kt}\,[\,D + Et\,]$ where $(A_1 + A_2) = D$ and $(A_1 - A_2)\, h = E$

This equation shows that initially the displacement increases due to the factor D + Et. But as time passes, the exponential term becomes relatively more important and the displacement returns continuously from the maximum value to zero rather faster than over damped. The motion becomes just aperiodic or non oscillatory. This is called critical damping. Time-displacement curve for critically damped harmonic motion is as shown below.



Application: It finds application in many pointer type instruments like galvanometer, where the pointer moves at once to take a correct position and stays at this position without any oscillation.

Case III : Under damped ($k < \omega_0$)

If the damping is very low, that is if $k < \omega_0$, $\sqrt{k^2-\omega_0^2} = \sqrt{-(\omega_0^2 - k^2)} = i\omega$

where $i = \sqrt{-1}$ and $\omega = \sqrt{(\omega_0^2 - k^2)}$ which is a real quantity.

Now the solution is $x = e^{-kt} [A_1 e^{i\omega t} + A_2 e^{-i\omega t}]$

$x = e^{-kt}[A_1 (cos\omega t + isin\omega t) + A_2 (cos\omega t - isin\omega t)]$

$x = e^{-kt} [(A_1 + A_2) cos\omega t + I (A_1 - A_2) sin\omega t]$

Since x is a real quantity, both $(A_1 + A_2)$ and $i(A_1 - A_2)$ must be real. So A_1 and A_2 are complex quantities.

If $(A_1 + A_2) = a_0 \sin\phi$ and $i(A_1 - A_2) = a_0 \cos\phi$,

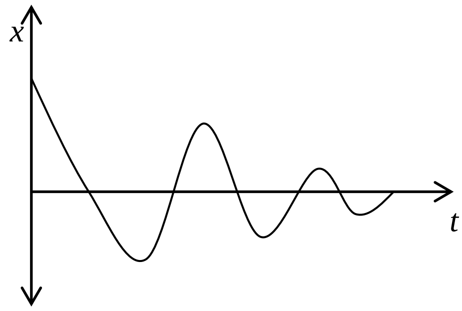
Then $x = a_0 e^{-kt}$ [cosωtsinϕ + sinωt cosϕ] or

$\boldsymbol{x = a_0 e^{-kt} \sin(\omega t + \phi)}$

This is the equation which represents a damped harmonic motion. This motion is oscillatory.

Amplitude of oscillation is $a_0 e^{-kt}$ which decreases exponentially with time.

Here it can be seen that angular frequency $\omega = \sqrt{(\omega_0^2 - k^2)}$ is less than ω_0 and period $T = \frac{2\pi}{\omega}$ is greater than $\frac{2\pi}{\omega_0}$. The time-displacement curve for under damped harmonic motion is as shown below.

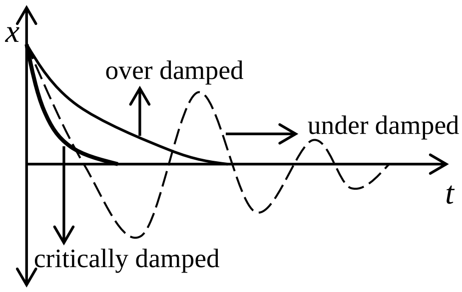


Effects of damping:

(i) A decrease in angular frequency.

(ii) An increase in period.

(iii) A decrease in amplitude exponentially with time.



Displacement –time graph of over damped, critical damped and under damped oscillations

Power Dissipation

In damped harmonic oscillator, work is to be done by the oscillating particle to overcome the damping force. The rate of dissipation of energy or power dissipation is defined as the ratio of energy loss in one period to time period. We know that total energy is proportional to the square of the amplitude.

Here amplitude A = $a_0 e^{-kt}$

Hence the energy at the instant 't' is $E_t = (a_0 e^{-kt})^2$ or $\mathbf{E_t = Ce^{-2kt}}$ where C is a constant.

The energy of the oscillator after one cycle (one period) is $E_{t+T} = C\, e^{-2k(t+T)}$

$$= Ce^{-2kt}\, e^{-2kT} = E_t e^{-2kT}$$

$\therefore$ Average power dissipation, $P = \frac{\text{Energy loss in one period}}{\text{Time period}} = \frac{E_t - E_{t+T}}{T} = \frac{E_t - E_t e^{-2kT}}{T}$

$P = \frac{E_t - E_t(1-2kT)}{T}$ (for small value of k).

$\mathbf{P = 2kE_t}$ or $P = \frac{E_t}{\tau}$ where $\tau = \frac{1}{2k}$ is called **relaxation time**.

Relaxation time (τ) is the time after which the energy reduces to $\left(\frac{1}{e}\right)^{th}$ of its initial value.

Quality factor

The quality factor is defines as 2π times the ratio of energy stored in the system to the energy lost per unit time. $Q = 2\pi \times \frac{\text{energy stored in system}}{\text{energy lost per unit time}}$

$= 2\pi \times \frac{E}{PT}$ where P is the power dissipated and T is known as periodic time.

$Q = 2\pi \times \frac{E}{\left(\frac{E}{\tau}\right)T} = \frac{2\pi}{T}\tau = \omega\tau$ where $\frac{2\pi}{T} = \omega$ (angular frequency)

Significance: Higher the value of Q, higher would be the value of relaxation time τ.

(i.e. as Q is high, damping is low.)

Forced Harmonic Oscillator

If an external periodic force is applied on a damped harmonic oscillator, the oscillator oscillates with the frequency of the applied force. Such an oscillator is called forced harmonic oscillator and its oscillations are called forced oscillations. For example, when we push a swing, we have to keep on pushing periodically to keep the swing in same oscillatory motion. Another example is an electrical oscillator.

Let the external periodic force be represented by $\mathbf{F = F_0 \sin pt}$ where F_0 be the maximum value of the force with frequency *p*. The forces acted upon this oscillator are

(i) a restoring force $-Cx$

(ii) a frictional force $-\gamma \frac{dx}{dt}$ (damping force)

(iii) external periodic force $F = F_0 \sin pt$ where $\frac{p}{2\pi}$ is the frequency of the driving force.

According to Newton's second law, F = ma ; F = $m\frac{d^2x}{dt^2}$

So, $m\frac{d^2x}{dt^2} = -\gamma\frac{dx}{dt} - Cx + F$

$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + Cx = F_0 \sin pt$

$\frac{d^2x}{dt^2} + \frac{\gamma}{m}\frac{dx}{dt} + \frac{C}{m}x = \frac{F_0}{m}\sin pt$

$\boldsymbol{\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega_0^2 x = f_0 \sin pt}$ → **(1)** where $2k = \frac{\gamma}{m}$, $\omega_0 = \sqrt{\frac{C}{m}}$ is the natural frequency in absence of damping and driven forces and $f_0 = \frac{F_0}{m}$. k is the damping constant , C is force constant.

Here $\frac{p}{2\pi}$ is the frequency of the applied force.

This is the differential equation for forced harmonic oscillator.

Solution of differential equation of FHO

Let us suppose that the solution of this differential equation be of the form,

$\boldsymbol{x = A\sin(pt-\theta)}$ → **(2)**

Here, A is the amplitude of the forced oscillations and θ represents the phase difference between the force and the resultant displacement of the system .

We have $\frac{dx}{dt} = A\cos(pt-\theta) \times p = Ap\cos(pt-\theta)$

Also, $\frac{d^2x}{dt^2} = -Ap\sin(pt-\theta) \times p = -Ap^2\sin(pt-\theta)$

Substituting these equation in eqn (1), we get,

$-Ap^2\sin(pt-\theta) + 2kAp\cos(pt-\theta) + \omega_0^2 A\sin(pt-\theta) = f_0\sin(pt-\theta+\theta)$

[$f_0 \sin pt$ may be written as $f_0\sin(pt-\theta+\theta)$]

$-Ap^2\sin(pt-\theta)+2kAp\cos(pt-\theta)+\omega_0^2 A\sin(pt-\theta) = f_0\sin(pt-\theta)\cos\theta + f_0\cos(pt-\theta)\sin\theta$

$A(\omega_0^2-p^2)\sin(pt-\theta) + 2kAp\cos(pt-\theta) = f_0\cos\theta\sin(pt-\theta) + f_0\sin\theta\cos(pt-\theta)$

Equating the coefficient of $\sin(pt-\theta)$ and $\cos(pt-\theta)$

$A(\omega_0^2-p^2) = f_0\cos\theta$ → (3) and $2kAp = f_0\sin\theta$ → (4)

Squaring and adding eqn (3) and eqn (4), $A^2(\omega_0^2-p^2)^2 + 4k^2A^2p^2 = {f_0}^2$

Or $A^2[(\omega_0^2-p^2)^2 + 4k^2p^2] = {f_0}^2$

i.e., $A = \frac{f_0}{\sqrt{(\omega_0^2-p^2)^2 + 4k^2p^2}}$ → (4)

This is the expression for **amplitude of forced oscillation**.

Now, to find **phase of forced oscillations**,

Dividing (4) by (3), $\tan\theta = \frac{2kp}{\omega_0^2-p^2}$

The **phase difference** $\theta = \tan^{-1}\frac{2kp}{\omega_0^2-p^2}$ → (5)

Substituting (4) and (5) in eqn (2), we have the solution as

$$x = \frac{f_0}{\sqrt{(\omega_0^2-p^2)^2 + 4k^2p^2}} \sin\left[pt - \tan^{-1}\frac{2kp}{\omega_0^2-p^2}\right] \quad \rightarrow \quad (6)$$

The complete solution becomes,

$$x = a_0e^{-kt}\sin(\omega t + \phi) + \frac{f_0}{\sqrt{(\omega_0^2-p^2)^2 + 4k^2p^2}} \sin\left[pt - \tan^{-1}\frac{2kp}{\omega_0^2-p^2}\right] \quad \rightarrow \quad (8)$$

First term represents the natural damped oscillation and the second term represents the forced oscillation. Initially both the vibrations will be present, but with the passage of time, the first term vanishes and the motion of the body will be completely represented by the second term.

So the solution is only $x = \frac{f_0}{\sqrt{(\omega_0^2-p^2)^2 + 4k^2p^2}} \sin\left[pt - \tan^{-1}\frac{2kp}{\omega_0^2-p^2}\right]$

Case I: Low driving frequency ($p < \omega_0$)

We have $A = \frac{f_0}{\sqrt{(\omega_0^2-p^2)^2 + 4k^2p^2}}$

When $p < \omega_0$, neglect p^2, then $A = \frac{f_0}{\omega_0^2} = \frac{F_0/m}{C/m}$ or $A = \frac{F_0}{C}$

Here amplitude depends only on force constant, but not on mass of the body or frequency of the force applied. The force and displacement are always in phase.

Case II: High driving frequency ($p > \omega_0$)

We have $A = \frac{f_0}{\sqrt{(\omega_0^2-p^2)^2 + 4k^2p^2}}$

When $p > \omega_0$, neglect $\omega_0{}^2$, and also for low damping, k^2 can be neglected, then $A = \frac{f_0}{p^2}$

The displacement lags behind the force by a phase by π.

Resonance

The phenomenon in which the amplitude of a forced harmonic oscillator becomes maximum at a particular driving frequency which is very close to the natural frequency, is known as **amplitude resonance**. The frequency of the driving force at which resonance occurs is known as resonant frequency (P_R).

Eg: Tuning of a radio (A tuned circuit in a radio receiver responds strongly to waves having frequencies near its resonant frequency and this fact is used to select a particular station and reject the others), musical instruments can be made to vibrate by bringing them in contact with vibrations which have the frequency equal to the natural frequency of the instruments, a vibrating rattle in a car that occurs only at a certain engine speed or wheel rotation speed etc.

Expression for resonant frequency

In case of forced vibrations, amplitude $A = \frac{f_0}{\sqrt{(\omega_0^2 - p^2)^2 + 4k^2p^2}}$

The amplitude is maximum ($A = A_{max}$), when $\sqrt{(\omega_0^2 - p^2)^2 + 4k^2p^2}$ is minimum.

i.e., when $\frac{d}{dp}\left[\sqrt{(\omega_0^2 - p^2)^2 + 4k^2p^2}\right] = 0$

i.e., when $-2(\omega_0^2 - p^2) \times 2p + 8k^2p = 0$

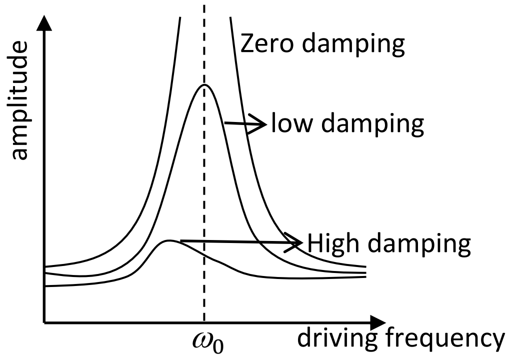
i.e., when $(\omega_0^2 - p^2) + 2k^2 = 0$

i.e., when $p^2 = \omega_0^2 - 2k^2$

i.e., when $\mathbf{P_R} = \sqrt{\boldsymbol{\omega_0^2 - 2k^2}}$ where P_R is resonant frequency.

When damping k is low, $p = \omega_0$,

then $A_{max} = \frac{f_0}{2kp} = \frac{f_0 \tau}{\omega_0}$ where $\tau = \frac{1}{2k}$



The figure shows the variation of amplitude with driving frequency.

When k = 0, the amplitude becomes infinite and there is no damping**.** But this does not occur as damping is never zero.

At resonance, the amplitude of oscillations is maximum.

For small values of k (low damping), the amplitude decreases very rapidly on either side of the resonant frequency than for higher value of k.

Sharpness of Resonance

The term sharpness of resonance refers to the rate of fall in amplitude with the change of driving frequency on either side of resonant frequency.

Resonance is said to be sharp, when for a small change of the driving frequency from the resonant frequency, there is a large change in the energy of vibration.

When the damping is low, the amplitude falls of a very rapidly on either side of resonant frequency and that the resonance is sharp. Example for sharp resonance is a sonometer wire with a tuning fork.

When the damping is high, the amplitude falls of very slowly on either side of resonant frequency and that the resonance is flat. Example for flat resonance is resonance of an air column.

Significance: Sharpness of resonance is defined by the Q factor which is related to how quickly the energy of the oscillating system decreases.

Quality factor (Q) at resonance

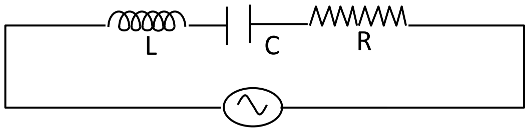
Q measures sharpness of resonance.

Quality factor at resonance is defined as the ratio of amplitude at resonance to the amplitude at zero driving frequency. $Q = \frac{A_{max}}{A_{p=0}} = \frac{f_0/2k\omega_0}{f_0/\omega_0^2} = \boldsymbol{\omega_0\tau}$

. $Q = \frac{\omega_0}{2k} = \frac{\sqrt{\frac{C}{m}}}{\frac{\gamma}{m}} = \frac{\sqrt{Cm}}{\gamma}$ Thus Q factor at resonance depends on the values of C, m and γ .

LCR circuit as an electrical analog of mechanical oscillator

Consider a series LCR circuit applied by an alternating emf, $E = E_0\ sinpt$.



$E = E_0\ sinpt$

Let 'q' be the charge in the conductor, and 'C' be the capacitance, then $V = \frac{q}{C}$

Let $I = \frac{dq}{dt}$ is the current in the circuit, then induced emf in the inductance is $L\frac{dI}{dt}$

According to Ohm's law, V = IR

The sum of potential difference across each circuit is equal to the applied voltage.

$$L\frac{dI}{dt} + IR + \frac{q}{C} = E_0\ sinpt$$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{c} = E_0\ sinpt$$

$\boldsymbol{\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = E_0\ sinpt}$ Here we see that the above voltage equation of LCR circuit is similar to the force equation of driven harmonic oscillator. $\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega_0^2\,x = f_0\,sin\,pt.$

The electric charge oscillates between a capacitor (C) and inductor (L) through resistor (R), similar to the mechanical oscillation of the oscillator. The resistance (R) causes the dissipation of electric energy where as damping causes the dissipation in mechanical oscillator.

Comparison of above two equations yield the following equivalence relations.

Quantity in mechanical oscillator	**Quantity in electrical oscillator**
Mass (m)	Inductor (L)
Displacement (x)	Charge (q)
Velocity v = $\frac{dx}{dt}$	Electric current I = $\frac{dq}{dt}$
Damping coefficient k	Electric resistance R
Force constant (C)	Reciprocal of capacitance $\frac{1}{C}$
Potential energy $\frac{1}{2}Cx^2$	Energy stored in capacitor = $\frac{1}{2}\frac{1}{C}q^2$ = $\frac{1}{2}Cv^2$
Kinetic energy $\frac{1}{2}mv^2$	Energy stored in inductor $\frac{1}{2}LI^2$
Resonant angular frequency, p = $\omega_0 = \sqrt{\frac{C}{m}}$ and Resonant frequency $\upsilon_0 = \frac{1}{2\pi}\sqrt{\frac{C}{m}}$ [since $\nu = \frac{\omega_0}{2\pi}$]	Resonant angular frequency of an LCR circuit $\omega_0 = \sqrt{\frac{1}{LC}}$ Resonant frequency $\upsilon_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$
Quality factor Q = $\omega_0\tau = \frac{\sqrt{Cm}}{\gamma}$	Quality factor $\frac{L\omega_0}{R}$ $\left[\frac{\omega_0}{R/L}\right]$

Potential energy at the extreme position is analogous to energy stored in the capacitor while kinetic energy at the mean position is analogous to energy in the inductor. In mechanical oscillator, energy is switched between potential and kinetic energies while in LCR circuit, electrical charge switched between capacitor and inductor. These results clearly show their close similarity.

WAVES

Wave motion is a form of disturbances which travel through a medium due to the repeated periodic motion of the particles of the medium about their mean positions.

Without transferring matter, only the disturbance is handed over from one particle to the next.

Example: Waves produced when a stone is dropped into a water tank.

There are two types of waves.

Elastic waves or mechanical waves: The waves which require a medium for propagation are known as elastic wave or mechanical wave.

Electromagnetic waves: The waves which do not require any medium for propagation is known as electromagnetic wave.

Types of wave motion

Transverse wave motion

The particles of the medium vibrate about their mean position in a direction perpendicular to the direction of propagation of the wave.

This type of waves travels in the form of crests and troughs.

The distance between adjacent crests or troughs constitutes one wave.

Transverse waves can be polarized.

Example: **light wave, waves in stretched string**.

Longitudinal wave motion

In longitudinal wave motion, particles of the medium vibrate about their mean position in the direction parallel to the direction of propagation of the wave.

This wave type travels in the form of compressions and rarefactions.

The distance between the adjacent compressions or rarefactions constitute one wave.

Longitudinal waves cannot be polarized.

Example: **Sound waves in air**.

In some cases, the waves are neither purely transverse nor purely longitudinal. For examples, ripples or surface waves on water. Again, waves may be one dimensional (transverse waves along a string and longitudinal waves along a spring) two dimensional (ripples on water) and three dimensional (sound waves travelling through space) according as they transport energy in one, two or three directions.

The maximum displacement of the wave is known as the **amplitude (A)** of the wave.

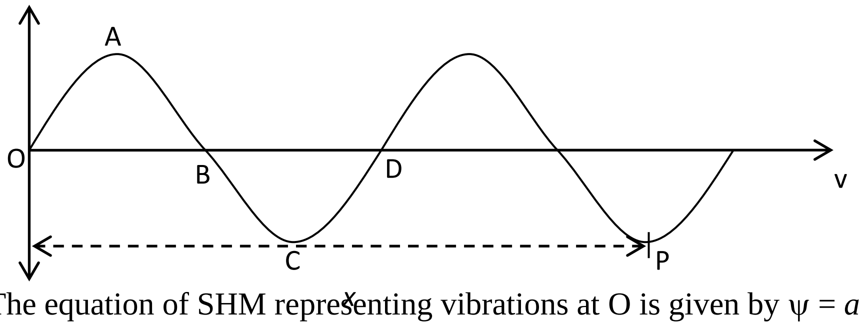
The distance between two consecutive points having the same state of vibration or having the same phase is called **wavelength (λ).**

The time required for the wave to travel a distance of one wavelength is called **period**.

Mathematical description of a wave: Any wave can be mathematically described by a function of position and time coordinate known as wave function (ψ). It describes the displacement of the particles at any position and time.

Equation of a plane progressive harmonic wave

A wave which propagate by transferring energy across any medium is known as progressive wave. Consider a wave travelling towards **positive direction** from the origin O with velocity v.



The equation of SHM representing vibrations at O is given by $\psi = a \sin\omega t$.

Consider a point P at a distance x from O. Let ϕ be the phase difference between vibrations at O and P.

Then the displacement at P, $\psi = a \sin(\omega t - \phi)$

We know that phase difference = $\frac{2\pi}{\lambda} \times$ path difference (i.e., $\phi = \frac{2\pi}{\lambda} \times x$)

So, $\psi = a \sin(\omega t - \frac{2\pi}{\lambda} x)$

Also, $\omega = 2\pi\nu$, then $\boldsymbol{\psi = a \sin(2\pi\nu t - \frac{2\pi}{\lambda} x)}$ [since v = $\nu\lambda$]

i.e., $\psi = a \sin \frac{2\pi}{\lambda} (\text{v}t - x)$ or $\psi = a \sin 2\pi (\frac{t}{T} - \frac{x}{\lambda})$

For a wave travelling towards **negative direction** from the origin O with velocity v, then

$\psi = a \sin(\omega t + \phi)$ or $\boldsymbol{\psi = a \sin \frac{2\pi}{\lambda} (\text{v}t + x)}$

One Dimensional Wave Equation

We have the general equation for harmonic wave travelling along positive direction of x-axis as,

$\psi = a \sin \frac{2\pi}{\lambda} (\text{v}t - x)$ → (1)

Differentiating (1) with respect to t, $\frac{\partial \psi}{\partial t} = a \cos \frac{2\pi}{\lambda} (\text{v}t - x) \times \frac{2\pi \text{v}}{\lambda}$ → (2)

Differentiating (1) with respect to x , $\frac{\partial \psi}{\partial x} = -a \cos \frac{2\pi}{\lambda} (\text{v}t - x) \times \frac{2\pi}{\lambda}$ → (3)

Comparing (2) and (3), we get, $\frac{\partial \psi}{\partial x} = -\frac{1}{\text{v}} \frac{\partial \psi}{\partial t}$ → (4)

Differentiating eqn (4) w.r.t. x, $\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) = -\frac{1}{\text{v}} \times \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial t}\right)$

$$\frac{\partial^2 \psi}{\partial x^2} = -\frac{1}{\text{v}} \times \frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial x}\right) = -\frac{1}{\text{v}} \times \frac{\partial}{\partial t}\left(-\frac{1}{\text{v}} \frac{\partial \psi}{\partial t}\right)$$

$\boldsymbol{\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{\text{v}^2} \frac{\partial^2 \psi}{\partial t^2}}$ This is the **differential equation of wave motion**.

Solution of the one dimensional wave equation

We have the differential equation of wave motion as $\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}$ → (1) where ψ is a function of *x* and *t*. So, $\psi_{(x,t)} = X_{(x)}T_{(t)}$ → (2)

Differentiating (2) twice with respect to *x* and *t*,

We get $\frac{\partial \psi}{\partial x} = T\frac{dX}{dx}$ and $\frac{\partial^2 \psi}{\partial x^2} = T\frac{d^2X}{dx^2}$

$\frac{\partial \psi}{\partial t} = X\frac{dT}{dt}$ and $\frac{\partial^2 \psi}{\partial t^2} = X\frac{d^2T}{dt^2}$

Substituting this in equation (1), $T\frac{d^2X}{d\,x^2} = \frac{1}{v^2} \times X\frac{d^2T}{dt^2}$ → (3)

Rearranging equation (3), $\frac{1}{X}\frac{d^2X}{d\,x^2} = \frac{1}{v^2} \times \frac{1}{T}\frac{d^2T}{dt^2}$ → (4)

Here LHS is a function of *x* and RHS is a function of *t* only.

Hence each side of the equation must be equal to a constant ($-K^2$).

{since change in *x* will not change the right side and change in *t* will not change the left side}

$\frac{1}{X}\frac{d^2X}{dx^2} = -k^2$ ⇒ $\frac{d^2X}{dx^2} + k^2X = 0$ → (5)

$\frac{1}{v^2T}\frac{d^2T}{d\,t^2} = -k^2$ ⇒ $\frac{d^2T}{d\,t^2} + k^2v^2T = 0$ → (6)

Since $k^2v^2 = \omega^2$ $\therefore \frac{d^2T}{dt^2} + \omega^2T = 0$ → (7)

(5) and (7) are the standard differential equation with solutions given as

$X(x) = A\exp(\pm ikx)$ and $T(x) = A\exp(\pm i\omega t)$ where A is a constant.

So equation (2) can be written as $\boldsymbol{\psi_{(x,t)} = A \exp[i(kx \pm \omega t)]}$ Here A is constant, ω is the angular frequency and *k* is the wave vector. This is the solution of one dimensional wave equation.

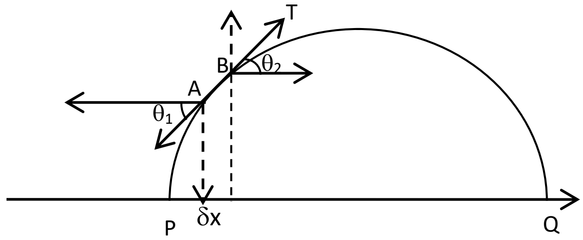
Three Dimensional Wave Equation and its Solution

The differential equation in three dimensional is $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}$

or $\boldsymbol{\nabla^2\psi = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}}$ where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

The solution of this equation is $\boldsymbol{\psi_{(x,y,z,t)} = A \exp[i(\vec{k}.\vec{r} \pm \omega t + \phi)]}$ Here A and ϕ are constants.

Transverse Vibrations of a Stretched String



Consider a flexible uniform string stretched between two points A and B by a constant tension T. Let the string lie along x-axis. Let it be plucked at the centre and is made to vibrate transversally. These vibrations are simple harmonic. Let a small element AB of length δx.

The magnitude of the tension will be same everywhere. (Since the string is perfectly flexible.) The tension T acts tangentially at every point.

At point A, tension T makes an angle θ_1 with horizontal and at point B, tension T makes an angle θ_2 with horizontal.

The net force acting in the y-direction is

$$F = T\sin\theta_2 - T\sin\theta_1 = T(\sin\theta_2 - \sin\theta_1)$$

Since θ is small, $\sin\theta = \tan\theta$

$\therefore$ $F = T(\tan\theta_2 - \tan\theta_1) \quad \rightarrow (1)$

But $\tan\theta_2 = \left(\frac{\partial y}{\partial x}\right)_{x+dx}$ at B (slope at B)

$\tan\theta_1 = \left(\frac{\partial y}{\partial x}\right)_{x}$ at A (slope at A)

$\therefore$ Equation (1) becomes, $F = T\left(\left(\frac{\partial y}{\partial x}\right)_{x+dx} - \left(\frac{\partial y}{\partial x}\right)_{x}\right)$

Applying Taylor's series,

$$F = T\left(\left(\frac{\partial y}{\partial x}\right)_{x+dx} - \left(\frac{\partial y}{\partial x}\right)_{x}\right) = T\frac{\partial^2 y}{\partial x^2}dx$$

If 'm' is the mass per unit length of the string. Then mass of the element = *mdx*

Force acting on the element = mass × acceleration

$$F = mdx \times \frac{\partial^2 y}{\partial t^2} \quad \text{or} \quad T\frac{\partial^2 y}{\partial x^2}dx = mdx \times \frac{\partial^2 y}{\partial t^2}$$

i.e., $\boldsymbol{\frac{\partial^2 y}{\partial x^2} = \frac{m}{T} \times \frac{\partial^2 y}{\partial t^2}}$ This is the wave equation in the case of waves in a stretched string.

Comparing with the one dimensional wave equation $\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}$, we get $\frac{1}{v^2} = \frac{m}{T}$

So, the expression for the velocity of transverse vibrations in a stretched string is velocity $\mathbf{v} = \sqrt{\frac{T}{m}}$

If a string of length 'l' vibrating in 'p' segment , length of each segment be $\frac{l}{p}$ and it corresponds to $\frac{\lambda}{2}$

.i.e., $\frac{l}{p} = \frac{\lambda}{2}$ or $\lambda = \frac{2l}{p}$ Then $\nu = \frac{v}{\lambda} = \frac{\sqrt{T/m}}{2l/P} = \frac{P}{2l}\sqrt{\frac{T}{m}}$

i.e., **frequency** $\boldsymbol{\nu = \frac{P}{2l}\sqrt{\frac{T}{m}}}$

When p = 1, $\nu_1 = \frac{1}{2l}\sqrt{\frac{T}{m}}$

This the fundamental frequency of the transverse vibrations in a stretched string.

When p =2, the string will be vibrating in two segments, then $l = \lambda$, $\nu_2 = \frac{1}{l}\sqrt{\frac{T}{m}}$.

This stage is called second mode of vibration or first overtone.

Laws of transverse vibrations of stretched string

The fundamental frequency $\nu = \frac{1}{2l}\sqrt{\frac{T}{m}}$

Hence (i) Frequency of transverse vibrations in a stretched string is inversely proportional to the length of the stretched string. $\nu \propto \frac{1}{l}$

(ii) Frequency of transverse vibrations in a stretched string is directly proportional to the square root of tension of the string. $\nu \propto \sqrt{T}$

(iii)) Frequency of transverse vibrations in a stretched string is inversely proportional to the square root of mass per unit length of the string. $\nu \propto \frac{1}{\sqrt{m}}$